New formulas for normalizing photon-added (-subtracted) two-mode squeezed thermal states*

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For the first time, we derive the compact forms of normalization factors for photon-added (-subtracted) two-mode squeezed thermal states by using the *P*-representation and the integration within an ordered product of operators (IWOP) technique. It is found that these two factors are related to the Jacobi polynomials. In addition, some new relationships for Jacobi polynomials are presented.

Keywords: two-mode squeezed thermal states, normalization, photon addition and subtraction, integration within an ordered product of operators technique

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1. Introduction

Quantum entanglement is a key resource in the quantum information process.[1] It is desired to generate a quantum state with higher entanglement. To realize this purpose, many schemes are proposed. Among them, photon-subtraction and photon-addition are effective methods.[2–4] For example, photon-subtraction/addition has been applied to improve the entanglement between Gaussian states,[5,6] loophole-free tests of Bell's inequality,[7,8] and quantum computing.[9] Recently, the photon-subtraction two-mode squeezed vacuum states (PS-TMSVSs) have attracted much attention from both experimentalists and theorists.[10–26] In general, it is necessary to obtain the normalization factor of PS-TMSVS for further investigating its properties,[27,28] such as its photon-number distribution, Wigner function, and decoherence.

On the other hand, the interaction between quantum systems and a thermal reservoir cannot be avoided, thus we often do not have enough information to completely specify the state of the system. In such a case, the system only can be described by mixed states, such as thermal states. Actually, the two-mode squeezed thermal states (TMSTSs) can be considered as the generalized bipartite Gaussian states.

In this paper, we shall introduce two kinds of quantum mixed states, i.e, photon-addition (PA) and photon-subtraction (PS) two-mode squeezed thermal states, and two kinds of generalized bipartite non-Gaussian states. Then we derive the analytical expressions of the normalization factors by using the *P*-representation of the thermal state and the technique of integration within an ordered product of operators (IWOP). We show that both normalization factors are related to the Jacobi polynomials in a compact form, which is a remarkable result. This method is concise and convenient. In addition, some new representations of generating functions for the Jacobi polynomials and some relationships are also obtained. The new generating functions can be directly used to calculate the higher-order squeezing for TMSTSs.

2. Two-mode squeezed thermal state and its normally ordering

The TMSTS is given by

$$\rho^S \equiv S(r)\,\rho_{\rm th1}\rho_{\rm th2}S^{\dagger}(r), \tag{1}$$

where $S(r)=\exp[r(a^{\dagger}b^{\dagger}-ab)]$ is the two-mode squeezing operator with squeezing parameter r, and $\rho_{\rm th1,2}$ is the density operator of a single-mode thermal state

$$\rho_{\rm th1,2}=\sum_{n=0}^{\infty}\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\,|n\rangle\langle n|, \tag{2}$$

with $\bar{n}$ being the average photon number of thermal state $\rho_{{\rm th}j}$ ($j=1,2$). For simplicity, we assume the average photon numbers of $\rho_{{\rm th}j}$ ($j=1,2$) to be identical.

Next, we derive the normal ordering of the TMSTS. Note the coherent state $|\alpha\rangle=\exp[-\frac{1}{2}|\alpha|^2+\alpha a^{\dagger}]|0\rangle$ and the transformation relationship $S(r)a^{\dagger}S^{\dagger}(r)=a^{\dagger}\cosh r-b\sinh r$ and $S(r)b^{\dagger}S^{\dagger}(r)=b^{\dagger}\cosh r-a\sinh r$, the two-mode squeezed coherent state $S|\alpha,\beta\rangle$ can be rewritten as

$$S|\alpha,\beta\rangle = {\rm sech}r\exp\left[-\frac{1}{2}(|\alpha|^2+|\beta|^2)-\alpha\beta\tanh r\right]$$

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$$\times \exp\left[\left(a^{\dagger}\alpha+b^{\dagger}\beta\right)\operatorname{sech}r+a^{\dagger}b^{\dagger}\tanh r\right]|00\rangle, \quad (3)$$

where $S(r)|00\rangle = \operatorname{sech}r \exp\left[a^{\dagger}b^{\dagger}\tanh r\right]|00\rangle$ and $\mathrm{e}^{\tau a}a^{\dagger}\mathrm{e}^{-\tau a} = a^{\dagger}+\tau$ as well as $\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-[A,B]/2} = \mathrm{e}^{B}\mathrm{e}^{A}\mathrm{e}^{[A,B]/2}$ (valid for $[A,[A,B]] = [B,[A,B]] = 0$) are used.

Then using the P-representation of density operator $\rho_{\mathrm{th}j}$, i.e.,[29]

$$\rho_{\mathrm{th}j} = \frac{1}{\bar{n}}\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{e}^{-\frac{1}{\bar{n}}|\alpha|^2}|\alpha\rangle\langle\alpha|, \quad (4)$$

and the vacuum projector $|00\rangle\langle 00| =: \exp[-a^{\dagger}a - b^{\dagger}b]:$ ($:\ :$ denotes the normally ordering) as well as the IWOP technique,[30,31] we obtain

$$\begin{aligned}\rho^{S} &= \frac{1}{\bar{n}^2}\int \frac{\mathrm{d}^2\alpha\mathrm{d}^2\beta}{\pi^2}\mathrm{e}^{-\frac{1}{\bar{n}}(|\alpha|^2+|\beta|^2)}S|\alpha,\beta\rangle\langle\alpha,\beta|S^{\dagger}\\ &= A_1 :\exp\left[A_2\left(a^{\dagger}b^{\dagger}+ab\right)-A_3\left(a^{\dagger}a+b^{\dagger}b\right)\right]:, \quad (5)\end{aligned}$$

where we have set

$$\begin{aligned}A_1 &= \frac{\operatorname{sech}^2 r}{(\bar{n}+1)^2-\bar{n}^2\tanh^2 r},\\ A_2 &= \frac{(2\bar{n}+1)\sinh r\cosh r}{(2\bar{n}+1)\cosh^2 r+\bar{n}^2},\\ A_3 &= \frac{\cosh^2 r+\bar{n}\cosh 2r}{(2\bar{n}+1)\cosh^2 r+\bar{n}^2}, \quad (6)\end{aligned}$$

and used the integration formula[32]

$$\int \frac{\mathrm{d}^2 z}{\pi}\mathrm{e}^{\zeta|z|^2+\xi z+\eta z^*} = -\frac{1}{\zeta}\mathrm{e}^{-\frac{\xi\eta}{\zeta}}, \quad \mathrm{Re}\zeta<0. \quad (7)$$

Equation (5) is the normally ordering form of the TMSTS to be used in our calculations below.

3. Normalization of photon-added TMSTS

For the two-mode case, the photon-added TMSTS (PA-TMSTS) can be generated by repeatedly operating the photon creation operators $a^{\dagger}$ and $b^{\dagger}$ on the TMSTS, i.e,

$$\rho^{SA} \equiv N_{m,n}^{-1}a^{\dagger m}b^{\dagger n}S(r)\rho_{\mathrm{th}1}\rho_{\mathrm{th}2}S^{\dagger}(r)a^{m}b^{n}, \quad (8)$$

where m,n are the added photon numbers of the modes (non-negative integers), and $N_{m,n}$ is the normalization of the PA-TMSTS to be determined by $\mathtt{Tr}\rho^{SA} = 1$.

By using Eq. (5), the PA-TMSTS reads

$$\rho^{SA} = \frac{A_1}{N_{m,n}} :a^{\dagger m}b^{\dagger n}\mathrm{e}^{A_2\left(a^{\dagger}b^{\dagger}+ab\right)-A_3\left(a^{\dagger}a+b^{\dagger}b\right)}a^{m}b^{n}:. \quad (9)$$

Thus using the completeness relationship of the coherent state $\int \mathrm{d}^2\alpha\mathrm{d}^2\beta|\alpha,\beta\rangle\langle\alpha,\beta|/\pi^2 = 1$ and Eq. (7), we can give the normalization factor $N_{m,n}$ as

$$\begin{aligned}N_{m,n} &= A_1\mathtt{Tr}\left[:a^{\dagger m}b^{\dagger n}\mathrm{e}^{A_2\left(a^{\dagger}b^{\dagger}+ab\right)-A_3\left(a^{\dagger}a+b^{\dagger}b\right)}a^{m}b^{n}:\right]\\ &= A_1\int\frac{\mathrm{d}^2\alpha\mathrm{d}^2\beta}{\pi^2}|\alpha|^{2m}|\beta|^{2n}\mathrm{e}^{A_2(\alpha^*\beta^*+\alpha\beta)-A_3(|\alpha|^2+|\beta|^2)}\\ &= \left.\frac{\partial^{2m+2n}}{\partial\tau^m\partial t^m\partial\tau'^n\partial t'^n}\mathrm{e}^{B_1\left(\tau t+\tau' t'\right)+B_2\left(\tau\tau'+tt'\right)}\right|_{t,\tau,t',\tau'=0}, \quad (10)\end{aligned}$$

where we have set $B_1 = \cosh^2 r+\bar{n}\cosh 2r$ and $B_2 = (2\bar{n}+1)\sinh r\cosh r$. Here we introduce a new expression for the generating function of the Jacobi polynomials in the form (see Appendix A for proof)

$$\begin{aligned}&\left.\frac{\partial^{2m+2n}}{\partial\tau^m\partial t^m\partial\tau'^n\partial t'^n}\mathrm{e}^{A\left(\tau' t'+\tau t\right)+B\left(\tau\tau'+t't\right)}\right|_{t,\tau,t',\tau'=0}\\ &= m!n!\begin{cases}A^{n-m}\left(B^2-A^2\right)^m P_m^{(n-m,0)}\left(\dfrac{B^2+A^2}{B^2-A^2}\right), & m\leqslant n,\\ A^{m-n}\left(B^2-A^2\right)^n P_n^{(m-n,0)}\left(\dfrac{B^2+A^2}{B^2-A^2}\right), & n\leqslant m,\end{cases} \quad (11)\end{aligned}$$

thus the normalization factor $N_{m,n}$ can be given (without a loss of generality assuming $m\leqslant n$) by

$$N_{m,n} = m!n!B_1^{n-m}\omega^m P_m^{(0,n-m)}\left(\frac{\upsilon}{\omega}\right), \quad (12)$$

where we have used the property of the Jacobi polynomials $P_m^{(\alpha,\beta)}(-x) = (-1)^m P_m^{(\beta,\alpha)}(x)$ and $\omega = \bar{n}^2+(2\bar{n}+1)\cosh^2 r$, $\upsilon = \bar{n}(\bar{n}+1)\cosh 4r+\left(\bar{n}+\cosh^2 r\right)\cosh 2r$. Equation (12) indicates that the normalization factor is related to the Jacobi polynomials, which is important for further analytically studying the statistical properties of the PA-TMSTS. Note that equation (12) exhibits an exchanging symmetry.

It is clear that when $m = n = 0$, equation(12) just reduces to the TMSTS due to $P_0^{(0,0)}(x) = 1$; while for $n\neq 0$ and $m = 0$, due to $P_0^{(0,n)}(x) = 1$, equation (12) becomes $N_{0,n} = n!B_1^n$. For the case of $m = n$, $N_{m,m}$ is related to the Legendre polynomial of parameter υ/ω because of $P_n^{(0,0)}(x) = P_n(x)$ and $P_0(x) = 1$. In addition, when $\bar{n} = 0$, we have $\omega = B_1 = \cosh^2 r$ and $\upsilon/\omega = \cosh 2r$, then equation (12) reads

$$N_{m,n}(\bar{n}=0) = m!n!\cosh^{2n}rP_m^{(0,n-m)}(\cosh 2r), \quad (13)$$

which is just the normalization of the two-mode photon-added squeezed vacuum state.[33]

4. Normalization of photon-subtraction TMSTS

The photon-subtraction TMSTS (PS-TMSTS) is defined as

$$\rho^{SS} \equiv M_{m,n}^{-1}a^{m}b^{n}S(r)\rho_{\mathrm{th}1}\rho_{\mathrm{th}2}S^{\dagger}(r)a^{\dagger m}b^{\dagger n}. \quad (14)$$

In order to determine the normalization factor $M_{m,n}^{-1}$, it is convenient to derive the anti-normal ordering of ρ^{S}.

Using Eqs. (5), (7) and the formula converting any single-mode operator $\hat{O}$ into its anti-normal ordering form[31]

$$\hat{O} =: \int\frac{\mathrm{d}^2 z}{\pi}\langle -z|\hat{O}|z\rangle\mathrm{e}^{|z|^2+z^*a-za^{\dagger}+a^{\dagger}a}:, \quad (15)$$

where $|z\rangle$ is a coherent state, and symbol $\vdots\,\vdots$ denotes the anti-normal ordering, (note that the order of Bose operators a and $a^\dagger$ within $\vdots\,\vdots$ can be permuted), we can obtain the anti-normal ordering form of the TMSTS

$$\rho^S = \tilde{A}_1 \vdots \exp\left[\tilde{A}_2\left(a^\dagger b^\dagger + ab\right) - \tilde{A}_3\left(a^\dagger a + b^\dagger b\right)\right] \vdots, \quad (16)$$

where the convergent condition $\sinh 2r < [2\bar{n}(\bar{n}+1)]/(2\bar{n}+1)$ is used, and we have set

$$\begin{aligned} \tilde{A}_1 &= \frac{1}{(\bar{n}+1)^2 - (2\bar{n}+1)\cosh^2 r}, \\ \tilde{A}_2 &= \frac{(2\bar{n}+1)\sinh r \cosh r}{(\bar{n}+1)^2 - (2\bar{n}+1)\cosh^2 r}, \\ \tilde{A}_3 &= \frac{\sinh^2 r + \bar{n}\cosh 2r}{(\bar{n}+1)^2 - (2\bar{n}+1)\cosh^2 r}. \end{aligned} \quad (17)$$

Equation (16) implies that the P function $P(\alpha,\beta)$ of the TMSTS is

$$P(\alpha,\beta) = \tilde{A}_1 \exp\left[\tilde{A}_2\left(\alpha^*\beta^* + \alpha\beta\right) - \tilde{A}_3(|\alpha|^2 + |\beta|^2)\right], \quad (18)$$

which leads to the P representation of density operator $S\rho_{\rm th1}\rho_{\rm th2}S^\dagger$, i.e.,

$$\rho^S = \int \frac{\mathrm{d}^2\alpha \mathrm{d}^2\beta}{\pi^2} P(\alpha,\beta)\,|\alpha,\beta\rangle\langle\alpha,\beta|. \quad (19)$$

In particular, for the case without squeezing, $r=0$, and equations (5) and (16) both reduce to

$$\begin{aligned} \rho^S(r=0) &= \frac{1}{(\bar{n}+1)^2} \colon \exp\left[-\frac{a^\dagger a + b^\dagger b}{\bar{n}+1}\right] \colon \\ &= \frac{1}{\bar{n}^2} \vdots \exp\left[-\frac{1}{\bar{n}}\left(a^\dagger a + b^\dagger b\right)\right] \vdots, \end{aligned} \quad (20)$$

as expected.[29] Here, we should mention that the normal and the anti-normal ordering forms of TMSTS are useful to higher-order squeezing and photon statistics[35,36] of TMSTS.

Now, we examine its normalization. Using Eq. (19), in a way similar to that derived in Eq. (12), we obtain

$$M_{m,n} = \left.\frac{\partial^{2m+2n}}{\partial\tau^m \partial t^m \partial\tau'^n \partial t'^n} \mathrm{e}^{\tilde{B}_1(t'\tau'+\tau t) + \tilde{B}_2(\tau\tau'+tt')}\right|_{t,\tau,t',\tau'=0}, \quad (21)$$

where $\tilde{B}_2 = B_2$ and $\tilde{B}_1 = \left(\sinh^2 r + \bar{n}\cosh 2r\right)$. Thus using the newly found generating function (11), we have

$$M_{m,n} = m!n! \begin{cases} \tilde{B}_1^{n-m}\left(\tilde{B}_2^2 - \tilde{B}_1^2\right)^m P_m^{(n-m,0)}\left(\frac{\tilde{\upsilon}}{\tilde{\omega}}\right), & m \leqslant n, \\ \tilde{B}_1^{m-n}\left(\tilde{B}_2^2 - \tilde{B}_1^2\right)^n P_n^{(m-n,0)}\left(\frac{\tilde{\upsilon}}{\tilde{\omega}}\right), & n \leqslant m, \end{cases} \quad (22)$$

where $\tilde{\upsilon} = (\bar{n}+1)(\bar{n}\cosh 4r - \cosh 2r) + \cosh^2 r \cosh 2r$ and $\tilde{\omega} = (2\bar{n}+1)\sinh^2 r - \bar{n}^2$. Equation (22) is just the normalization factor of the PS-TMSTS, a concise expression relating to the Jacobi polynomials. In particular, when $\bar{n}=0$, i.e., for the photon-subtraction two-mode squeezing vacuum state, $M_{m,n}$ reduces to the result in Ref. [37]. Using the normalized factors obtained above together with the results in Ref. [38], we can directly give the explicit expressions involving the normalization factors of photon-number distributions of PS-TMSTS and PA-TMSTS.

5. Some new relationships for Jacobi polynomials

From Eq.(11), it is easily found that

$$\begin{aligned} &\left.\frac{\partial^{2m+2n}}{\partial\tau^m \partial t^m \partial\tau'^n \partial t'^n} \mathrm{e}^{\left(At'+B\tau\right)\tau' + \left(A\tau+Bt'\right)t}\right|_{t,\tau,t',\tau'=0} \\ &= \left.\frac{\partial^{m+n}}{\partial\tau^m \partial t'^n}\left(A\tau+Bt'\right)^m\left(At'+B\tau\right)^n\right|_{\tau,t'=0}, \end{aligned} \quad (23)$$

thus we have

$$\begin{aligned} &\left.\frac{\partial^{m+n}}{\partial\tau^m \partial t'^n}\left(A\tau+Bt'\right)^m\left(At'+B\tau\right)^n\right|_{\tau,t'=0} \\ &= m!n! \begin{cases} A^{n-m}\left(B^2-A^2\right)^m P_m^{(n-m,0)}\left(\frac{B^2+A^2}{B^2-A^2}\right), & m \leqslant n, \\ A^{m-n}\left(B^2-A^2\right)^n P_n^{(m-n,0)}\left(\frac{B^2+A^2}{B^2-A^2}\right), & n \leqslant m, \end{cases} \end{aligned} \quad (24)$$

which is a new relationship about the generating function of Jacobi polynomials. In particular, when $m=n$, equation (24) becomes

$$\begin{aligned} &\left.\frac{\partial^{2m}}{\partial\tau^m \partial t^m}\left[\left(At+B\tau\right)\left(A\tau+Bt\right)\right]^m\right|_{\tau,t'=0} \\ &= (m!)^2\left(B^2-A^2\right)^m P_m\left(\frac{B^2+A^2}{B^2-A^2}\right), \end{aligned} \quad (25)$$

where $P_m(x)$ is the m-th Legendre polynomial. Equation (25) is just a new formula for the generating function of Legendre polynomial $P_m(x)$, which is different from the new form found in Refs. [17] and [39]. In fact, one can check Eq. (24) by directly expanding all the exponential items and comparing them with the standard expression of the Legendre polynomial.

In addition, for the case of $\bar{n}=0$, i.e., photon-addition TMSVS, we can take $A \to \cosh^2 r = \frac{x+1}{2}$ and $B \to \sinh r\cosh r = \frac{1}{2}\sqrt{x^2-1}$, $x = \cosh 2r$, then use $P_m^{(\alpha,\beta)}(-x) = (-1)^m P_m^{(\beta,\alpha)}(x)$, thus equations (24) and (25) become

$$\begin{aligned} &\left.\frac{\partial^{m+n}}{\partial\tau^m \partial t'^n}\left(\tau + t'\sqrt{\frac{x-1}{x+1}}\right)^m\left(t' + \tau\sqrt{\frac{x-1}{x+1}}\right)^n\right|_{\tau,t'=0} \\ &= m!n! \begin{cases} \frac{2^m}{(x+1)^m} P_m^{(0,n-m)}(x), & m \leqslant n, \\ \frac{2^n}{(x+1)^n} P_m^{(0,n-m)}(x), & n \leqslant m, \end{cases} \end{aligned} \quad (26)$$

$$\left.\frac{\partial^{2m}}{\partial\tau^m \partial t^m}\left(t + \sqrt{\frac{x-1}{x+1}}\tau\right)^m\left(\tau + \sqrt{\frac{x-1}{x+1}}t\right)^m\right|_{\tau,t'=0}$$

$$= \frac{2^m (m!)^2}{(x+1)^m} P_m(x). \tag{27}$$

Equations (26) and (27) are two new expressions about the generating functions of the Legendre and Jacobi polynomials.

On the other hand, according to the spirit of Ref. [40], noticing Eq. (10), we can construct

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{\tau^m t^n}{m!n!} N_{m,n} \\
&= A_1 \mathrm{Tr}\left[: \mathrm{e}^{A_2\left(a^\dagger b^\dagger + ab\right) - (A_3-\tau)a^\dagger a - (A_3-t)b^\dagger b} : \right] \\
&= \frac{A_1}{t\tau - tA_3 - \tau A_3 + A_1}.
\end{aligned} \tag{28}$$

In particular, when $\bar{n} = 0$, which leads to $A_1 = \mathrm{sech}^2 r$, $A_2 = \tanh r$, and $A_3 = 1$, equation (28) reduces to

$$\sum_{m,n=0}^{\infty} \frac{\tau^m t^n}{m!n!} N_{m,n} = \frac{1}{(t\tau - t - \tau)\cosh^2 r + 1}. \tag{29}$$

Combining with Eq. (13), we have ($x = \cosh 2r$)

$$\begin{aligned}
&\sum_{n>m=0}^{\infty} \tau^m \left(\frac{x+1}{2} t\right)^n P_m^{(0,n-m)}(x) \\
&+ \sum_{m\geqslant n=0}^{\infty} t^n \left(\frac{x+1}{2}\tau\right)^m P_n^{(0,m-n)}(x) \\
&= \frac{2}{(t\tau - t - \tau)(x+1) + 2},
\end{aligned} \tag{30}$$

which is a new generating function formula of the Jacobi polynomials. In particular, when $n = 0$, we do not need the summation over t, which means setting $t = 0$ in Eq. (30), using $P_0^{(0,m)}(x) = 1$, from Eq. (30) we have $\sum_{m=0}^{\infty} \left(\frac{x+1}{2}\tau\right)^m = 1/[1-(1+x)\tau/2]$ as expected.

6. Conclusion

In this paper, we introduced two kinds of non-Gaussian states generated by photon-addition to and photon-subtraction from the TMSTS which can be considered generalized bipartite Gaussian states. Then using the IWOP technique and the P-representation of the thermal state, we derived analytically the compact expressions for the normalization factors of these two kinds of non-Gaussian states. It is found that the two factors are related to the Jacobi polynomials in a compact form. In addition, some new relationships are presented for Jacobi polynomials' generating functions. Our method is concise and convenient. The new generating functions can also be directly used to calculate the higher-order squeezing for TMSTSs. In addition, this method can also be used to deal with the single-mode (mixed) case.[41,42]

Appendix A: A new expression for generating the function of Jacobi polynomials

In this appendix, we will prove Eq. (11). Rewriting

$$H \equiv \frac{\partial^{2m+2n}}{\partial\tau^m \partial t^m \partial\tau'^n \partial t'^n} \mathrm{e}^{A\left(\tau' t' + \tau t\right) + B\left(\tau\tau' + t' t\right)} \Big|_{t,\tau,t',\tau'=0}, \tag{A1}$$

and expanding the exponential items, we have

$$\begin{aligned}
H &= \sum_{l,j,k,s=0}^{\infty} \frac{A^{l+j} B^{s+k}}{l!\, j!\, k!\, s!} \\
&\quad \times \frac{\partial^{2m+2n}}{\partial\tau^m \partial t^m \partial\tau'^n \partial t'^n} \tau^{k+j} t^{j+s} \tau'^{l+k} t'^{l+s} \Big|_{t,\tau,t',\tau'=0} \\
&= \sum_{s=0}^{\min[m,n]} \frac{(m!n!)^2 A^{n+m}}{s!s!(n-s)!(m-s)!} \left(\frac{B^2}{A^2}\right)^s.
\end{aligned} \tag{A2}$$

Comparing Eq. (A2) with the standard expression of the Jacobi polynomials[32]

$$\begin{aligned}
P_m^{(\alpha,\beta)}(x) &= \left(\frac{x-1}{2}\right)^m \sum_{k=0}^{m} \binom{m+\alpha}{k} \\
&\quad \times \binom{m+\beta}{m-k} \left(\frac{x+1}{x-1}\right)^k,
\end{aligned} \tag{A3}$$

we can find that taking $m \leqslant n$ and $y = \left(B^2 + A^2\right) / \left(B^2 - A^2\right)$,

$$\begin{aligned}
H &= \sum_{s=0}^{m} \frac{(m!n!)^2 A^{n+m}}{s!s!(n-s)!(m-s)!} \left(\frac{B^2}{A^2}\right)^s \\
&= m!n! \left(\frac{y-1}{2}\right)^{-m} A^{m+n} \left[\left(\frac{y-1}{2}\right)^m \right. \\
&\quad \left. \times \sum_{k=0}^{\min[m,n]} \frac{m!n!}{k!k!(n-k)!(m-k)!} \left(\frac{y+1}{y-1}\right)^k \right] \\
&= m!n! A^{n-m} \left(B^2 - A^2\right)^m P_m^{(n-m,0)}(y).
\end{aligned} \tag{A4}$$

In a similar way, for $n \leqslant m$, we have

$$H = m!n! A^{m-n} \left(B^2 - A^2\right)^n P_n^{(m-n,0)}(y). \tag{A5}$$

Thus we prove Eq. (11).

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