Finite-time consensus of heterogeneous multi-agent systems*

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We investigate the finite-time consensus problem for heterogeneous multi-agent systems composed of first-order and second-order agents. A novel continuous nonlinear distributed consensus protocol is constructed, and finite-time consensus criteria are obtained for the heterogeneous multi-agent systems. Compared with the existing results, the stationary and kinetic consensuses of the heterogeneous multi-agent systems can be achieved in a finite time respectively. Moreover, the leader can be a first-order or a second-order integrator agent. Finally, some simulation examples are employed to verify the efficiency of the theoretical results.

Keywords: heterogeneous multi-agent system, finite-time consensus, nonlinear consensus protocol

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1. Introduction

As one of the most typical collective behaviors of multi-agent systems, consensus, which means that the outputs of all spatially distributed agents converge to a common desired state by implementing appropriate distributed protocols, has attracted more and more attention from many researchers in various fields, such as physics, artificial intelligence, and automatic control. Consensus algorithms have broad applications in the formation control of autonomous vehicles,[1] flocking,[2] and the rendezvous control of agents.[3]

So far, by using the matrix theory,[4,5] the frequency-domain analysis method,[6,7] the Lyapunov direct method, etc.,[8,9] consensus problems of multi-agent systems have been studied in detail, many consensus algorithms have been proposed, and the consensus criteria have been obtained for the first-order, the second-order, and the high-order multi-agent systems. However, most of the existing results are mainly given for the homogeneous multi-agent systems, in which all the agents have the same dynamics. In real engineering applications, the dynamics of the agents are always different due to various restrictions, but little attention has been paid to the consensus problem of the heterogeneous multi-agent systems which consist of agents with different dynamics. Lee and Spong[10] studied the consensus of continuous time heterogeneous multi agents with non-uniform communication delays, and obtained the delay independent consensus conditions based on a frequency-domain analysis. Liu *et al.*[11] studied discrete-time heterogeneous multi-agent systems, two stationary consensus algorithms were constructed, and the sufficient consensus criteria for the agents with bounded communication delays were obtained based on the properties of non-negative matrices.

Another important topic in the study of the consensus problem is the convergence rate. In most literature, consensus algorithms for multi-agent systems are asymptotic, which means that the convergence rate is at best exponential with an infinite settling time. Besides a faster convergence rate, the system under a finite-time control usually has better disturbance rejection properties.[12,13] Several kinds of finite-time consensus protocols have been developed for the first-order[14–16] and the second-order multi-agent systems.[17,18]

Inspired by these facts, we find that it is significant and necessary to study the finite-time consensus algorithms for heterogeneous multi-agent systems. However, it is worthy to note that the extension of finite-time consensus algorithms from the first-order case to the second-order one is nontrivial, not to mention the difficulty of analyzing the consensus for the heterogeneous multi-agent systems. Although there are some results about heterogeneous systems,[19] to the best of our knowledge, there are no results about the finite-time consensus of the heterogeneous multi-agent systems.

The finite-time consensus of the heterogeneous multi-agent systems is studied in this paper. A novel nonlinear consensus protocol is proposed for solving the consensus problem of the heterogeneous multi-agent systems composed of first-order and second-order agents. Sufficient consensus criteria are obtained. Moreover, the leader does not need to be a first-order integrator agent only as in Ref. [19]. And the stationary and the kinetic consensuses of the heterogeneous multi-agent systems can be respectively achieved in finite time with different initial states.

The following notations will be used throughout this paper. The $\mathbf{R}$ and $\mathbf{R}^+$ stand for the sets of real number and positive real number, respectively, $\mathbf{R}^n$ denotes the n-dimensional

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real vector space, $\mathbf{R}^{n\times n}$ is the set of $n\times n$ matrices. and $\mathbf{l}_n$ ($\mathbf{0}_n$) is a vector with all its elements being one (zero).

2. Preliminaries and lemmas

2.1. Graph theory

For multi-agent systems, we assume that each agent is a node, and the information exchange among n agents can be modeled by a undirected weighted graph $G=\{V, E, \boldsymbol{A}\}$, where the node indexes belong to a finite index set $\Gamma=\{1, 2, \ldots, n\}$, $V=\{v_i|i\in\Gamma\}$ is the set of agents, $E\subseteq V\times V$ is the set of edges, and $\boldsymbol{A}$ is the corresponding weighted adjacency matrix. The adjacency matrix $\boldsymbol{A}=[a_{ij}]\in\mathbf{R}^{n\times n}$ is defined such that $a_{ij}>0$ if $(v_j, v_i)\in E$, $a_{ij}=0$ if $(v_j, v_i)\notin E$, and $a_{ii}=0$ for all $i\in\Gamma$. The set of neighbors of agent v_i is denoted as $N_i=\{v_j:\ (v_j, v_i)\in E\}$. The degree of agent v_i is defined as $\deg(v_i)=d_i=\sum_{j=1}^{n}a_{ij}=\sum_{j\in N_i}a_{ij}$. Then the degree matrix of graph G is $\boldsymbol{D}=\mathrm{diag}\{d_1, \ldots, d_n\}$, and the Laplacian matrix is $\boldsymbol{L}=\boldsymbol{D}-\boldsymbol{A}$.

2.2. Some lemmas and the Lyapunov theory for finite-time stability

Lemma 1[20] Suppose function φ: $\mathbf{R}^2\to\mathbf{R}$ satisfies $\varphi(x_i, x_j)=-\varphi(x_j, x_i)$, $i,\ j\in\Gamma$, $i\neq j$. Then for any undirected graph G and a set of numbers $y_1, y_2, \ldots, y_N$,

$$\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}y_i\varphi(x_j, x_i)=-\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ij}(y_j-y_i)\varphi(x_j, x_i).$$

Lemma 2[21] For $x_i\in\mathbf{R}$, $i=1, \ldots, n$, $0<p\leq 1$,

$$\left(\sum_{i=1}^{n}|x_i|\right)^p\leq\sum_{i=1}^{n}|x_i|^p\leq n^{1-p}\left(\sum_{i=1}^{n}|x_i|\right)^p.$$

Lemma 3 Suppose function φ: $\mathbf{R}^2\to\mathbf{R}^+$ satisfies $\varphi(x_i, x_j)=\varphi(x_j, x_i)$, $i,\ j\in\Gamma$, $i\neq j$, then for any undirected graph G and a set of numbers $y_1, y_2, \ldots, y_N$,

$$\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}|y_i|\varphi(x_j, x_i)\geq\frac{1}{2}\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}|y_j-y_i|\varphi(x_j, x_i).$$

Proof From the definition of the undirected graph and the assumption, we can obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}|y_i|\varphi(x_j, x_i)\\
&=\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ij}|y_i|\varphi(x_j, x_i)+\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ij}|y_i|\varphi(x_j, x_i)\\
&=\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ij}|y_i|\varphi(x_j, x_i)+\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ji}|y_j|\varphi(x_i, x_j)\\
&=\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ij}|y_i|\varphi(x_j, x_i)+\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ij}|y_j|\varphi(x_j, x_i)\\
&=\frac{1}{2}\sum_{(v_i, v_j)\in E}a_{ij}(|y_i|+|y_j|)\varphi(x_j, x_i)\\
&\geq\frac{1}{2}\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}|y_j-y_i|\varphi(x_j, x_i).
\end{aligned}$$

Lemma 4[22] For a connected undirected graph G, the Laplacian matrix $\boldsymbol{L}$ of G has the property

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{x}=\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(x_j-x_i)^2=\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i}a_{ij}(x_j-x_i)^2$$

for any $\boldsymbol{x}=[x_1, \ldots, x_n]^{\mathrm{T}}\in\mathbf{R}^n$, which implies that $\boldsymbol{L}$ is positive semi-definite. And 0 is a simple eigenvalue of $\boldsymbol{L}$ with $\mathbf{1}_n$ being the associated eigenvector. Assume that the eigenvalues of $\boldsymbol{L}$ are denoted by $0, \lambda_2, \ldots, \lambda_n$ satisfying $0\leq\lambda_2\leq\cdots\leq\lambda_n$, then the second smallest eigenvalue $\lambda_2>0$. Furthermore, if $\mathbf{1}_n^{\mathrm{T}}\boldsymbol{x}=0$, then $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{x}\geq\lambda_2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$.

It is well-known that a sufficient condition for the existence of a unique solution of a nonlinear differential equation $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})$ is that the function $\boldsymbol{f}(\boldsymbol{x})$ is locally Lipschitz continuous. The solution of such a nonlinear differential equation can have at most an asymptotic convergence rate. Since the finite-time stability guarantees that every system state reaches the system origin in a finite time, it has a much stronger requirement than the asymptotic stability. The following lemma presents sufficient conditions for the finite-time stability.

Lemma 5[23] For the non-Lipschitz continuous nonlinear system $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{f}(0)=0$, $\boldsymbol{x}(0)=\boldsymbol{x}_0$, $\boldsymbol{x}\in\mathbf{R}^n$, there exist a positive definite continuous function $V(\boldsymbol{x}):\ U\to\mathbf{R}$, real numbers $c>0$ and $\alpha\in(0, 1)$, and an open neighborhood $U_0\subset U$ of the origin such that $\dot{V}(\boldsymbol{x})+c(V(\boldsymbol{x}))^{\alpha}\leq 0$, $\boldsymbol{x}\in U_0\backslash\{0\}$. Then $V(\boldsymbol{x})$ approaches 0 in a finite time. In addition, the finite settling time T satisfies $T(\boldsymbol{x}_0)\leq(V(\boldsymbol{x}_0))^{1-\alpha}/c(1-\alpha)$.

3. Main result

For the simplicity of presentation, we assume that the states of all agents are in a one-dimensional space. However, the results of this paper are still valid for multiple high-dimensional agents by the introduction of the Kronecker product and appropriately rewriting the protocol.

Suppose that the heterogeneous multi-agent system consists of first-order and second-order integrator agents. The number of agents is n, and the number of the second-order integrator agents is m, $(m<n)$. Without loss of generality, we assume that the second-order integrator agents are labeled from 1 to m, and the dynamics are given by

$$\begin{cases}\dot{x}_i(t)=v_i(t),\\ \dot{v}_i(t)=u_i(t),\ \ i\in\Gamma_m,\end{cases}\tag{1}$$

where $x_i\in\mathbf{R}$, $v_i\in\mathbf{R}$, and $u_i\in\mathbf{R}$ are the position, the velocity, and the control input of agent i ($i\in\Gamma_m$, $\Gamma_m=\{1, 2, \ldots, m\}$),

respectively. The first-order agent dynamics is given as follows:

$$\dot{x}_i(t) = u_i(t), \quad i \in \Gamma_{n-m}, \tag{2}$$

where $x_i \in \mathbf{R}$ and $u_i \in \mathbf{R}$ are the position and the control input of agent i ($i \in \Gamma_{n-m}$, $\Gamma_{n-m} = \{m+1, m+2, \ldots, n\}$), respectively.

The dynamics of the leader indexed by 0 is described as follows. If the leader is a second-order integrator agent,

$$\begin{cases} \dot{x}_0 = v_0, & x_0 \in \mathbf{R}, \\ \dot{v}_0 = u_0, & v_0 \in \mathbf{R}, \end{cases}$$

if the leader is a first-order integrator agent,

$$\dot{x}_0 = u_0, \quad x_0 \in \mathbf{R}.$$

Remark 1 Compared with Ref. [11], in which the heterogeneous multi-agent system asymptotically converges to a stationary consensus, we consider a leader–follower heterogeneous multi-agent system, and the leader can be a kinetic agent or a stationary one. Moreover, all the followers can reach a consensus with the leader in a finite time. This means that the situation in this paper is more closed to the engineering reality.

Remark 2 Compared with Ref. [19], in which the leader needs to be a first-order integrator agent, the leader in this paper can be a first-order integrator agent or a second-order one. This will be shown in the simulation results in Section 4.

In this paper, for simplicity, we assume that the pinned gain of agent i is denoted by $\boldsymbol{B}$ and $\boldsymbol{C}$, where

$$\boldsymbol{B} = \mathrm{diag}\{b_1, b_2, \ldots, b_m\}, \quad \boldsymbol{C} = \mathrm{diag}\{c_{m+1}, c_{m+2}, \ldots, c_n\},$$

with

$$b_i = \begin{cases} 1, & \text{if agent } i \text{ is connected to the leader,} \\ 0, & \text{otherwise,} \end{cases} \quad i \in \Gamma_m,$$

$$c_i = \begin{cases} 1, & \text{if agent } i \text{ is connected to the leader,} \\ 0, & \text{otherwise,} \end{cases} \quad i \in \Gamma_{n-m}.$$

We design the nonlinear consensus protocol for the heterogeneous multi-agent system (1), (2) as follows:

$$u_i(t) = \begin{cases} -\Big(\sum\limits_{j\in N_i} a_{ij}\psi\left(\left|x_j - x_i\right|\right)\left|x_j - x_i\right|^{\alpha-1} \\ \quad + \sum\limits_{j\in N_i} a_{ij}\psi\left(\left|v_j - v_i\right|\right)\left|v_j - v_i\right|^{\alpha-1} \\ \quad + b_i\left|v_i - v_0\right|\Big)\mathrm{sign}\left(x_i - x_0 + v_i - v_0\right) \\ \quad + \dot{v}_0 - \left(v_i - v_0\right), \quad i \in \Gamma_m, \\ \sum\limits_{j\in N_i} a_{ij}\psi\left(x_j - x_i\right)\left|x_j - x_i\right|^{\alpha-1} \\ \quad + \dot{x}_0 - c_i\left(x_i - x_0\right), \qquad i \in \Gamma_{n-m}, \end{cases} \tag{3}$$

where $N_i = N_i^{\mathrm{f}} \cup N_i^{\mathrm{s}}$, N_i^{f} and N_i^{s} are the first-order and the second-order neighboring agents of agent i, respectively. And the interaction function satisfies the following assumption.

Assumption 1 Function $\psi(y)$ satisfies $y \cdot \psi(y) > 0$, $\forall y \in \mathbf{R}\backslash\{0\}$, $\psi(0) = 0$. And $\psi(y) \geq \beta \cdot y^q$, $\forall y \in \mathbf{R}$, where $\beta > 0$, $q = q_1/q_2 < 1$, and q_1 and q_2 are positive odd integers.

Remark 3 To achieve the goal of consensus to the leader's state, only a small fraction of agents (at least one agent) are required to know the leader's state. However, the consensus protocols in Ref. [19] indicate that each agent needs to know the state information of all the other agents.

Remark 4 For the simplicity of presentation, we assume that the leader is a second-order integrator agent in nonlinear consensus protocol (3). Certainly, the results are still valid for the case that the leader is a first-order integrator agent (letting v_0 be zero in Eq. (3)).

Theorem 1 Suppose that the graph G is undirected and connected, and $\max\{1, -q\} < \alpha < 2 - q$, then the heterogeneous multi-agent system (1), (2) can achieve a consensus in a finite time with the nonlinear consensus protocol (3).

Proof Define the error vector

$$\bar{x}_i = x_i - x_0, \quad (i \in \Gamma), \quad \bar{v}_i = v_i - v_0, \quad (i \in \Gamma_m). \tag{4}$$

According to Eqs. (1)–(4), we have

$$\begin{cases} \dot{\bar{x}}_i(t) = \bar{v}_i(t), & i \in \Gamma_m, \\ \dot{\bar{v}}_i(t) = -\Big(\sum\limits_{j\in N_i} a_{ij}\psi\left(\left|\bar{x}_j - \bar{x}_i\right|\right)\left|\bar{x}_j - \bar{x}_i\right|^{\alpha-1} \\ \quad + \sum\limits_{j\in N_i} a_{ij}\psi\left(\left|\bar{v}_j - \bar{v}_i\right|\right)\left|\bar{v}_j - \bar{v}_i\right|^{\alpha-1} + b_i\left|\bar{v}_i\right|\Big) \\ \quad \times \mathrm{sign}\left(\bar{x}_i + \bar{v}_i\right) - \bar{v}_i, \quad i \in \Gamma_m, \\ \dot{\bar{x}}_i(t) = \sum\limits_{j\in N_i} a_{ij}\psi\left(\bar{x}_j - \bar{x}_i\right)\left|\bar{x}_j - \bar{x}_i\right|^{\alpha-1} - c_i\bar{x}_i, \quad i \in \Gamma_{n-m}. \end{cases} \tag{5}$$

Take the following Lyapunov function for Eq. (5):

$$V = \frac{1}{2}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S} = \frac{1}{2}\sum_{i\in\Gamma} s_i^2 = \frac{1}{2}\sum_{i\in\Gamma_m} s_i^2 + \frac{1}{2}\sum_{i\in\Gamma_{n-m}} s_i^2,$$

where $\boldsymbol{S} = (s_1, s_2, \ldots, s_n)^{\mathrm{T}}$, $s_i = \bar{x}_i + \bar{v}_i$, $i \in \Gamma$, and $\bar{v}_i = 0$, $(i \in \Gamma_{n-m})$, that is

$$s_i = \begin{cases} \bar{x}_i + \bar{v}_i, & i \in \Gamma_m, \\ \bar{x}_i, & i \in \Gamma_{n-m}. \end{cases}$$

Now consider the time derivative of V along the trajectory of the heterogeneous system, we have

$$\begin{aligned} \dot{V} &= \sum_{i\in\Gamma_m} s_i\dot{s}_i + \sum_{i\in\Gamma_{n-m}} s_i\dot{s}_i \\ &= \sum_{i\in\Gamma_m} s_i\left(\bar{v}_i + \dot{\bar{v}}_i\right) + \sum_{i\in\Gamma_{n-m}} \bar{x}_i\dot{\bar{x}}_i \\ &= -\sum_{i\in\Gamma_m}\left|s_i\right|\Bigg(\sum_{j\in N_i} a_{ij}\psi\left(\left|\bar{x}_j - \bar{x}_i\right|\right)\left|\bar{x}_j - \bar{x}_i\right|^{\alpha-1} \\ &\quad + \sum_{j\in N_i} a_{ij}\psi\left(\left|\bar{v}_j - \bar{v}_i\right|\right)\left|\bar{v}_j - \bar{v}_i\right|^{\alpha-1} + b_i\left|\bar{v}_i\right|\Bigg) \\ &\quad + \sum_{i\in\Gamma_{n-m}} \bar{x}_i\left(\sum_{j\in N_i} a_{ij}\psi\left(\bar{x}_j - \bar{x}_i\right)\left|\bar{x}_j - \bar{x}_i\right|^{\alpha-1} - c_i\bar{x}_i\right). \end{aligned}$$

From Assumption 1 and Lemma 1, we have

$$\begin{aligned}\dot{V} &\le -\beta\sum_{i\in\Gamma_m}|s_i|\left(\sum_{j\in N_i}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha-1+q}+\sum_{j\in N_i}a_{ij}\left|\bar{v}_j-\bar{v}_i\right|^{\alpha-1+q}\right)-\frac{\beta}{2}\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha+q}-\sum_{i\in\Gamma_m}b_i|s_i|\,|\bar{v}_i|-\sum_{i\in\Gamma_{n-m}}c_i\bar{x}_i^2\\ &\le 0,\end{aligned}$$

which means that V is decreasing. Since $N_i=N_i^{\mathrm{f}}\cup N_i^{\mathrm{s}}$, from Lemma 2, we have

$$\begin{aligned}\dot{V} &\le -\beta\sum_{i\in\Gamma_m}|s_i|\left(\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha-1+q}+\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha-1+q}\right.\\ &\quad\left.+\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|0-\bar{v}_i\right|^{\alpha-1+q}+\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{v}_j-\bar{v}_i\right|^{\alpha-1+q}\right)\\ &\quad-\sum_{i\in\Gamma_m}b_i|s_i|\,|\bar{v}_i|-\frac{\beta}{2}\sum_{i\in\Gamma_{n-m}}\left(\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha+q}+\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha+q}\right)-\sum_{i\in\Gamma_{n-m}}c_i\bar{x}_i^2\\ &\le -\beta\sum_{i\in\Gamma_m}|s_i|\left(\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha-1+q}+\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|0-\bar{v}_i\right|^{\alpha-1+q}\right)-\beta\sum_{i\in\Gamma_m}|s_i|\left(\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha-1+q}+\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{v}_j-\bar{v}_i\right|^{\alpha-1+q}\right)\\ &\quad-\frac{\beta}{2}\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha+q}-2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{x}_j+\bar{v}_j-\bar{x}_i\right|^{\alpha+q}-2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|-\bar{v}_j\right|^{\alpha+q}\\ &\quad-\sum_{i\in\Gamma_m}b_i|s_i|\,|\bar{v}_i|-\sum_{i\in\Gamma_{n-m}}c_i\bar{x}_i^2\\ &\le -\beta\sum_{i\in\Gamma_m}|s_i|\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left(\left|\bar{x}_j-\bar{x}_i\right|+\left|0-\bar{v}_i\right|\right)^{\alpha-1+q}-\beta\sum_{i\in\Gamma_m}|s_i|\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left(\left|\bar{x}_j-\bar{x}_i\right|+\left|\bar{v}_j-\bar{v}_i\right|\right)^{\alpha-1+q}\\ &\quad-\frac{\beta}{2}\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha+q}-2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{x}_j+\bar{v}_j-\bar{x}_i\right|^{\alpha+q}\\ &\le -\beta\sum_{i\in\Gamma_m}|s_i|\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|\bar{x}_j-\bar{x}_i+0-\bar{v}_i\right|^{\alpha-1+q}-\beta\sum_{i\in\Gamma_m}|s_i|\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{x}_j-\bar{x}_i+\bar{v}_j-\bar{v}_i\right|^{\alpha-1+q}\\ &\quad-\frac{\beta}{2}\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|\bar{x}_j-\bar{x}_i\right|^{\alpha+q}-2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|\bar{x}_j+\bar{v}_j-\bar{x}_i\right|^{\alpha+q}\\ &\le -\beta\sum_{i\in\Gamma_m}|s_i|\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|s_j-s_i\right|^{\alpha-1+q}-\beta\sum_{i\in\Gamma_m}|s_i|\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|s_j-s_i\right|^{\alpha-1+q}-\frac{\beta}{2}\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{f}}}a_{ij}\left|s_j-s_i\right|^{\alpha+q}\\ &\quad-2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i^{\mathrm{s}}}a_{ij}\left|s_j-s_i\right|^{\alpha+q}.\end{aligned}$$

Using Lemmas 2 and 3, we have

$$\begin{aligned}\dot{V} &\le -\beta\sum_{i\in\Gamma_m}|s_i|\sum_{j\in N_i}a_{ij}\left|s_j-s_i\right|^{\alpha-1+q}\\ &\quad-2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i}a_{ij}\left|s_j-s_i\right|^{\alpha+q}\\ &\le -\frac{\beta}{2}\sum_{i\in\Gamma_m}\sum_{j\in N_i}a_{ij}\left|s_j-s_i\right|^{\alpha+q}\\ &\quad-2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma_{n-m}}\sum_{j\in N_i}a_{ij}\left|s_j-s_i\right|^{\alpha+q}\\ &= -2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma}\sum_{j\in N_i}a_{ij}\left|s_j-s_i\right|^{\alpha+q}\\ &= -2^{\frac{\alpha+q}{2}-2}\beta\sum_{i\in\Gamma}\sum_{j\in N_i}\left[a_{ij}^{\frac{2}{\alpha+q}}\left(s_j-s_i\right)^2\right]^{\frac{\alpha+q}{2}}\\ &\le -2^{\frac{\alpha+q}{2}-2}\beta\left[\sum_{i\in\Gamma}\sum_{j\in N_i}a_{ij}^{\frac{2}{\alpha+q}}\left(s_j-s_i\right)^2\right]^{\frac{\alpha+q}{2}}.\end{aligned}$$

Let $\bar{\boldsymbol{A}}=[a_{ij}^{2/(\alpha+q)}]_{n\times n}\in\mathbf{R}^{n\times n}$, $\boldsymbol{L}_{\bar{\boldsymbol{A}}}$ is the Laplacian matrix of graph $G(\bar{\boldsymbol{A}})$, $\lambda_2(L_{\bar{\boldsymbol{A}}})$ is the second smallest eigenvalue of $\boldsymbol{L}_{\bar{\boldsymbol{A}}}$, and $\lambda=\lambda_2(L_{\bar{A}})$. From Lemma 4, we have

$$\begin{aligned}\dot{V} &\le -2^{\frac{\alpha+q}{2}-2}\beta\left(2\boldsymbol{S}^{\mathrm{T}}\boldsymbol{L}_{\bar{\boldsymbol{A}}}\boldsymbol{S}\right)^{\frac{\alpha+q}{2}}\\ &\le -2^{\frac{\alpha+q}{2}-2}\beta\left(4\lambda_2(\boldsymbol{L}_{\bar{\boldsymbol{A}}})V\right)^{\frac{\alpha+q}{2}}\\ &\le -2^{3(\alpha+q)/2-2}\beta\lambda^{\frac{\alpha+q}{2}}V^{\frac{\alpha+q}{2}}.\end{aligned}$$

Then, we obtain

$$\dot{V}+2^{3(\alpha+q)/2-2}\beta\lambda^{\frac{\alpha+q}{2}}V^{\frac{\alpha+q}{2}}\le 0.$$

Consequently, the consensus of the heterogeneous multi-agent system can be achieved in a finite time according to Lemma 5. Moreover, we can obtain that the finite settling time satisfies the inequality

$$T\le\frac{V(0)^{1-\frac{\alpha+q}{2}}}{2^{3(\alpha+q)/2-2}\beta\lambda^{\frac{\alpha+q}{2}}\left(1-\frac{\alpha+q}{2}\right)}.$$

The proof is completed.

4. Simulation examples

In this section, we provide simulation examples to illustrate the effectiveness of our theoretical results.

Here we consider heterogeneous multi-agent systems with one leader indexed by 0 and six followers indexed from 1 to 6. Suppose that vertices 1–3 are the second-order integrator agents and vertices 4–6 are the first-order integrator agents.

Example 1 The communication topology is given in Fig. 1. The leader is a second-order agent whose dynamics is described by

$$\begin{cases} \dot{x}_0 = v_0, \ \ x_0 \in \mathbf{R}, \\ \dot{v}_0 = u_0, \ \ v_0 \in \mathbf{R}. \end{cases} \tag{6}$$

We assume that the initial state is $\boldsymbol{x}(0) = (0,\ 1,\ 3,\ 1.5,\ -1,\ 2)^{\mathrm{T}}$ with $\boldsymbol{v}(0) = (-1,\ 0,\ 1)^{\mathrm{T}}$. Suppose that $\psi(y) = \mathrm{e}^{y}\mathrm{sign}(y)$ and $\alpha = 1.5$.

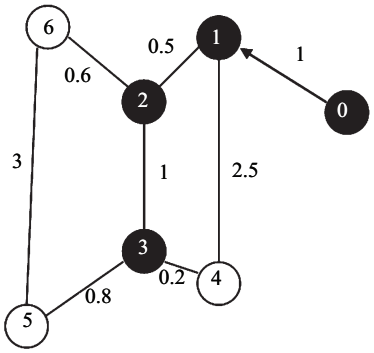


Fig. 1. Communication topology with a second-order leader.

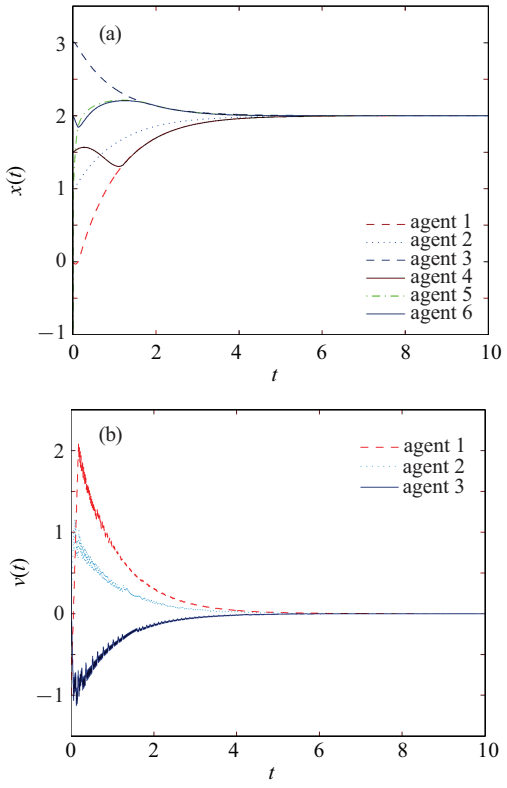


Fig. 2. (color online) Simulation results of (a) agent positions and (b) agent velocities with consensus protocol (3) and $x_0 = 2$, $v_0 = 0$.

Figures 2 and 3 show the finite-time consensuses of the heterogeneous multi-agent system (1), (2) with the nonlinear consensus protocol (3) for $x_0 = 2$, $v_0 = 0$ and $x_0 = \sin 0.1t$, $v_0 = 0.1\cos 0.1t$, respectively.

Example 2 The communication topology is given in Fig. 4. The leader is a first-order agent whose dynamics is described by

$$\dot{x}_0 = u_0, \ \ x_0 \in \mathbf{R}. \tag{7}$$

The same assumption of example 1 is used. Figures 5 and 6 show the finite-time consensuses of the heterogeneous multi-agent system (1), (2) with the nonlinear consensus protocol (3) for $x_0 = 2.5$ and $x_0 = \sin 0.1t$, respectively.

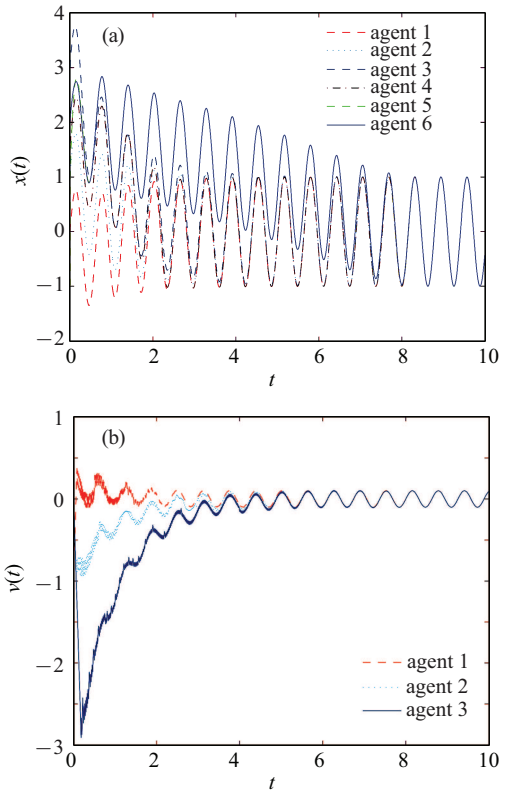


Fig. 3. (color online) Simulation results of (a) agent positions and (b) agent velocities with consensus protocol (3) and $x_0 = \sin 0.1t$, $v_0 = 0.1\cos 0.1t$.

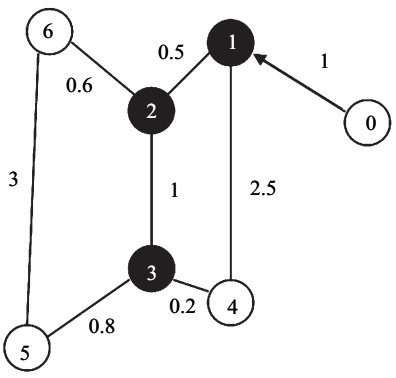


Fig. 4. Communication topology with a first-order leader.

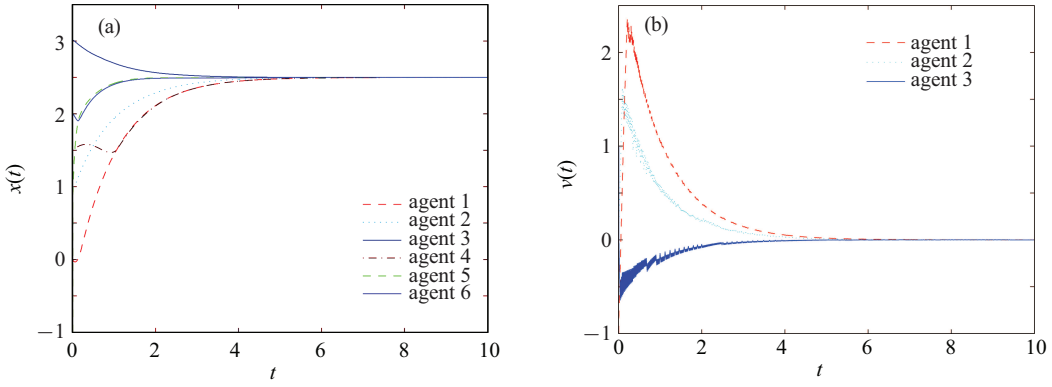


Fig. 5. (color online) Simulation results of (a) agent positions and (b) agent velocities with consensus protocol (3) and $x_0 = 2.5$.

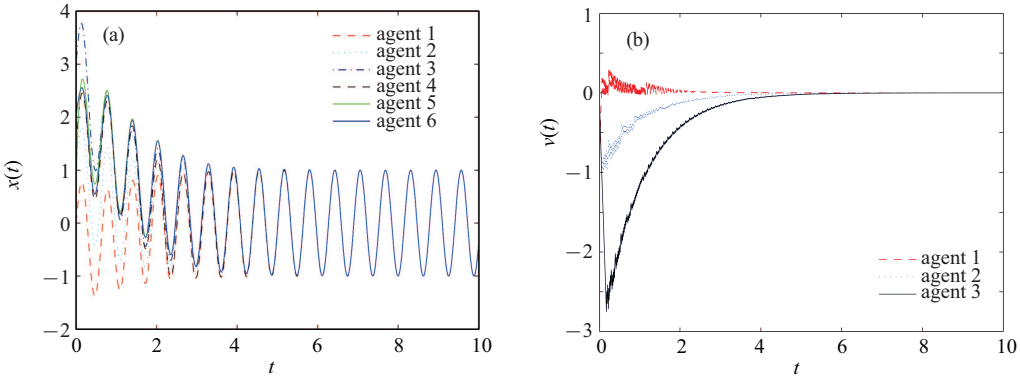


Fig. 6. (color online) Simulation results of (a) agent positions and (b) agent velocities with consensus protocol (3) and $x_0 = \sin 0.1t$.

From these two examples, we know that the finite-time consensus problem of the heterogeneous multi-agent system (1), (2) with the nonlinear consensus protocol (3) can be solved no matter whether the leader is a second-order agent or a first-order one. Figures 2 and 5 respectively show that the stationary consensuses of the heterogeneous multi- agent systems with a second-order leader and a first-order leader can be solved in a finite time. From Figs. 3 and 6, we know that the heterogeneous multi-agent system can converge to its leader when the leader is a kinetic second-order agent or a kinetic first-order agent.

5. Conclusion

In this paper, we discuss the finite-time consensus problem of the heterogeneous multi-agent systems that are composed of first-order and second-order integrator agents. By using a novel nonlinear protocol, sufficient criteria are obtained, which ensure the heterogeneous multi-agent systems to reach consensuses with their leaders in a finite time. The leader in this paper can be a first-order agent or a second-order agent, and the leader can also be a stationary one or a kinetic one. Simulation results are given to illustrate the effectiveness of our theoretical results.

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