Ballistic diffusion induced by non-Gaussian noise*

Qin Li(覃 莉)† and Li Qiang(李 强)

College of Science, Northwest A&F University, Yangling 712100, *China*

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In this letter, we have analyzed the diffusive behavior of a Brownian particle subject to both internal Gaussian thermal and external non-Gaussian noise sources. We discuss two time correlation functions $C(t)$ of the non-Gaussian stochastic process, and find that they depend on the parameter q, indicating the departure of the non-Gaussian noise from Gaussian behavior: for $q \leq 1$, $C(t)$ is fitted very well by the first-order exponentially decaying curve and approaches zero in the long-time limit, whereas for $q > 1$, $C(t)$ can be approximated by a second-order exponentially decaying function and converges to a non-zero constant. Due to the properties of $C(t)$, the particle exhibits a normal diffusion for $q \leq 1$, while for $q > 1$ the non-Gaussian noise induces a ballistic diffusion, i.e., the long-time mean square displacement of the free particle reads $\langle [x(t) - \langle x(t) \rangle]^2 \rangle \propto t^2$.

Keywords: non-Gaussian noise, ballistic diffusion, correlation function

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1. Introduction

In recent years, there is a growing interest in studying processes that occur in disordered media and other systems which show anomalous diffusive behaviors.[1] Diffusion is one of the most fundamental mechanisms for the transport of energy, mass, and information. Physically, it is the process in which many systems reach uniformity and equilibrium. Due to its importance, it has been the focus of extensive research in many different disciplines of natural science. Diffusive dynamics are usually analyzed using the mean square displacement of the particles, which behaves in general as

$$\langle [x(t) - \langle x(t) \rangle]^2 \rangle_{t \longrightarrow \infty} \propto t^{\alpha}, \tag{1}$$

where $\langle \cdots \rangle$ is an ensemble average. The exponent α classifies different types of diffusion: subdiffusion for $0 < \alpha < 1$, normal diffusion for $\alpha = 1$, and superdiffusion for $1 < \alpha \leq 2$. For $\alpha = 2$, the process is called ballistic diffusion, or fast superdiffusion, which has been proved to be a nonergodic process.[2] Superdiffusion has been observed in a number of systems ranging from early discoveries in intermittent chaotic systems, fluid particles in fully developed turbulence, and millennial climate changes.[3] Various models and mechanisms have been proposed to explain the deviation from normal diffusion. For example, Morgado *et al.*[4] modified the spectral density of the thermal bath by removing the low-frequency part of the acoustic modes, leading to a strong superdiffusion. Most of the models of noise-induced superdiffusion, so far, have dealt with Gaussian white or colored noise sources.[5–11]

In this work, we focus on the variation of diffusive behavior of a particle induced by non-Gaussian noise as well as Gaussian white noise in an over-damped limit. There is a growing interest in studying dynamic systems driven by non-Gaussian noise in recent years. This is motivated by the fact that non-Gaussian noise with random amplitudes following the power-law distribution is quite ubiquitous in natural phenomena.[12] For example, experimental results for crayfish and rat skins offer a strong indication that there could be non-Gaussian noise in these sensory systems.[13,14] A simple mechanism has been proposed to generate non-Gaussian noise.[15] Studies on the role of non-Gaussian noise on several "noise-induced phenomena",[16–25] like stochastic resonance, resonant trapping, and noise-induced transitions, have shown the existence of strong effects on the system's response. Moreover, in many physical and biological systems, a deviation of real statistics of fluctuations from Gaussian law, leading to anomalous diffusion, is observed.[1,6,26] These studies motivated us to analyze the diffusive behavior of a particle driven by non-Gaussian noise.

This paper is organized as follows. Section 2 considers a stochastic system driven by an internal Gaussian thermal noise and an external non-Gaussian noise source η, in which η is generated by the specific function which was proposed by Borland,[15,27] and has been adopted in several studies.[18,28–31] The departure from Gaussian behavior is controlled by parameter q: for $q = 1$ we have a Gaussian distribution, and different non-Gaussian distributions for $q > 1$ or $q < 1$. Then we numerically obtain two time correlation functions of the process η. In Section 3, we theoretically analyze the diffusive behavior of the particle, and then continue discussing numerical simulations that support and extend the theoretical results. Finally, in Section 4, we draw some general conclusions.

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†Corresponding author. E-mail: qinli@nwsuaf.edu.cn

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2. Model and process of η

To begin with, we consider a Brownian particle of mass m driven by random forces $\xi(t)$ and $\eta(t)$, whose motion can be described by the following Langevin equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -m\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \xi(t) + \eta(t), \tag{2}$$

where x is the coordinate of the particle, γ the friction constant, and $\xi(t)$ corresponds to the Gaussian thermal noise of a zero mean and its two time correlation is given by

$$\langle \xi(t)\xi(t')\rangle = 2\gamma k_{\mathrm{B}}T\delta(t-t'), \tag{3}$$

where k_{B} is the Boltzmann constant, and T is the temperature of the heat bath. In what follows, we will use dimensionless units $k_{\mathrm{B}} = m = 1$. By use of the remaining noise term $\eta(t)$ in Eq. (2), we have considered the collective effect of the non-thermal environment.[32] We assume that its two time correlation is independent of damping in the thermal bath and therefore Eq. (2) describes a thermodynamically open system. However, by the non thermal environment we mean the degrees of freedom in the complex system which strongly interact with the tagged particle. It should be noted that the non-thermal environment is also embedded in the same thermal environment. Due to non-linear dynamics, the non-thermal noise may be non-Gaussian and colored in characteristics. We assume that both the noises $\xi(t)$ and $\eta(t)$ are stationary and $\xi(t)$ is independent of $\eta(t)$, so that $\langle \xi(t)\eta(t)\rangle = 0$.

The colored non-Gaussian noise can be generated from the solution of the following Langevin equation:[15]

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = -\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta) + \frac{\sqrt{2D}}{\tau}\zeta(t), \tag{4}$$

where D is related to the the intensity of the noise, τ is the correlation time, $\zeta(t)$ is a stand Gaussian white noise having characteristics given by $\langle\zeta(t)\rangle = 0$ and $\langle\zeta(t)\zeta(t')\rangle = \delta(t-t')$. The potential $V_q(\eta)$ is

$$V_q(\eta) = \frac{D}{\tau(q-1)}\ln\left[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right]. \tag{5}$$

For $q = 1$, the process η coincides with a Gaussian colored noise, namely the Ornstein–Uhlenbeck process (with a correlation time equal to τ), while for $q \neq 1$ it is a non-Gaussian process. As shown in Refs. [18] and [20], the probability distribution function (PDF) of η is only well defined for $q < 3$, whereas for $q \geq 3$ the PDF is not normalizable and cannot be accepted as a true probability function. The final expression for the PDF depends on q

$$P_q(\eta) = \frac{1}{Z_q}\left[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right]^{-1/(q-1)}. \tag{6}$$

Within the range $1 < q < 3$, equation (6) yields a long-tail distribution and decays in the manner of a power law (slower than a Gaussian distribution). For $q < 1$, the sign inside the brackets is changed, and the PDF has a bounded support with a cutoff at $|\eta| = \eta_{\mathrm{c}} \equiv \sqrt{2D/\tau(1-q)}$. The Z_q is a normalization constant.

The distribution $P_q(\eta)$ is an even function, and therefore the mean value is $\langle\eta\rangle = 0$. It can be easily verified that the second moment $\langle\eta^2\rangle$ is finite for $q < 5/3$, and diverges for $q \in [5/3, 3)$. More precisely, it is[20]

$$\langle\eta^2\rangle = \begin{cases} \dfrac{2D}{\tau(5-3q)}, & q < \dfrac{5}{3}, \\ \infty, & q \in [\dfrac{5}{3}, 3). \end{cases} \tag{7}$$

Based on the properties of the noise η summarized above, in our following analysis, we shall keep $q < 5/3$.

Now in order to determine the diffusive behavior of the system, we consider two time correlation function of the stochastic process $C(t) = \langle\eta(t+t')\eta(t')\rangle$, in the stationary regime $t' \to \infty$. In the case $q = 1$, process η is an Ornstein–Uhlenbeck noise, and the correlation function is easily obtained as[32]

$$C(t)_{q=1} = \frac{D}{\tau}\exp\left(-\frac{t}{\tau}\right). \tag{8}$$

In the case $q \neq 1$, it is not possible to obtain an analytical expression for $C(t)$, but it is not a problem to reach the central result of the present problem since it does not need a specific form of two time correlation function of the noise used. For analysis of the diffusive behavior of the particle, one may use the exact correlation function based on the numerical solution of Eq. (2).

By using the Monte Carlo method, we have observed that first-order exponential decay of the correlations is still valid for $q < 1$. Namely, it can be written as

$$C(t)_{q<1} = \frac{D_{\mathrm{ng}}}{\tau_{\mathrm{ng}}}\exp\left(-\frac{t}{\tau_{\mathrm{ng}}}\right), \tag{9}$$

where

$$D_{\mathrm{ng}} = \frac{4D}{(5-3q)^2}, \tag{10}$$

$$\tau_{\mathrm{ng}} = \frac{2\tau}{5-3q}. \tag{11}$$

where D_{ng} is the associated effective multiplicative noise intensity, and τ_{ng} the effective noise correlation time. It is obvious that $\tau_{\mathrm{ng}} \to \tau$ and $D_{\mathrm{ng}} \to D$ when $q \to 1$, $\tau_{\mathrm{ng}} < \tau$, and $D_{\mathrm{ng}} < D$ for $q < 1$. It should be noted here that the above approximation is consistent with the prediction which was based on the Path Integral approach.[18]

In Fig. 1, $C(t)$ for different values q when $q \leq 1$ is plotted (the dotted curve corresponds to the numerical result and the

solid curve shows the fitted function). It can be seen that $C(t)$ fits with the first-order exponentially decaying curve very well. Figure 1 also shows that the pre-exponential factor increases with increasing q, and is much greater for Gaussian noise than for non-Gaussian noise, which implies that the noise strength of non-Gaussian noise for $q < 1$ is smaller than that of Gaussian noise.

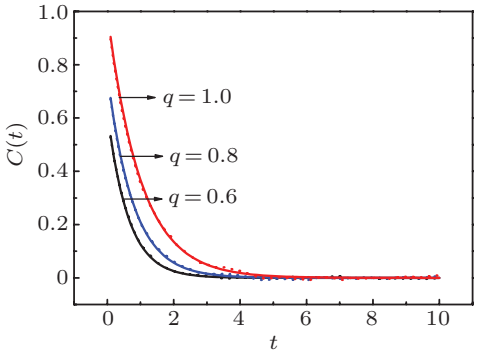


Fig. 1. (color online) Plot of numerically calculated $C(t)$ for different values q when $q \leq 1$. The parameters used are $D = 1.0$ and $\tau = 1.0$.

First-order exponential decay of the correlations fails for $q > 1$. On the other hand, $C(t)$ can be approximated by a second-order exponentially decaying function. Namely,

$$C(t)_{q>1} = A_0 + A_1 \exp\left(-\frac{t}{\tau_1}\right) + A_2 \exp\left(-\frac{t}{\tau_2}\right), \quad (12)$$

where

$$A_0 = \left(\frac{q-1}{5-3q}\right)^3 \frac{D}{6\tau}, \quad (13)$$

A_1, A_2, τ_1, and τ_2 are constants depending on parameters q, D, and τ. The τ_1 and τ_2 denotes correlation times, respectively.

In Fig. 2, we plot $C(t)$ for different values of q when $q > 1$. The numerical curve is fitted well by the second-order exponentially decaying function. Taking $q = 1.5$ for example, in the given parameter, its correlation function converges to a constant $A_0 = 1.33333$ and has a correlation time of 25.27893 and 3.561922, respectively. As the noise tends to exhibit Gaussian behavior, the numerical curve corresponds to a first-order exponentially decaying function.[33]

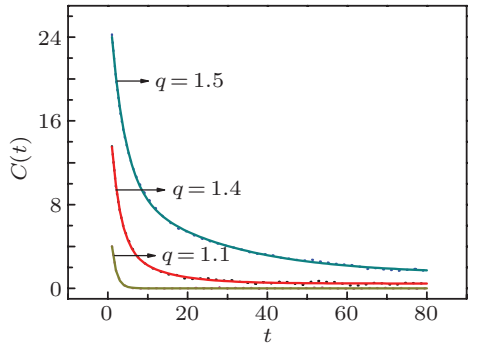


Fig. 2. (color online) Plot of numerically calculated $C(t)$ for different values of q when $q > 1$. The parameters used are $D = 8.0$ and $\tau = 1.0$. Simulations are indicated by dots and the fitted function by a full line.

It is exciting to find that the correlation function of the process η for $q > 1$ tends to a non-zero constant in the long time limit. To the best of our knowledge, although $C(t)$ has been studied,[18,30,33] it is the first time that this property has been found.

3. Diffusive behavior of the system

Using the Laplace analysis, we obtain the solution of Eq. (2) as follows:

$$x(t) = x_0 + \int_0^t H(t-s)\xi(s)\,\mathrm{d}s + v_0 H(t) + \int_0^t H(t-s)\eta(s)\,\mathrm{d}s, \quad (14)$$

where the response function $H(t)$ is given by

$$H(t) = \frac{1}{\gamma} - \frac{1}{\gamma}\exp(-\gamma t). \quad (15)$$

Then the mean square displacement of the particle can be obtained as

$$\langle [x(t) - \langle x(t)\rangle]^2\rangle = \int_0^t H(t-s)\,\mathrm{d}s \int_0^t H(t-s')\langle \xi(s)\xi(s')\rangle\,\mathrm{d}s' + \int_0^t H(t-s)\,\mathrm{d}s \int_0^t H(t-s')\langle \eta(s)\eta(s')\rangle\,\mathrm{d}s'. \quad (16)$$

Substituting Eqs. (8) and (9) into Eq. (16), we obtain that for $q \leq 1$

$$\langle [x(t) - \langle x(t)\rangle]^2\rangle_{t\to\infty} = \frac{2\gamma + 2D_{\mathrm{ng}}}{\gamma^2} t + m, \quad (17)$$

where

$$m = -\frac{D_{\mathrm{ng}}}{\tau_{\mathrm{ng}}}\frac{1}{\gamma^3}\frac{1}{\gamma + 1/\tau_{\mathrm{ng}}} - \frac{2D_{\mathrm{ng}}}{\gamma^3} - \frac{3 + 2D_{\mathrm{ng}}\tau_{\mathrm{ng}}}{\gamma^2}. \quad (18)$$

It is seen from Eq. (17) that $\langle [x(t) - \langle x(t)\rangle]^2\rangle_{t\to\infty} \propto t$. Therefore, for $q \leq 1$ the particle exhibits a normal diffusion. This is due to the fact that the correlation function $\langle \eta(t)\eta(t')\rangle$ in Eq. (16) is equal to zero in the long-time limit for $q \leq 1$.

Substituting Eq. (12) into Eq. (16), we can obtain

$$\langle [x(t) - \langle x(t)\rangle]^2\rangle_{t\to\infty} = \frac{A_0}{\gamma^2} t^2 + C_1 t - C_2, \quad (19)$$

for $q > 1$, where

$$C_1 = -\frac{2A_0}{\gamma^3} + \frac{2A_1\tau_1 + 2A_2\tau_2}{\gamma^2}, \quad (20)$$

$$C_2 = A_1\frac{1}{\gamma^3}\frac{1}{\gamma + \frac{1}{\tau_1}} + A_2\frac{1}{\gamma^3}\frac{1}{\gamma + \frac{1}{\tau_2}} - \frac{A_0}{\gamma^4} + \frac{2A_1\tau_1 + 2A_2\tau_2}{\gamma^3} + \frac{2A_1\tau_1^2 + 2A_2\tau_2^2}{\gamma^2}. \quad (21)$$

From Eq. (19) we can see that $\langle [x(t) - \langle x(t)\rangle]^2\rangle$ is proportional to t^2 in the long time limit. Namely, the ballistic diffusion is

observed for $q > 1$. The reason why one can expect the ballistic behavior to emerge for $q > 1$ is that $C(t)$ for $q > 1$ converges to a non-zero constant A_0. Furthermore, the pre-factor of the ballistic diffusion depends on A_0/γ^2.

In order to test our theoretical predictions and to compare the diffusive behavior of $q > 1$ with that of $q \leq 1$, we resort to numerical simulations of the system indicated by Eq. (2). In Figs. 3 and 4, we present some numerical examples. As shown by Eqs. (17) and (19), for $q \leq 1$ the system exhibits a normal diffusion, while for $q > 1$ the non-Gaussian noise induces a ballistic diffusion. Furthermore, one can see from Fig. 4 that for $q > 1$ the opening of the parabola decreases with increasing values of q, indicating that the higher the degree of departure from the Gaussian behavior is, the bigger the pre-factor of the ballistic diffusion is.

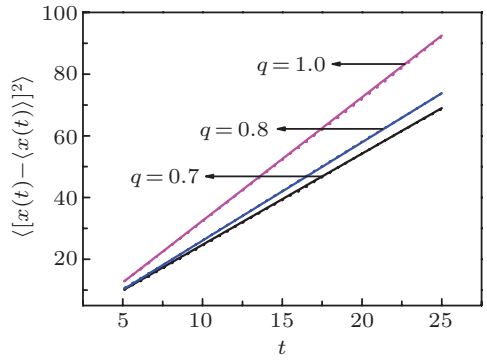


Fig. 3. (color online) The mean square displacement of the particle for $q \leq 1$. Simulations are indicated by dots and theory by a full line. The parameters are $D = 1$, $\tau = 1$, and $\gamma = 1$.

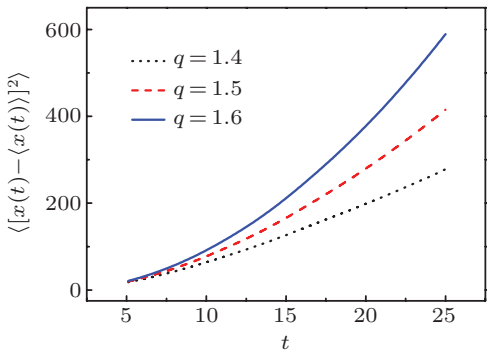


Fig. 4. (color online) The mean square displacement of the particle for $q > 1$. The parameters are the same as those in Fig. 3.

Figures 5 and 6 depict the mean square displacement of the system at $q = 1.5$ for different τ and D, respectively. It can be seen that the pre-factor of the ballistic diffusion strongly depends on the parameters τ and D: decreases with increasing τ and increases with increasing D. This is because fluctuation of η becomes more intense with increasing D/τ.

In Fig. 7, we plot the second moment of velocity of the particle. It can be seen that the process we studied is asymptotic stationary.

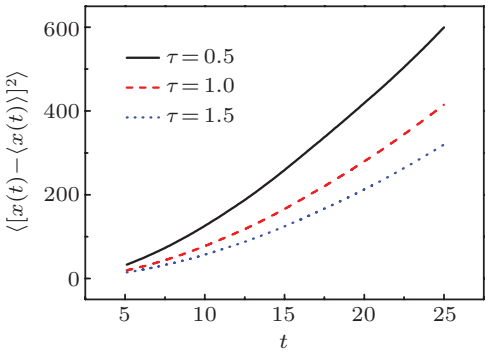


Fig. 5. (color online) The mean square displacement of the particle for different τ at a fixed q. The parameters are $q = 1.5$, $D = 1$, and $\gamma = 1$.

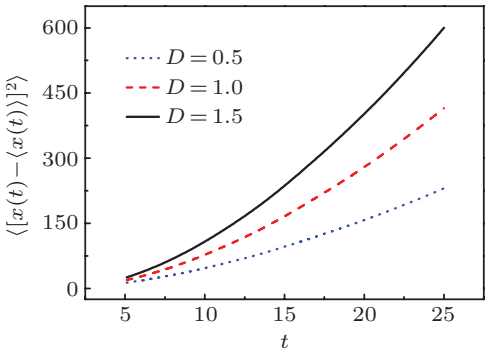


Fig. 6. (color online) The mean square displacement of the particle for different D at $q = 1.5$. The parameters are $\tau = 1$ and $\gamma = 1$.

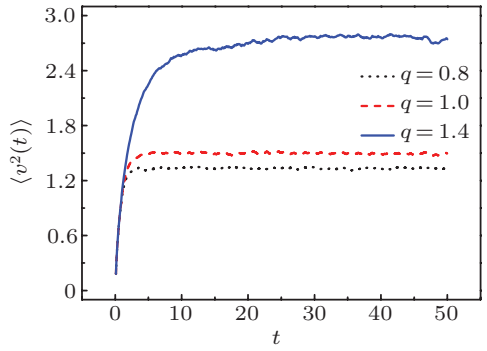


Fig. 7. (color online) The second moment of velocity of the particle for different q. The parameters are $D = 1$, $\tau = 1$, $\gamma = 1$, and $\{v_0^2\} = 0.0$.

4. Summary

To summarize, we have analyzed the diffusive behavior of a system subject to both internal thermal and external non-thermal noises. For the internal noise, we have considered the usual Gaussian distribution but the external noise exhibits non-Gaussian characteristics. We have observed two time correlation functions of the stochastic process $C(t) = \langle \eta(t+t')\eta(t')\rangle$ in the stationary regime $t' \to \infty$, and found that for $q \leq 1$, $C(t)$ is fitted very well by the first-order exponentially decaying curve, in which the pre-exponential factor increases with q, while for $q > 1$, it can be approximated by a second-order exponentially decaying function. Moreover, it is exciting to find that $C(t)$ for $q < 1$ is equal to zero in the long-time limit, whereas for $q > 1$ it converges to a non-zero constant. Due

to the properties of $C(t)$, the diffusive behavior of the system depends on parameter q indicating the departure of the non-Gaussian noise from the Gaussian behavior: the system exhibits a normal diffusion for $q \leq 1$, while for $q > 1$ ballistic diffusion is observed. The pre factor of the ballistic diffusion has a strong dependence on parameters q, τ and D.

We believe that the mechanisms found in this study have a broad application. It is expected that ballistic diffusion might become a useful operational mechanism to achieve a high efficiency for the molecular motors and other systems in addition to what have been discussed in Ref. [34].

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