Novel loop-like solitons for the generalized Vakhnenko equation*

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A non-traveling wave solution of a generalized Vakhnenko equation arising from the high-frequent wave motion in a relaxing medium is derived via the extended Riccati mapping method. The solution includes an arbitrary function of an independent variable. Based on the solution, two hyperbolic functions are chosen to construct new solitons. Novel single-loop-like and double-loop-like solitons are found for the equation.

Keywords: generalized Vakhnenko equation, extended Riccati mapping method, nontraveling wave solution, loop-like soliton

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1. Introduction

In this paper, we study the following generalized Vakhnenko equation (GVE):

$$\frac{\partial}{\partial x}\left(\mathscr{D}^2 u+\frac{1}{2}pu^2\right)+p\mathscr{D}u=0,\ \ \mathscr{D}:=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}, \qquad (1)$$

where p is an arbitrary non-zero constant.

The GVE (1) can be traced to the celebrated Vakhnenko equation (VE)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0, \qquad (2)$$

which was initially presented by Vakhnenko in Ref. [1] to model the high-frequent wave motion in a relaxing medium.[2] In Refs. [1] and [3], the traveling wave solutions of the VE were derived. The most remarkable feature of VE (2) is that it possesses a single-loop soliton solution. In Refs. [4] and [5], a two-loop soltion solution was found for the VE (2) by using the Hirota's bilinear method and the inverse scattering transform method, respectively. The N-loop soliton solution to VE (2) was obtained by employing Hirota's method.[6] Some dynamic behaviors of the loop soliton solution for VE (2) were investigated in Ref. [7]. Some traveling wave solutions of VE (2) were obtained by using the (ω/g) method.[8] The homotopy analysis method was applied to obtain a one-loop soliton solution expressed by a series of exponential functions for VE (2).[9] Recently, Ma and Li derived the nontraveling wave solutions of VE (2) by using the (G'/G)-expansion method, and interesting periodic and soliton solutions were found.[10]

Besides VE (2), the traveling wave solutions were also studied for its extended forms, such as the (2+1)-dimensional Vakhnenko equation, GVE (1), the extended GVE, and the modified GVE.[11–19]

Due to parameter p involved in GVE (1), it can describe a type of propagating high-frequent wave in a relaxing medium.[11,12] However, it is more difficult to obtain the nontraveling wave solutions of the GVE (1) than those of the VE (2).

The Riccati mapping method is a concise and efficient symbolic computation tool to find exact solutions for nonlinear evolution equations, which was initially proposed by Fan as a class of the auxiliary equation method.[20] After that, the extended Riccati mapping method has been widely used to construct non-traveling wave solutions, generally in the variable separation form.[21–26] Since those solutions include arbitrary functions of independent variables, it is convenient to excite various soliton structures which can simulate nonlinear physical phenomena.[27–32]

The main purpose of this paper is to apply the extended Riccati mapping method to construct the nontraveling wave solution of the GVE (1), and then to explore some interesting loop-like solitons.

The paper is arranged as follows. In Section 2, a nontraveling wave solution involving an arbitrary function $f(X)$ is calculated for GVE (1). Two novel loop-like solitons are studied in Section 3. Some conclusions are then drawn in Section 4.

2. Nontraveling wave solution of GVE (1)

We now start to look for a non-traveling wave solution of the GVE (1) by using the extended Riccati mapping method.

Firstly, following Vakhnenko *et al.* in Refs. [4] and [5],

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we introduce two new independent variables X and T as

$$x = T + \int_{-\infty}^{X} U(X',T)\,\mathrm{d}X' + x_0, \quad t = X, \tag{3}$$

where $u(x,t) = U(X,T)$, and x_0 is a constant. Then we introduce a new function W defined as

$$W(X,T) = \int_{-\infty}^{X} U(X',T)\,\mathrm{d}X'. \tag{4}$$

From Eqs. (3) and (4), we have

$$W_X(X,T) = U(X,T), \quad W_T = \int_{-\infty}^{X} U_T(X',T)\,\mathrm{d}X', \tag{5}$$

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = (1+W_T)\frac{\partial}{\partial x}. \tag{6}$$

Thus, the GVE (1) can be transformed to a new nonlinear evolution equation (NEE)

$$W_{XXXT} + pW_XW_{XT} + pW_{XX} + pW_{XX}W_T = 0. \tag{7}$$

Integrating once for variable X in Eq. (7) and letting the integration constant be zero yields

$$W_{XXT} + pW_XW_T + pW_X = 0. \tag{8}$$

We suppose that the non-traveling wave solution of Eq. (8) can be expressed as

$$W(X,T) = \sum_{i=0}^{n} a_i(X,T)\phi^i(q), \tag{9}$$

where $a_i(X,T)$ $(i = 0,1,2,\ldots,n)$ are functions of the indicated variables to be determined, where $a_n \neq 0$, $q = q(X,T)$ is an arbitrary function of X and T, and $\phi(q)$ satisfies the following Riccati ordinary differential equation:

$$\phi' = \sigma + \phi^2, \tag{10}$$

with σ being an arbitrary constant.

Remark 1 When nonlinear function $q = q(X,T)$ in Eq. (9) is simplified to the linear traveling wave transformation $q = kX + VT$ (k,V are constants), $a_i(X,T)$ $(i = 0,1,2,\ldots,n)$ are also simplified to constants to be determined. As a result, the non-traveling wave solution can degenerate to the traveling wave solution.

Remark 2 The solution obtained under the nonlinear transformation $q = q(X,T)$ is called the non-traveling wave solution.

Applying the homogeneous balance principle[32] for Eq. (8), we get $n+3 = 2(n+1) \Rightarrow n = 1$. Thus the solution of Eq. (8) can be rewritten as

$$W(X,T) = a_0 + a_1\phi(q), \tag{11}$$

where $a_0 = a_0(X,T)$, $a_1 = a_1(X,T) \neq 0$, and $q = q(X,T)$.

Hereafter we adopt the following notations: $a_{iX} = \frac{\partial a_i(X,T)}{\partial X}$, $a_{iT} = \frac{\partial a_i(X,T)}{\partial T}$, $q_X = \frac{\partial q(X,T)}{\partial X}$, $q_T = \frac{\partial q(X,T)}{\partial T}$, $i = 0,1$.

From Eq. (11), W_X, W_{XXT}, W_XW_T can be calculated. Substituting W_X, W_{XXT}, W_XW_T into Eq. (8), collecting the terms of ϕ with the same power, then letting each coefficient be zero, we derive a set of over-determined partial differential equations about a_0, a_1

$$\phi^4 : \ 6a_1q_X^2q_T + pa_1^2q_Xq_T = 0, \tag{12}$$

$$\phi^3 : \ p(a_1a_{1X}q_T + a_1a_{1T}q_X) + 2(a_1q_X^2)_T + 2q_T[(a_1q_X)_X + a_{1X}q_X] = 0, \tag{13}$$

$$\phi^2 : \ (a_1q_X)_{XT} + (a_{1X}q_X)_T + q_T(a_{1XX} + 2\sigma a_1q_X^2) + 6\sigma a_1q_X^2q_T + p[a_{1X}a_{1T} + a_1q_T(a_{0X} + \sigma a_1q_X) + a_1q_X(a_{0T} + \sigma a_1q_T)] + pa_1q_X = 0, \tag{14}$$

$$\phi : \ a_{1XXT} + 2\sigma(a_1q_X^2)_T + 2\sigma q_T[(a_1q_X)_X + a_{1X}q_X] + p[a_{1T}(a_{0X} + \sigma a_1q_X) + a_{1X}(a_{0T} + \sigma a_1q_T)] + pa_{1X} = 0, \tag{15}$$

$$\phi^0 : \ a_{0XXT} + \sigma(a_1q_X)_{XT} + \sigma(a_{1X}q_X)_T + \sigma q_T(a_{1XX} + 2\sigma a_1q_X^2) + p(a_{0X} + \sigma a_1q_X) \times (a_{0T} + \sigma a_1q_T) + p(a_{0X} + \sigma a_1q_X) = 0. \tag{16}$$

In this paper, we consider q in the following variable separation form:

$$q(X,T) = f(X) + VT, \tag{17}$$

where V is a constant.

After solving Eqs. (12)–(16), we can obtain a solution with respect to a_0, a_1 as follows:

$$\begin{aligned} a_0(X,T) &= -T + \int \left[\frac{4\sigma}{p}f'^2(X) - \frac{1}{p}\frac{f'''(X)}{f'(X)}\right]\mathrm{d}X, \\ a_1(X,T) &= -\frac{6}{p}f'(X). \end{aligned} \tag{18}$$

Thus, a class of hyperbolic function solution can be obtained for the GVE (1)

$$\begin{cases} x = \displaystyle\int \left(-\frac{f'''(X)}{pf'(X)} + \frac{4\sigma}{p}f'^2(X)\right)\mathrm{d}X - \frac{6}{p}f'(X)\phi(q) + x_0, \\ t = X, \\ u(x,t) = \displaystyle -\frac{f'''(X)}{pf'(X)} - \frac{2\sigma}{p}f'^2(X) \\ \qquad\quad \displaystyle -\frac{6}{p}f''(X)\phi(q) - \frac{6}{p}f'^2(X)\phi^2(q), \end{cases} \tag{19}$$

where x_0 is a constant, and $\phi(q)$ satisfies

$$\phi(q) = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}q), & \sigma < 0, \\ \sqrt{\sigma}\tanh(\sqrt{\sigma}q), & \sigma > 0, \end{cases} \tag{20}$$

with $q = f(X) + VT$.

3. Loop-like soliton of GVE (1)

Thanks to the arbitrary function $f(X)$ in solution (19), it is convenient to explore new soliton solutions. In this section, we try two hyperbolic functions to excite new solitons for the GVE (1).

3.1. Single-loop-like soliton

We take $f(X)$ in solution (19) as follows:

$$f(X) = a\tanh(kX), \tag{21}$$

where a and k are real constants.

Substituting Eq. (21) into Eq. (19), we have

$$\begin{cases} x = -\dfrac{4k^2}{p}X + \dfrac{2k}{p}(3+2\sigma a^2)\tanh(kX) \\ \quad -\dfrac{4\sigma a^2 k}{3p}\tanh^3(kX) - \dfrac{6ak}{p}\operatorname{sech}^2(kX)\phi(q), \\ t = X, \\ u(x,t) = -\dfrac{4k^2}{p} + \dfrac{6k^2}{p}\operatorname{sech}^2(kX) - \dfrac{2\sigma a^2 k^2}{p}\operatorname{sech}^4(kX) \\ \quad +\dfrac{12ak^2}{p}\tanh(kX)\operatorname{sech}^2(kX)\phi(q) \\ \quad -\dfrac{6}{p}a^2k^2\operatorname{sech}^4(kX)\phi^2(q), \end{cases} \tag{22}$$

where $\phi(q)$ is determined by Eq. (20).

Solution (22) describes a novel single-loop-like soliton, which is illustrated in Figs. 1–3. Figure 1 shows the plots of the soliton in the x–t–u coordinate system. Figures 2 and 3 show the soliton of solution (19) in the x–u plane by setting different parameters T and a, respectively. The single-loop-like soliton is obviously different from the well-known single-loop-like one studied in Refs. [15], [17], and [18], as the soliton shapes in Figs. 2(b)–2(d) are unsymmetrical.

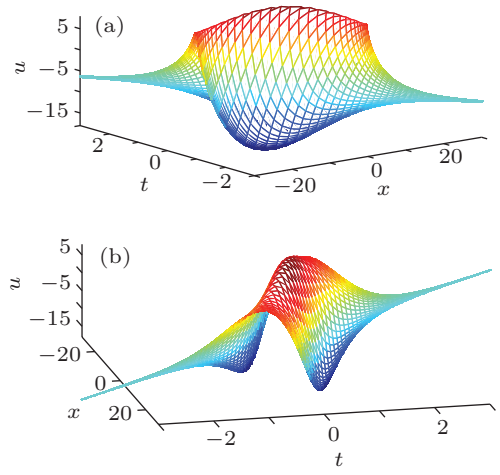


Fig. 1. (color online) Plots of solution (22) at different coordinate views ((a), (b)), where $V = 1$, $k = 1$, $p = 0.5$, $\sigma = -3$, $a = 0.8$.

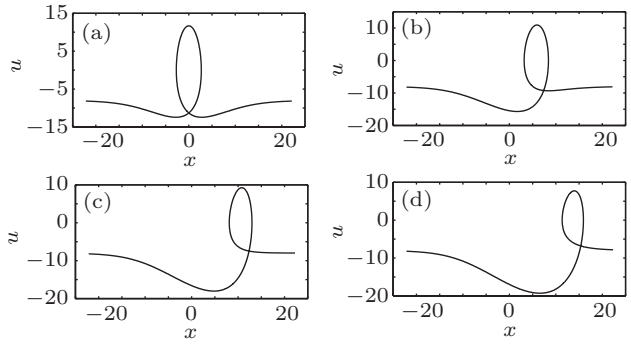


Fig. 2. Plots of solution (22) for (a) $T = 0$, (b) $T = 0.3$, (c) $T = 0.6$, (d) $T = 0.9$, where $V = 1$, $k = 1$, $p = 0.5$, $\sigma = -3$, $a = 0.8$.

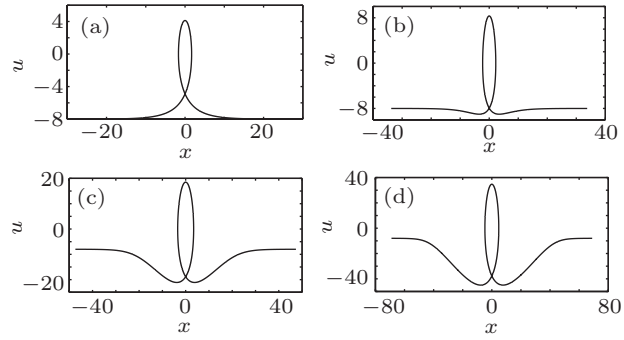


Fig. 3. Plots of solution (22) for (a) $a = 0.1$, (b) $a = 0.6$, (c) $a = 1.1$, (d) $a = 1.6$, where $V = 1$, $k = 1$, $p = 0.5$, $\sigma = -3$, $T = 0$.

3.2. Double-loop-like soliton

The $f(X)$ in solution (19) can be chosen as follows:

$$f(X) = a\operatorname{sech}(kX). \tag{23}$$

Proceeding as before, we can observe a new double-loop-like soliton, as demonstrated in Figs. 4 and 5. The soliton is shown from different views in Fig. 4. The soliton plots in Fig. 5 appear to be interesting.

It is easily seen that the single-loop-like soliton in the previous subsection is mathematically a two-valued function, while the double-loop-like one is a four-valued function.

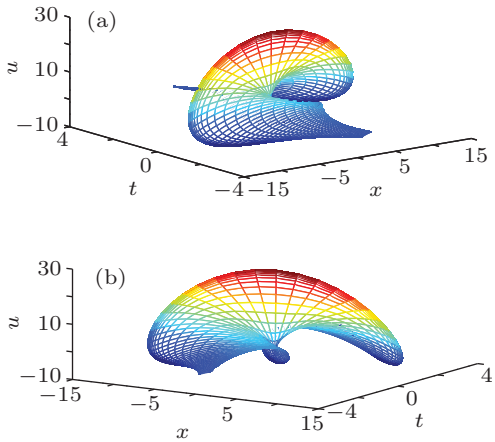


Fig. 4. (color online) Plots of solution (19) with Eq. (23) at different coordinate views ((a), (b)), where $V = 1$, $k = 1$, $p = 0.5$, $\sigma = -3$, $a = 0.8$.

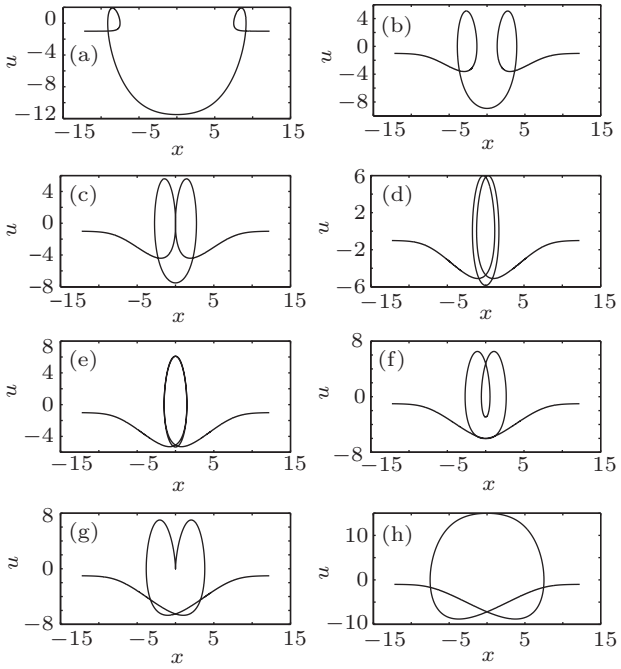


Fig. 5. Plots of solution (19) with Eq. (23) for (a) $T = 0$, (b) $T = 0.9$, (c) $T = 1.035$, (d) $T = 1.15$, (e) $T = 1.18$, (f) $T = 1.3$, (g) $T = 1.42$, (h) $T = 2$, where $V = 1$, $k = 1$, $p = 1$, $\sigma = -3$, $a = -1.6$.

The parameter a in $f(X)$ given by Eqs. (21) and (23) can also determine the shape of the soliton. However, the shape is still loop-like whatever the changes.

Remark 3 Due to the difficulty of the computation of the indefinite integration in Eq. (19), it is hard to find proper $f(X)$. However, with the help of Maple, we find that the trigonometric function sine and cosine are also suitable.

4. Conclusion

There have been many papers for constructing exact non-traveling wave solutions for (2+1)-dimensional and (3+1)-dimensional NEEs. Loop-like and folded solitons were often observed in high-dimensional systems.[33,35] However, it is extremely difficult to look for non-traveling wave solutions in (1+1)-dimensional NEEs (see also Ref. [36]). We construct a non-traveling wave solution of a generalized Vakhnenko equation by using the extended Riccati mapping method. Based on the solution and choosing two hyperbolic functions, new solitons are found for the GVE (1).

There are only a small number of the (1+1)-dimensional NEEs possessing loop-like solitons.[7,33,37,38] Fortunately, the VE and its extended or modified versions (such as GVE (1)) have loop-like solitons. In this paper, two types of loop-like solitons for GVE (1), namely, single-loop-like and double-loop-like solitons, are successful found. We believe that the work here is valuable and may be helpful in nonlinear acoustics.

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