# Average consensus of multi-agent systems with communication time delays and noisy links*

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In this paper, we consider the average-consensus problem with communication time delays and noisy links. We analyze two different cases of coupling topologies: fixed and switching topologies. By utilizing the stability theory of the stochastic differential equations, we analytically show that the average consensus could be achieved almost surely with the perturbation of noise and the communication time delays even if the time delay is time-varying. The theoretical results show that multi-agent systems can tolerate relatively large time delays if the noise is weak, and they can tolerate relatively strong noise if the time delays are low. The simulation results show that systems with strong noise intensities yield slow convergence.

**Keywords:** average consensus, multi-agent systems, time delay, white noise

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## 1. Introduction

Recently, the distributed coordinated control of multi-agent systems has attracted a great deal of attention in many fields such as biology, physics, robotics, and control engineering. This is partly due to its broad applications in many areas including cooperative control of unmanned air vehicles, formation control, attitude alignment of clusters of satellites, and congestion control in communication networks.[1–3] A critical problem in the distributed coordinated control of multiple agents is to find control laws such that all agents can reach an agreement regarding a certain quantity of interest which depends on the states of all agents. This problem is usually called the consensus problem. For some consensus problems, the common value to which the states converge is also required to be the average of the initial states of the group. It is often called the average-consensus problem.[4]

In the past decade, numerous interesting results have been obtained for the average-consensus problem. In Ref. [4], the average-consensus problem in networks of dynamic agents with fixed and switching topologies was investigated. The authors proved that a linear protocol can ensure an average consensus if the network is strongly connected and balanced. In Ref. [5], the authors extended the results of Ref. [4] to the case of directed networks with switching interaction topologies. Nonlinear consensus problems for continuous-time multi-agent systems and discrete-time multi-agent systems were studied in Refs. [6] and [7]. The consensus seeking over random networks was considered in Refs. [8] and [9]. The finite-time consensus problem for networks of dynamic agents was studied in Ref. [10]. In Ref. [11], the authors studied the constrained consensuses problem for a group of agents with disconnected topologies. The consensus problem of the second-order multi-agent system has been studied extensively.[12–14] The impulsive consensus for the second-order multi-agent networks was considered in Ref. [14]. In recent years, a number of researchers have investigated the consensus problems from various perspectives.[15–21]

In multi-agent systems, communication time delays may arise naturally, e.g., because of agents moving, the congestion of the communication channels, and the asymmetry of the interactions. Therefore, the stability analysis for multi-agent systems with time delays has become an attractive subject of research in the past few years.[22–28] Some sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the global asymptotic stability for multi-agent systems with constant and time-varying delays.

The variation of the network topology is usually due to link failures or creations in networks with mobile nodes. By investigating recent results, we find that the variation of the communication topology is also a key factor that influences the stability of the multi-agent system. Cases with communication time delays and switching topologies have been treated.[22–26]

Since noise is ubiquitous in both natural and man-made systems, the motion of an agent group is inevitably subjected to noise and disturbance in the environment, which could affect the agent on sensing its neighbor agent's value. Recently, consensus problems in noisy environments have attracted the attention of some researchers.[29–34] In Refs. [29] and [30],

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the authors considered the consensus seeking of discrete-time multi-agent networks. The average-consensus problem for networks of continuous-time integrator agents under fixed and directed topologies was considered in Ref. [31]. In Ref. [32], the author studied the consensus problem in a mean square for multi-agent systems with measurement noise and time delays. Most researches in the above literature considered consensus problems with random measurement noise, where each agent had only noisy measurements of its neighbors' states. In fact, noise can also affect the coupling between agents. In noisy environments, the link weights between two agents are stochastically perturbed by white noise. In Ref. [33], the average-consensus problem with noisy links and fixed topologies has been investigated. In Ref. [34], the leader–following consensus problem with noisy links and switching topologies has also been investigated. However, as far as we know, the average-consensus problem for multi-agent systems with communication time delays and noisy links has not been well studied.

Motivated by the above discussion, the average-consensus problem of multi-agent systems with noisy links and communication time delays is considered in this paper. The aim of this paper is to provide answers to the following questions: Can average-consensus be achieved with communication time delays and noisy links? Besides the numerical evidence, are there any analytical conditions that can be rigorously established? Based on a Lyapunov functional and the linear matrix inequality (LMI) approach, sufficient conditions are obtained, which guarantee that all agents asymptotically achieve an average consensus with a probability one. The work in this paper is closely related to Refs. [32] and [33], where the consensus problems in noisy environments have been considered. It is worth pointing out that different from Ref. [32], in which the mean square consensus problem with measurement noises and time delays was studied, we consider the almost sure consensus problem with coupling noises and time delays. The measurement noise can affect the agent's sensing of the other agent's state, while the coupling noise can affect the network topology. Unlike Ref. [33], where a static network without communication time delays was considered, we consider a network with communication time delays and switching topologies.

The rest of the paper is organized as follows. Section 2 contains the problem formulation and some preliminaries. The main results of this paper are presented in Section 3. In Section 4, two illustrative numerical examples are given. Finally, conclusions are drawn in Section 5.

Throughout this paper unless specified, we let $\boldsymbol{W}(t) = (w_1(t),\ldots,w_n(t))$ be an $n$-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathscr{F},P)$ with a natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$, $w(t)$ is a one-dimensional Brownian motion. Accordingly, $\dot{w}$ is the one-dimensional white noise. The $|\cdot|$ denotes the Euclidean norm. If $\boldsymbol{A}$ is a vector or matrix, its transpose is denoted as $\boldsymbol{A}^{\mathrm{T}}$. The $\boldsymbol{A} < 0$ means that $\boldsymbol{A}$ is negatively definite. If $\boldsymbol{A}$ is a matrix, $\|\boldsymbol{A}\|$ is its operator norm, and $\lambda(\boldsymbol{A})$ is the eigenvalue of $\boldsymbol{A}$. The $\boldsymbol{I}_n$ denotes an $n\times n$ identity matrix, $\mathbf{1} = (1,\ldots,1)^{\mathrm{T}} \in \mathbf{R}^n$ is a vector.

## 2. Problem statement

The algebraic graph theory is a natural framework for analyzing the consensus problems. If each agent is regarded as a node in a graph, then their coupling topology is conveniently described by a simple graph. Let $\mathscr{G} = (\mathscr{V},\mathscr{E},A)$ be a weighted digraph of order $n$ with the set of nodes $\mathscr{V} = \{1,2,\ldots,n\}$, the set of edges $\mathscr{E} \subset \mathscr{V}\times\mathscr{V}$, and a weighted adjacency matrix $\boldsymbol{A} = [a_{ij}] \in \mathbf{R}^{n\times n}$ with nonnegative elements. The node indexes belong to a finite set $\mathscr{I} = \{1,2,\ldots,n\}$. An edge of $\mathscr{G}$ is denoted as $(i,j)$, which starts from node $i$ and ends at node $j$. Then element $a_{ij}$ associated with the edge of the digraph is positive, i.e., $a_{ij} > 0 \Leftrightarrow (i,j) \in \mathscr{E}$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathscr{I}$. The set of neighbors of node $i$ is denoted by $\mathscr{N}_i = \{j \in \mathscr{V} : (i,j) \in \mathscr{E}\}$ . A path in a digraph is a sequence $i_0,i_1,\ldots,i_f$ of different nodes such that $(i_{j-1},i_j)$ is an edge for $j = 1,2,\ldots,f$, $f \in \mathbf{Z}^+$. A digraph $\mathscr{G}$ is strongly connected if there exists a path between any two distinct nodes. Moreover, if $\sum_{j\in\mathscr{N}_i} a_{ij} = \sum_{j\in\mathscr{N}_i} a_{ji}$ for all $i = 1,2,\ldots,n$, then the digraph $\mathscr{G}$ is called balanced. A diagonal matrix $\boldsymbol{D} = \mathrm{diag}\{d_1,\ldots,d_n\} \in \mathbf{R}^{n\times n}$ is the degree matrix of $\mathscr{G}$, whose diagonal element is $d_i = \sum_{j\in\mathscr{N}_i} a_{ij}$ for $i = 1,2,\ldots,n$. Then the Laplacian of the weighted digraph $\mathscr{G}$ is defined as $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A} \in \mathbf{R}^{n\times n}$.

Let $x_i \in \mathbf{R}$ denote the value of node $i$, we refer to $\mathscr{G}_{\boldsymbol{x}} = (\mathscr{G},\boldsymbol{x})$ with $\boldsymbol{x} = (x_1,\ldots,x_n)^{\mathrm{T}}$ as a network with value $\boldsymbol{x} \in \mathbf{R}^n$ and topology $\mathscr{G}$. Nodes $i$ and $j$ agree in a network if and only if $x_i = x_j$. In addition, the nodes of a network reach a consensus if and only if $x_i = x_j$ for all $i,j \in \mathscr{I}, i \neq j$. We assume that each node of a group is a dynamic agent with dynamics

$$\dot{x}_i = f(x_i,u_i), \quad i \in \mathscr{I}, \tag{1}$$

where $u_i$ is the coupling input. For any initial state $\boldsymbol{x}_0$, if $\boldsymbol{x}(t)$ converges to an equilibrium point $\boldsymbol{x}^*$ with $x_i^* = x_j^*$ for all $i,j \in \mathscr{I}$ as $t \to \infty$, then system (1) is considered to solve a consensus problem, especially, if $x_i^* = \mathrm{Ave}(\boldsymbol{x}_0) = \frac{1}{n}\sum_{i=1}^n x_i(0)$, system (1) solves the average-consensus problem.

Suppose that agent $i$ receives a message sent by its neighbor $j$ after a time delay $\tau$, the following model was proposed in Ref. [4] to reach an average consensus for undirected graph $\mathscr{G}$:

$$\dot{x}_i = \sum_{j\in\mathscr{N}_i} a_{ij}(x_j(t-\tau) - x_i(t-\tau)), \quad i \in \mathscr{I}. \tag{2}$$

As mentioned above, noise is unavoidable in the real world, the motion of the agent group is subject to environmental noise. In noisy environments, the link weights between

two agents are stochastically perturbed by the white noise. So it is natural to assume that weights $a_{ij}$ are stochastically perturbed with $a_{ij} \to a_{ij} + \sigma_{ij}\dot{w}$ for all $a_{ij} > 0$. Thus system (2) is stochastically perturbed into the following Itô stochastic differential equation:

$$\begin{aligned} \mathrm{d}x_i = & \sum_{j\in\mathscr{N}_i} a_{ij}(x_j(t-\tau)-x_i(t-\tau))\mathrm{d}t \\ & + \sum_{j\in\mathscr{N}_i} \sigma_{ij}(x_j(t-\tau)-x_i(t-\tau))\mathrm{d}w, \quad i\in\mathscr{I}, \end{aligned} \tag{3}$$

where $\boldsymbol{A}_\sigma = [\sigma_{ij}] \in \mathbf{R}^{n\times n}$ is a matrix representing the intensity of noise. To highlight the presence of noise, it is natural to assume that $\sigma_{ij} > 0$ if $j \in N_i$, else $\sigma_{ij} = 0$. Let $\tilde{d}_i = \sum_{j\in\mathscr{N}_i}\sigma_{ij}$, $\boldsymbol{D}_\sigma = \mathrm{diag}\{\tilde{d}_1,\ldots,\tilde{d}_n\}$, and $\boldsymbol{G} = \boldsymbol{D}_\sigma - \boldsymbol{A}_\sigma$. It is obvious that $\boldsymbol{G}$ can be regarded as a Laplacian matrix of the digraph $\bar{\mathscr{G}} = (\mathscr{V}, \mathscr{E}, \boldsymbol{A}_\sigma)$.

With matrices $\boldsymbol{L}$ and $\boldsymbol{G}$, system (3) can be rewritten in a matrix form

$$\mathrm{d}\boldsymbol{x} = -\boldsymbol{L}\boldsymbol{x}(t-\tau)\mathrm{d}t - \boldsymbol{G}\boldsymbol{x}(t-\tau)\mathrm{d}w, \tag{4}$$

where $\boldsymbol{L}$ is the Laplacian of digraph $\mathscr{G}$, and $\boldsymbol{G}$ is the Laplacian of digraph $\bar{\mathscr{G}}$.

The following concepts and lemmas play an important role in the proof of the main results.

**Definition 1**[35] A square matrix $\boldsymbol{M} \in \mathbf{R}^{n\times n}$ is said to be positive and stable if its eigenvalues have positive real parts.

**Lemma 1**[4] If digraph $\mathscr{G}$ is strongly connected, then its Laplacian matrix $\boldsymbol{L}$ has a zero eigenvalue with right eigenvector $\mathbf{1}$, and the other $n-1$ eigenvalues have positive real parts.

**Lemma 2**[4] If digraph $\mathscr{G}$ is balanced, then $\mathbf{1}$ is the right eigenvector of the Laplacian of $\mathscr{G}$ associated with the zero eigenvalue, i.e., $\mathbf{1}^{\mathrm{T}}\boldsymbol{L} = 0$.

**Lemma 3**[35] Digraph $\mathscr{G}$ is strongly connected if and only if its Laplacian matrix $\boldsymbol{L}$ is irreducible.

**Lemma 4** For any strongly connected and balanced digraph $\mathscr{G}$ with the Laplacian matrix $\boldsymbol{L} \in \mathbf{R}^{n\times n}$, there exists a real orthogonal matrix $\boldsymbol{U}$ with $\boldsymbol{U}_1 \in \mathbf{R}^{n\times(n-1)}$ as its first $n-1$ columns and $\frac{1}{\sqrt{n}}\mathbf{1}$ as its last column such that

$$\begin{aligned} \boldsymbol{U}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{U} &= \begin{pmatrix} \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{L}\boldsymbol{U}_1 & \mathbf{0}_{(n-1)\times 1} \\ \mathbf{0}_{1\times(n-1)} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{\boldsymbol{L}} & \mathbf{0}_{(n-1)\times 1} \\ \mathbf{0}_{1\times(n-1)} & 0 \end{pmatrix}, \end{aligned}$$

where $\bar{\boldsymbol{L}} \in \mathbf{R}^{(n-1)\times(n-1)}$ is positive and stable.

**Proof** Note that $\frac{1}{\sqrt{n}}\mathbf{1}$ is the eigenvector of $\boldsymbol{L}$ associated with the zero eigenvalue, this Lemma can be proven easily from Lemmas 2 and 3. The proof is quite trivial and omitted here.

**Lemma 5**[35] For a given symmetric matrix $\boldsymbol{S}$ with the form $\boldsymbol{S} = [\boldsymbol{S}_{ij}]$, $\boldsymbol{S}_{11} \in \mathbf{R}^{r\times r}$, $\boldsymbol{S}_{12} \in \mathbf{R}^{r\times(n-r)}$, $\boldsymbol{S}_{22} \in \mathbf{R}^{(n-r)\times(n-r)}$, then, $\boldsymbol{S} < 0$ if and only if $\boldsymbol{S}_{11} < 0$, $\boldsymbol{S}_{22} - \boldsymbol{S}_{21}\boldsymbol{S}_{11}^{-1}\boldsymbol{S}_{12} < 0$ or $\boldsymbol{S}_{22} < 0$, $\boldsymbol{S}_{11} - \boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{21} < 0$.

Next, we introduce an operator $\mathscr{L}$ in the stochastic differential delay equations.[36] Consider the following $n$-dimensional stochastic differential delay equation:

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))\mathrm{d}t + g(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))\mathrm{d}\boldsymbol{W}.$$

For each $\boldsymbol{V} \in \boldsymbol{C}^{2,1}(\mathbf{R}^n\times\mathbf{R}_+, \boldsymbol{R}_+)$, we define an operator $\mathscr{L}\boldsymbol{V}$ from $\boldsymbol{R}^n\times\mathbf{R}^n\times\mathbf{R}_+$ to $\mathbf{R}$ by

$$\mathscr{L}\boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial t} + \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\cdot\boldsymbol{f} + \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{g}^{\mathrm{T}}\frac{\partial^2\boldsymbol{V}}{\partial^2\boldsymbol{x}}\cdot\boldsymbol{g}\right],$$

where $\partial\boldsymbol{V}/\partial\boldsymbol{x} = (\partial\boldsymbol{V}/\partial x_1,\ldots,\partial\boldsymbol{V}/\partial x_n)$, and $\partial^2\boldsymbol{V}/\partial^2\boldsymbol{x} = (\partial^2\boldsymbol{V}/\partial x_i\partial x_j)_{n\times n}$.

## 3. Main results

In this section, we consider the average consensus problems for networks with a fixed topology and those with a switching topology.

### 3.1. Networks with fixed topology

Consider system (3) with a fixed topology. If digraphs $\mathscr{G}$ and $\bar{\mathscr{G}}$ are balanced, then from Lemma 2 we have $\mathbf{1}^{\mathrm{T}}\boldsymbol{L} = \mathbf{1}^{\mathrm{T}}\boldsymbol{G} = 0$. Thus $\alpha = \mathrm{Ave}(x)$ is an invariant quantity, i.e.,

$$\dot{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\dot{x}_i = -\frac{1}{n}(\mathbf{1}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{x}(t-\tau) + \mathbf{1}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{x}(t-\tau)\dot{w}) = 0. \tag{5}$$

This allows the decomposition of $x$ in the form $\boldsymbol{x} = \boldsymbol{e} + \alpha\mathbf{1}$, where $\boldsymbol{e}$ is a disagreement vector and satisfies $\sum_{i=1}^{n} e_i$=0. Therefore the disagreement dynamics are given by

$$\mathrm{d}\boldsymbol{e} = -\boldsymbol{L}\boldsymbol{e}(t-\tau)\mathrm{d}t - \boldsymbol{G}\boldsymbol{e}(t-\tau)\mathrm{d}w. \tag{6}$$

Clearly, $\boldsymbol{e}(t,0) = 0$ is a trivial solution of system (6). Moreover, the consensus could be achieved with probability one provided that the trivial solution of system (6) is almost surely globally asymptotically stable, i.e., for any initial value $\boldsymbol{e}_0 \in \mathbf{R}^n$, $|\boldsymbol{e}(t,t_0,e_0)| \to 0$ is almost sure as $t \to \infty$.

**Lemma 6** If digraphs $\mathscr{G}$ and $\bar{\mathscr{G}}$ are strongly connected and balanced, then the disagreement system (6) is equivalent to the following $(n-1)$-dimensional system:

$$\mathrm{d}\boldsymbol{\delta} = -\bar{\boldsymbol{L}}\boldsymbol{\delta}(t-\tau)\mathrm{d}t - \bar{\boldsymbol{G}}\boldsymbol{\delta}(t-\tau)\mathrm{d}w, \tag{7}$$

where $\boldsymbol{\delta} = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{e}$, $\bar{\boldsymbol{L}} = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{L}\boldsymbol{U}_1$, $\bar{\boldsymbol{G}} = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{G}\boldsymbol{U}_1$, and $\boldsymbol{U}_1$ is the first $n-1$ columns of a real orthogonal matrix $\boldsymbol{U}$ as shown in Lemma 4.

**Proof** Let $\bar{\boldsymbol{e}} = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{e}$ be an orthogonal transformation, then $|\bar{\boldsymbol{e}}| = |\boldsymbol{e}|$. If digraphs $\mathscr{G}$ and $\bar{\mathscr{G}}$ are balanced, then we have $\sum_{i=1}^{n} e_i$=0. Thus,

$$\bar{\boldsymbol{e}} = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{e} = [\boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{e} \;\; 0]^{\mathrm{T}} \triangleq [\boldsymbol{\delta} \;\; 0]^{\mathrm{T}}, \tag{8}$$

where $\boldsymbol{\delta} = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{e} \in \mathbf{R}^{n-1}$.

Then from system (6), we have

$$\mathrm{d}\bar{e} = -\boldsymbol{U}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{U}\bar{e}(t-\tau)\mathrm{d}t - \boldsymbol{U}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{U}\bar{e}(t-\tau)\mathrm{d}w. \tag{9}$$

Noting that digraphs $\mathscr{G}$ and $\bar{\mathscr{G}}$ are also strongly connected, we thus have the following equations from Lemma 4:

$$\begin{aligned}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{U} &= \begin{pmatrix} \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{L}\boldsymbol{U}_1 & \boldsymbol{0}_{(n-1)\times 1} \\ \boldsymbol{0}_{1\times(n-1)} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{\boldsymbol{L}} & \boldsymbol{0}_{(n-1)\times 1} \\ \boldsymbol{0}_{1\times(n-1)} & 0 \end{pmatrix}, \\ \boldsymbol{U}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{U} &= \begin{pmatrix} \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{G}\boldsymbol{U}_1 & \boldsymbol{0}_{(n-1)\times 1} \\ \boldsymbol{0}_{1\times(n-1)} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{\boldsymbol{G}} & \boldsymbol{0}_{(n-1)\times 1} \\ \boldsymbol{0}_{1\times(n-1)} & 0 \end{pmatrix}, \end{aligned} \tag{10, 11}$$

where $\bar{\boldsymbol{L}}, \bar{\boldsymbol{G}} \in \mathbf{R}^{(n-1)\times(n-1)}$ are positive stable. Hence, system (6) is equivalent to the $(n-1)$-dimensional system (7).

**Theorem 1** Consider a directed network of agents with constant time-delay $\tau$ and a fixed topology. Assume that digraphs $\mathscr{G}$ and $\bar{\mathscr{G}}$ are strongly connected and balanced. Then, system (3) almost surely solves the average-consensus problem if for some positive numbers $\varepsilon_1, \varepsilon_2$, there exist positively definite matrices $\boldsymbol{P}, \boldsymbol{Q} \in \mathbf{R}^{(n-1)\times(n-1)}$ satisfying

$$\begin{pmatrix} -\boldsymbol{P}\bar{\boldsymbol{L}} - \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} & \boldsymbol{P}\bar{\boldsymbol{L}} & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} & -\varepsilon_2\boldsymbol{I}_{n-1} & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times(n-1)} & \boldsymbol{0}_{(n-1)\times(n-1)} & -\boldsymbol{Q} + \varepsilon_1\tau\boldsymbol{I}_{n-1} + \bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{G}} \end{pmatrix} < 0, \tag{12}$$

where $\varepsilon_1 > K_1\varepsilon_2$, and $K_1 = 2\tau\|\bar{\boldsymbol{L}}\|^2 + 2\|\bar{\boldsymbol{G}}\|^2$.

**Proof** From Lemma 6, if digraphs $\mathscr{G}$ and $\bar{\mathscr{G}}$ are strongly connected and balanced, then system (3) almost surely solves the average consensus problem if and only if the trivial solution of the $(n-1)$-dimensional system (7) is almost surely globally asymptotically stable.

Next, we will focus on the convergence analysis of system (7). Define a positively definite quadratic Lyapunov functional

$$\begin{aligned}\boldsymbol{V} &= \boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{\delta}(t) + \int_{t-\tau}^{t}\boldsymbol{\delta}^{\mathrm{T}}(s)\boldsymbol{Q}\boldsymbol{\delta}(s)\mathrm{d}s \\ &\quad + \varepsilon_1\int_{-\tau}^{0}\int_{t+\theta}^{t}|\boldsymbol{\delta}(s-\tau)|^2\mathrm{d}s\mathrm{d}\theta, \end{aligned} \tag{13}$$

where $\boldsymbol{P}$, $\boldsymbol{Q} \in \mathbf{R}^{(n-1)\times(n-1)}$ are positively definite matrices, and $\varepsilon_1$ is a positive number. By the Itô formula,[36] we can calculate $\mathrm{d}\boldsymbol{V}$ along the trajectories of system (7), then we have

$$\mathrm{d}\boldsymbol{V} = \mathscr{L}\boldsymbol{V}\mathrm{d}t - 2\boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{P}\bar{\boldsymbol{G}}\boldsymbol{\delta}(t-\tau)\mathrm{d}w, \tag{14}$$

where

$$\begin{aligned}\mathscr{L}\boldsymbol{V} &= -2\boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{P}\bar{\boldsymbol{L}}\boldsymbol{\delta}(t) + 2\boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{P}\bar{\boldsymbol{L}}(\boldsymbol{\delta}(t) - \boldsymbol{\delta}(t-\tau)) \\ &\quad + \boldsymbol{\delta}^{\mathrm{T}}(t-\tau)\bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{G}}\boldsymbol{\delta}(t-\tau) \\ &\quad + \boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\delta}(t) - \boldsymbol{\delta}^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\boldsymbol{\delta}(t-\tau) + \varepsilon_1\tau|\boldsymbol{\delta}(t-\tau)|^2 \\ &\quad - \varepsilon_1\int_{t-\tau}^{t}|\boldsymbol{\delta}(s-\tau)|^2\mathrm{d}s. \end{aligned} \tag{15}$$

Due to the fact that for any $\varepsilon_2 > 0$,

$$\begin{aligned}&2\boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{P}\bar{\boldsymbol{L}}(\boldsymbol{\delta}(t) - \boldsymbol{\delta}(t-\tau)) \\ &\le \varepsilon_2^{-1}\boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{P}\bar{\boldsymbol{L}}\bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\delta}(t) + \varepsilon_2|\boldsymbol{\delta}(t) - \boldsymbol{\delta}(t-\tau)|^2, \end{aligned} \tag{16}$$

we have

$$\begin{aligned}\mathscr{L}\boldsymbol{V} &\le \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Xi}\boldsymbol{\zeta}(t) + \varepsilon_2|\boldsymbol{\delta}(t) - \boldsymbol{\delta}(t-\tau)|^2 \\ &\quad - \varepsilon_1\int_{t-\tau}^{t}|\boldsymbol{\delta}(s-\tau)|^2\mathrm{d}s, \end{aligned} \tag{17}$$

where

$$\boldsymbol{\Xi} = \begin{pmatrix} -\boldsymbol{P}\bar{\boldsymbol{L}} - \bar{\boldsymbol{L}}^{\mathrm{T}}P + \varepsilon_2^{-1}\boldsymbol{P}\bar{\boldsymbol{L}}\bar{\boldsymbol{L}}^{\mathrm{T}}P + \boldsymbol{Q} & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times(n-1)} & -\boldsymbol{Q} + \varepsilon_1\tau\boldsymbol{I}_{n-1} + \bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{G}}, \end{pmatrix},$$

and $\boldsymbol{\zeta}^{\mathrm{T}}(t) = (\boldsymbol{\delta}^{\mathrm{T}}(t), \boldsymbol{\delta}^{\mathrm{T}}(t-\tau))$.

Note that

$$\begin{aligned}&\boldsymbol{\delta}(t) - \boldsymbol{\delta}(t-\tau) \\ &= \int_{t-\tau}^{t}\dot{\boldsymbol{\delta}}(s)\mathrm{d}s \\ &= -\int_{t-\tau}^{t}\bar{\boldsymbol{L}}\boldsymbol{\delta}(s-\tau)\mathrm{d}s - \int_{t-\tau}^{t}\bar{\boldsymbol{G}}\boldsymbol{\delta}(s-\tau)\mathrm{d}w(s). \end{aligned} \tag{18}$$

From the Cauchy–Schwarz inequality, it is easy to compute that

$$E|\boldsymbol{\delta}(t) - \boldsymbol{\delta}(t-\tau)|^2 \le K_1 E\int_{t-\tau}^{t}|\boldsymbol{\delta}(s-\tau)|^2\mathrm{d}s, \tag{19}$$

where $K_1 = 2\tau\|\bar{\boldsymbol{L}}\|^2 + 2\|\bar{\boldsymbol{G}}\|^2$.

If we choose $\varepsilon_1 > K_1\varepsilon_2$ and can find two positively definite matrices $\boldsymbol{P}$, $\boldsymbol{Q}$ for $\boldsymbol{\Xi} < 0$, then setting $-\lambda = \lambda_{\max}(\boldsymbol{\Xi})$ $(\lambda > 0)$, we obtain from inequalities (17) and (19) that

$$\begin{aligned}E(\mathscr{L}\boldsymbol{V}) &\le -\lambda E|\boldsymbol{\delta}(t)|^2 - \lambda E|\boldsymbol{\delta}(t-\tau)|^2 \\ &\quad - (\varepsilon_1 - K_1\varepsilon_2)E\int_{t-\tau}^{t}|\boldsymbol{\delta}(s-\tau)|^2\mathrm{d}s. \end{aligned} \tag{20}$$

From expressions (13) and (20), it is obvious that a positive constant $\alpha$ exists such that

$$\begin{aligned}\alpha E|\delta(t)|^2 &\leq E(\boldsymbol{V}(t)) \leq E(\boldsymbol{V}(t_0)) + E\int_{t_0}^{t} \mathscr{L}\boldsymbol{V}(s)\mathrm{d}s \\ &\leq E(\boldsymbol{V}(t_0)) - \lambda\int_{t_0}^{t} E|\delta(s)|^2\mathrm{d}s. \end{aligned} \tag{21}$$

Then, from the Gronwall inequality, we have

$$E|\delta(t)|^2 \leq \frac{E(\boldsymbol{V}(t_0))}{\alpha}\exp\left[-\frac{\lambda}{\alpha}(t-t_0)\right], \tag{22}$$

which yields the mean-square exponential stability of system (7). It is well-known that for a linear system, the mean-square exponential stability implies the almost sure exponential stability.

From Lemma 5, $\boldsymbol{\Xi} < 0$ is equivalent to the following matrix inequalities:

$$-\boldsymbol{P}\bar{\boldsymbol{L}} - \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} + \varepsilon_2^{-1}\boldsymbol{P}\bar{\boldsymbol{L}}\bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} < 0, \tag{23}$$

$$-\boldsymbol{Q} + \varepsilon_1\tau\boldsymbol{I}_{n-1} + \bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{G}} < 0, \tag{24}$$

and the matrix inequality (23) is equivalent to

$$\begin{pmatrix} -\boldsymbol{P}\bar{\boldsymbol{L}} - \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} & \boldsymbol{P}\bar{\boldsymbol{L}} \\ \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} & -\varepsilon_2\boldsymbol{I}_{n-1} \end{pmatrix} < 0. \tag{25}$$

Hence, the trivial solution of the $(n-1)$-dimensional system (7) is almost surely globally asymptotically stable if the matrix inequality (12) holds. This completes the proof.

**Remark 1** Since $\bar{\boldsymbol{L}}$ is positive stable, then from the Lyapunov theorem, there exists a positively definite matrix $\boldsymbol{P}$ such that $-\boldsymbol{P}\bar{\boldsymbol{L}} - \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} = -\boldsymbol{I}_{n-1}$. Let $\boldsymbol{Q} = \beta\boldsymbol{I}_{n-1}$ with $\beta > 0$, then we can easily transform the matrix inequalities (23) and (24) into the following testable matrix inequalities:

$$-(1-\beta)\boldsymbol{I}_{n-1} + \varepsilon_2^{-1}\boldsymbol{P}\bar{\boldsymbol{L}}\bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} < 0, \tag{26}$$

$$-(\beta - \varepsilon_1\tau)\boldsymbol{I}_{n-1} + \bar{\boldsymbol{G}}^{\mathrm{T}}\bar{\boldsymbol{G}} < 0. \tag{27}$$

Clearly, the matrix inequalities (26) and (27) hold if

$$\lambda_{\max}(\boldsymbol{P}\bar{\boldsymbol{L}}\bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P}) < \varepsilon_2(1-\beta), \tag{28}$$

$$\tau < \frac{1}{\varepsilon_1}(\beta - \lambda_{\max}(\bar{\boldsymbol{G}}^{\mathrm{T}}\bar{\boldsymbol{G}})). \tag{29}$$

This suggests that the matrix inequality (12) is exactly solvable.

**Remark 2** The inequality (29) means that system (3) can tolerate relatively high time delays if the noise is weak, and it can tolerate relatively strong noise if the time delays are low.

If the time delay is varying with time in system (4), i.e.,

$$\mathrm{d}\boldsymbol{x} = -\boldsymbol{L}\boldsymbol{x}(t-\tau(t))\mathrm{d}t - \boldsymbol{G}\boldsymbol{x}(t-\tau(t))\mathrm{d}w, \tag{30}$$

we have the following result.

**Theorem 2** Consider a directed network of agents with time-varying delay $\tau(t)$ and a fixed topology. Assume that digraphs $\mathscr{G}$ and $\bar{\mathscr{G}}$ are strongly connected and balanced. Then, system (30) almost surely solves the average-consensus problem if for some positive numbers $\varepsilon_1, \varepsilon_2, \gamma, \rho$, and $\tau(t) < \gamma$, $\dot{\tau} \leq \rho < 1$, there exist positively definite matrices $\boldsymbol{P}$, $\boldsymbol{Q} \in \mathbf{R}^{(n-1)\times(n-1)}$ satisfying

$$\begin{pmatrix} -\boldsymbol{P}\bar{\boldsymbol{L}} - \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} & \boldsymbol{P}\bar{\boldsymbol{L}} & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \bar{\boldsymbol{L}}^{\mathrm{T}}\boldsymbol{P} & -\varepsilon_2\boldsymbol{I}_{n-1} & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times(n-1)} & \boldsymbol{0}_{(n-1)\times(n-1)} & -(1-\rho)\boldsymbol{Q} + \varepsilon_1\gamma\tau\boldsymbol{I}_{n-1} + \bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{G}} \end{pmatrix} < 0, \tag{31}$$

where $\varepsilon_1(1-\rho) > K_2\varepsilon_2$, and $K_2 = 2\gamma\|\bar{\boldsymbol{L}}\|^2 + 2\|\bar{\boldsymbol{G}}\|^2$.

**Proof** In fact, this theorem can be proved following the same lines of the proof of Theorem 1. The main ideas used in that proof essentially remain valid even if the delay is time-varying in the equations.

### 3.2. Networks with switching topology

Consider a network of mobile agents that communicate with each other and need to agree upon a certain objective of interest or perform a synchronization. Since the nodes of the network are moving, it is not hard to imagine that some of the existing communication links can fail simply due to the existence of an obstacle between the agents. The opposite situation can arise when new links between nearby agents are created because the agents come to an effective range of detection with respect to each other. In terms of the network topology, this means that a certain number of edges are added or removed from the graph. Here, we are interested in investigating whether it is still possible to reach a consensus in the case of a network with a switching topology. In this case, we consider the following hybrid system:

$$\mathrm{d}\boldsymbol{x} = -\boldsymbol{L}_\kappa\boldsymbol{x}(t-\tau)\mathrm{d}t - \boldsymbol{G}_\kappa\boldsymbol{x}(t-\tau)\mathrm{d}w, \quad \kappa = s(t) \in \Gamma, \tag{32}$$

where $\boldsymbol{L}_\kappa$ and $\boldsymbol{G}_\kappa$ are the Laplacians of digraphs $\mathscr{G}_\kappa$ and $\bar{\mathscr{G}}_\kappa$, respectively, and $s(t) : [0, +\infty) \to \Gamma$ is a switching signal that selects elements from a finite collection of digraphs $\Gamma = \{\mathscr{G}_1, \mathscr{G}_2, \ldots\}$. When each element of $\Gamma$ is balanced, then under the arbitrary switching signal $s$, $\alpha = \mathrm{Ave}(x)$ is also an invariant quantity. This allows the decomposition of any solution $\boldsymbol{x}(t)$ of system (32) in the form $\boldsymbol{x} = \boldsymbol{e} + \alpha\boldsymbol{1}$. Therefore, the disagreement system induced by system (32) takes the following form:

$$\mathrm{d}\boldsymbol{e} = -\boldsymbol{L}_\kappa\boldsymbol{e}(t-\tau)\mathrm{d}t - \boldsymbol{G}_\kappa\boldsymbol{e}(t-\tau)\mathrm{d}w. \tag{33}$$

**Theorem 3** Consider a directed network of agents with constant time-delay $\tau$ and a switching topology. Assume that digraphs $\mathscr{G}_\kappa$ and $\bar{\mathscr{G}}_\kappa$ are strongly connected and balanced for any arbitrary switching signal $s$. Then system (32) almost surely solves the average-consensus problem if for some positive numbers $\varepsilon_1, \varepsilon_2$, there exist positively definite matrices $\boldsymbol{P},\ \boldsymbol{Q} \in \mathbf{R}^{(n-1)\times(n-1)}$ satisfying

$$\begin{pmatrix} -\boldsymbol{P}\bar{\boldsymbol{L}}_\kappa - \bar{\boldsymbol{L}}_\kappa^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} & \boldsymbol{P}\bar{\boldsymbol{L}}_\kappa & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \bar{\boldsymbol{L}}_\kappa^{\mathrm{T}}\boldsymbol{P} & -\varepsilon_2\boldsymbol{I}_{n-1} & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times(n-1)} & \boldsymbol{0}_{(n-1)\times(n-1)} & -\boldsymbol{Q} + \varepsilon_1\tau\boldsymbol{I}_{n-1} + \bar{\boldsymbol{G}}_\kappa^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{G}}_\kappa \end{pmatrix} < 0, \tag{34}$$

where $\varepsilon_1 > K_1\varepsilon_2$, $K_1 = \max\limits_{\kappa\in\Gamma}\{2\tau\|\bar{\boldsymbol{L}}_\kappa\|^2 + 2\|\bar{\boldsymbol{G}}_\kappa\|^2\}$, $\bar{\boldsymbol{L}}_\kappa = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{L}_\kappa\boldsymbol{U}_1$, $\bar{\boldsymbol{G}}_\kappa = \boldsymbol{U}_1^{\mathrm{T}}\boldsymbol{G}_\kappa\boldsymbol{U}_1$, and $\boldsymbol{U}_1$ are the first $n-1$ columns of a real orthogonal matrix $\boldsymbol{U}$ as shown in Lemma 4.

**Proof** The Lyapunov function $\boldsymbol{V}(\boldsymbol{\delta})$ in Eq. (13) does not depend on the network topology. Moreover, topologies $\mathscr{G}_\kappa$ and $\bar{\mathscr{G}}_\kappa$ are strongly connected and balanced for any arbitrary switching signal. Then from Lemma 6, system (33) is equivalent to the following $(n-1)$-dimensional system:

$$\mathrm{d}\boldsymbol{\delta} = -\bar{\boldsymbol{L}}_\kappa\boldsymbol{\delta}(t-\tau)\mathrm{d}t - \bar{\boldsymbol{G}}_\kappa\boldsymbol{\delta}(t-\tau)\mathrm{d}w. \tag{35}$$

Thus, $\boldsymbol{V}(\boldsymbol{\delta})$ is also a valid Lyapunov function for the stability analysis of the switching system (35). The following proof is similar to that of Theorem 1 and omitted here.

If the time delay is time-varying in system (32), i.e.,

$$\mathrm{d}\boldsymbol{x} = -\boldsymbol{L}_\kappa\boldsymbol{x}(t-\tau(t))\mathrm{d}t - \boldsymbol{G}_\kappa\boldsymbol{x}(t-\tau(t))\mathrm{d}w, \tag{36}$$

we have the following result.

**Theorem 4** Consider a directed network of agents with time-varying delay $\tau(t)$ and a switching topology. Assume that digraphs $\mathscr{G}_\kappa$ and $\bar{\mathscr{G}}_\kappa$ are strongly connected and balanced for any arbitrary switching signal $s$. Then system (36) almost surely solves the average-consensus problem if for some positive numbers $\varepsilon_1, \varepsilon_2, \gamma, \rho$, and $\tau(t) < \gamma$, $\dot{\tau} \le \rho < 1$, there exist positively definite matrices $\boldsymbol{P},\ \boldsymbol{Q} \in \mathbf{R}^{(n-1)\times(n-1)}$ satisfying

$$\begin{pmatrix} -\boldsymbol{P}\bar{\boldsymbol{L}}_\kappa - \bar{\boldsymbol{L}}_\kappa^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} & \boldsymbol{P}\bar{\boldsymbol{L}}_\kappa & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \bar{\boldsymbol{L}}_\kappa^{\mathrm{T}}\boldsymbol{P} & -\varepsilon_2\boldsymbol{I}_{n-1} & \boldsymbol{0}_{(n-1)\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times(n-1)} & \boldsymbol{0}_{(n-1)\times(n-1)} & -(1-\rho)\boldsymbol{Q} + \varepsilon_1\gamma\tau\boldsymbol{I}_{n-1} + \bar{\boldsymbol{G}}_\kappa^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{G}}_\kappa \end{pmatrix} < 0, \tag{37}$$

where $\varepsilon_1(1-\rho) > K_2\varepsilon_2$, and $K_2 = \max\limits_{\kappa\in\Gamma}\{2\gamma\|\bar{\boldsymbol{L}}_\kappa\|^2 + 2\|\bar{\boldsymbol{G}}_\kappa\|^2\}$.

**Remark 3** Since $\bar{\boldsymbol{L}}_\kappa$ and $\bar{\boldsymbol{G}}_\kappa$ are time-varying matrices, the matrix equalities (34) and (37), which can be easily solved by using numerical software, should be satisfied for all possible graphs of $\Gamma$.

## 4. Simulation results

Numerical simulations are given to illustrate the theoretical results obtained in the previous sections. In all the simulations, all equations are integrated with a fixed time step 0.001, and the initial values of all nodes are randomly taken from interval $[-1,\ 1]$, but satisfying $\sum_i x_i(0) = 0$.

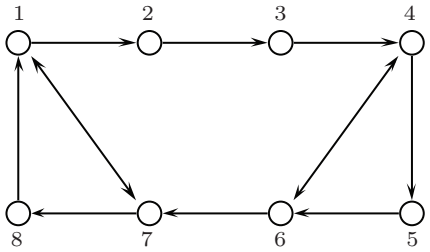



**Fig. 1.** An example of a strongly connected and balanced digraph with 8 nodes.

**Example 1** Consider a directed network with fixed topology $\mathscr{G}$ as shown in Fig. 1. It is obvious that digraph $\mathscr{G}$ is strongly connected and balanced. Let $\boldsymbol{G} = \sigma_0\boldsymbol{L}$ ($\sigma_0 > 0$ representing the noise intensity), hence digraph $\bar{\mathscr{G}}$ is also strongly connected and balanced. For simplicity, take $\varepsilon_1 = 1.01K_1$ and $\varepsilon_2 = 1$. The matrix $\bar{\boldsymbol{L}}$ is given by

$$\bar{\boldsymbol{L}} = \begin{pmatrix} 2.8393 & -0.0079 & -0.3199 & 0.4300 & -0.0841 & -1.4193 & -0.5824 \\ 0 & 0.5804 & 0.7786 & 0.5440 & 0.3230 & -0.2150 & -0.4601 \\ 0 & -0.4721 & 0.5804 & 0.5938 & -0.2350 & -0.0923 & 0.2164 \\ 0 & 0 & 0 & 2.0000 & 1.1547 & 0.4963 & -1.0426 \\ 0 & 0 & 0 & 0 & 2.0000 & -0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 0 & 2.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0000 \end{pmatrix}.$$

For a given $\sigma_0$ (or $\tau$), applying Theorem 1, we can get the maximum delay bounds (or noise intensity bounds). For a given $\sigma_0 = 0.1$, applying the LMI toolbox in Matlab, we find that the maximum delay bound is $\tau = 0.123$. It is not hard to obtain $K_1 = 3.2167$ and $\varepsilon_1 = 3.2488$, and the corresponding positively definite matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are

$$\boldsymbol{P} = \begin{pmatrix} 0.3101 & 0.0067 & -0.0056 & -0.0198 & 0.0199 & 0.0856 & 0.0187 \\ 0.0067 & 0.6204 & -0.0923 & -0.1057 & -0.0303 & 0.0762 & 0.0698 \\ -0.0056 & -0.0923 & 0.8372 & -0.1688 & 0.1685 & 0.0692 & -0.1537 \\ -0.0198 & -0.1057 & -0.1688 & 0.4830 & -0.1591 & -0.0946 & 0.1311 \\ 0.0199 & -0.0303 & 0.1685 & -0.1591 & 0.6251 & 0.0480 & -0.0863 \\ 0.0856 & 0.0762 & 0.0692 & -0.0946 & 0.0480 & 0.6017 & -0.0174 \\ 0.0187 & 0.0698 & -0.1537 & 0.1311 & -0.0863 & -0.0174 & 0.6139 \end{pmatrix},$$

$$\boldsymbol{Q} = \begin{pmatrix} 0.6773 & 0.0125 & -0.0314 & 0.0022 & 0.0198 & 0.0242 & -0.0104 \\ 0.0125 & 0.4289 & -0.0090 & 0.0471 & -0.0402 & -0.0132 & 0.0382 \\ -0.0314 & -0.0090 & 0.4183 & -0.0177 & 0.0109 & 0.0330 & 0.0036 \\ 0.0022 & 0.0471 & -0.0177 & 0.6337 & -0.0126 & 0.0075 & 0.0175 \\ 0.0198 & -0.0402 & 0.0109 & -0.0126 & 0.7117 & -0.0191 & 0.0345 \\ 0.0242 & -0.0132 & 0.0330 & 0.0075 & -0.0191 & 0.7122 & 0.0027 \\ -0.0104 & 0.0382 & 0.0036 & 0.0175 & 0.0345 & 0.0027 & 0.7142 \end{pmatrix}.$$

Figure 2 shows the state trajectories of system (3) with time delays of $\tau = 0.1$ and $\tau = 0.123$. As shown in Figs. 2(a) and 2(b), the average consensus is achieved asymptotically for the cases with time-delay $\tau \leq 0.123$ and $\sigma_0 = 0.1$.

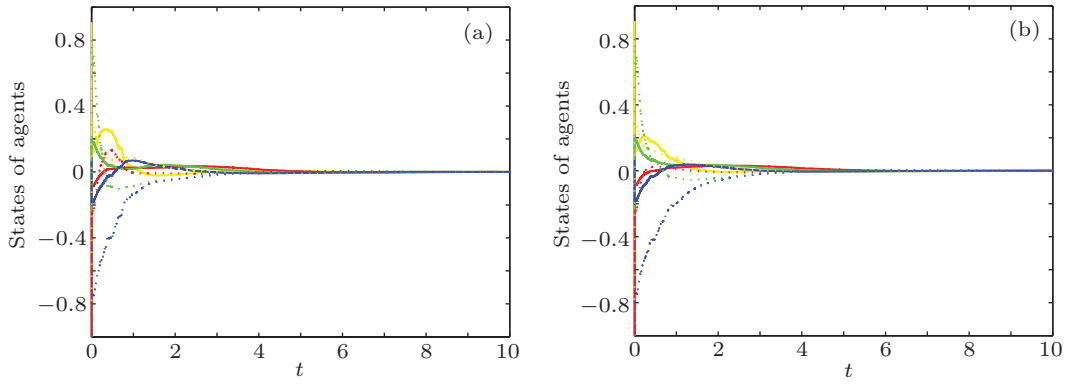



**Fig. 2.** (color online) State trajectories of system (3) when (a) $\tau = 0.1$ and (b) $\tau = 0.123$, where $\sigma_0 = 0.1$.

The matrix inequality (12) is dependent on both $\tau$ and $\sigma_0$. From Remark 2, it is expected that the consensus could be achieved for a case with a relatively large noise intensity $\sigma_0$ and a relatively low time delay $\tau$. Actually, this is the case for $\tau = 0.05$ , the maximum noise intensity is $\sigma_0 = 0.446$. For the case of $\tau = 0.05$ and $\sigma_0 = 0.446$, it is not hard to obtain $K_1 = 6.0201$ and $\varepsilon_1 = 6.0803$, and the corresponding positively definite matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are

$$\boldsymbol{P} = \begin{pmatrix} 0.1199 & -0.0043 & 0.0134 & 0.0139 & 0.0103 & 0.0355 & 0.0055 \\ -0.0043 & 0.4946 & -0.0569 & -0.1438 & 0.0521 & 0.0978 & -0.0111 \\ 0.0134 & -0.0569 & 0.6956 & -0.1090 & 0.1103 & 0.0120 & -0.1165 \\ 0.0139 & -0.1438 & -0.1090 & 0.2368 & -0.0703 & -0.0209 & 0.0678 \\ 0.0103 & 0.0521 & 0.1103 & -0.0703 & 0.3635 & -0.0314 & 0.0897 \\ 0.0355 & 0.0978 & 0.0120 & -0.0209 & -0.0314 & 0.2750 & -0.0523 \\ 0.0055 & -0.0111 & -0.1165 & 0.0678 & 0.0897 & -0.0523 & 0.3337 \end{pmatrix},$$

$$\boldsymbol{Q} = \begin{pmatrix} 0.5220 & -0.0053 & -0.0190 & 0.0643 & 0.0265 & -0.0191 & -0.0384 \\ -0.0053 & 0.3752 & 0.0077 & -0.0197 & 0.0199 & 0.0164 & -0.0142 \\ -0.0190 & 0.0077 & 0.4023 & 0.0066 & 0.0056 & 0.0012 & -0.0057 \\ 0.0643 & -0.0197 & 0.0066 & 0.4808 & 0.0549 & 0.0547 & -0.0348 \\ 0.0265 & 0.0199 & 0.0056 & 0.0549 & 0.6930 & -0.0461 & 0.1476 \\ -0.0191 & 0.0164 & 0.0012 & 0.0547 & -0.0461 & 0.6074 & -0.0626 \\ -0.0384 & -0.0142 & -0.0057 & -0.0348 & 0.1476 & -0.0626 & 0.6527 \end{pmatrix}.$$

Clearly, we can observe that the average consensus is achieved asymptotically for the cases of $\tau = 0.05$ and $\sigma_0 \leq 0.446$ in Figs. 3(a) and 3(b).

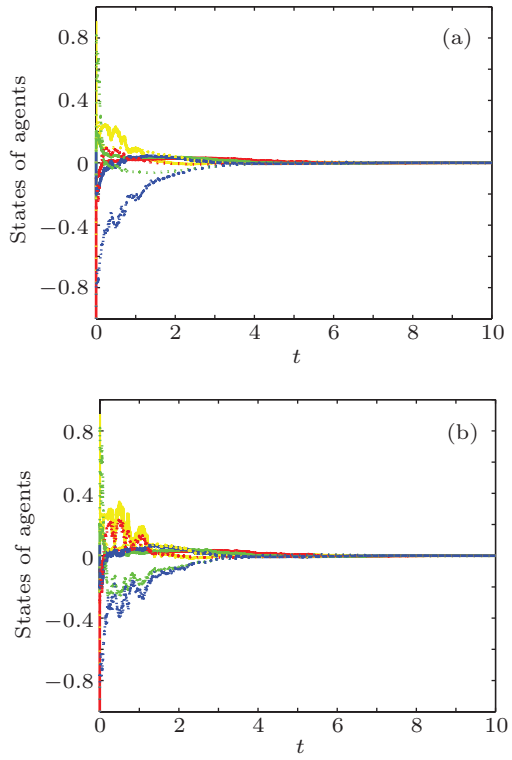



**Fig. 3.** (color online) State trajectories of system (3) when (a) $\sigma_0 = 0.3$ and (b) $\sigma_0 = 0.446$, where $\tau = 0.05$.

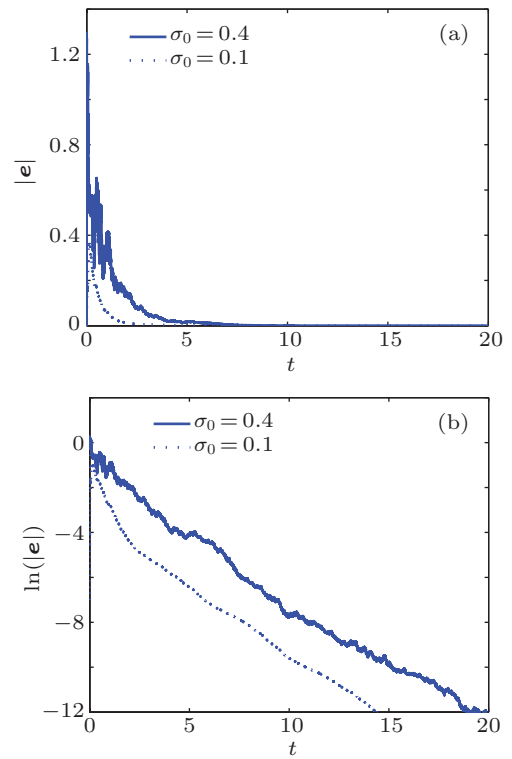



**Fig. 4.** (color online) (a) The variance of $|\boldsymbol{e}(t)|$ with time $t$ of system (6) when $\tau = 0.1$, $\sigma_0 = 0.1$ (dashed line), and $\sigma_0 = 0.4$ (solid line). (b) The corresponding logarithmic plots.

For the case with a time-varying delay, applying Theorem 2, we can also calculate the maximum time-delay bounds and the maximum noise intensity bounds. The simulation results (not shown) are also consistent with Theorem 2. From Figs. 2 and 3, we can see that the average consensus can be achieved with the perturbation of noise. Moreover, Figures 4(a) and 4(b) show that the speed of convergence in the environment with a large noise intensity is lower than that in the environment with a relatively small noise intensity.

**Example 2** In the switching topology case, the assumption of Theorems 3 and 4 about digraphs $\mathcal{G}_\kappa$ and $\bar{\mathcal{G}}_\kappa$ is exceedingly conservative. For simplicity, take $\boldsymbol{G}_\kappa = \sigma_0 \boldsymbol{L}_\kappa$. The following numerical example shows that the average consensus can be achieved when no elements of $\Gamma$ are connected.

Here we consider a directed network with switching topology $\{\mathcal{G}_a, \mathcal{G}_b, \mathcal{G}_c, \mathcal{G}_d\}$ with the property that none of them is strongly connected. That is each graph contains pairs of nodes that do not have a path between them. However, the union of vertices over all four graphs is strongly connected. All four graphs are shown in Fig. 5. All graphs in this figure are balanced. The hybrid system starts at the discrete state $\mathcal{G}_a$ and switches after $\Delta t_s = 0.001$ to the next state with the following switching order: $\{\mathcal{G}_a, \mathcal{G}_b, \mathcal{G}_c, \mathcal{G}_d, \mathcal{G}_a, \mathcal{G}_b, \mathcal{G}_c, \mathcal{G}_d, \ldots\}$.

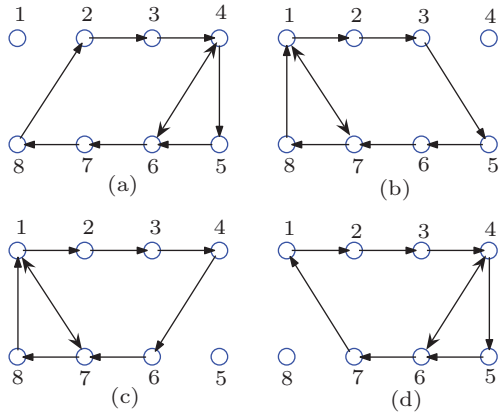



**Fig. 5.** Four examples of balanced digraphs. Panels (a)–(d) are digraphs $\mathcal{G}_a$ through $\mathcal{G}_d$.

Take $\tau(t) = 0.1|\sin(t)|$ and $\sigma_0 = 0.1$. The state trajectories and the group disagreement $|\boldsymbol{e}|$ of system (36) are shown in Figs. 6(a) and 6(b), respectively. Clearly, the group disagreement is decreasing, and the average consensus is achieved asymptotically. The simulation results show that a network with a switching topology could propagate sufficient information to allow the synchronization of its agents, despite an instantaneously disconnected topology.

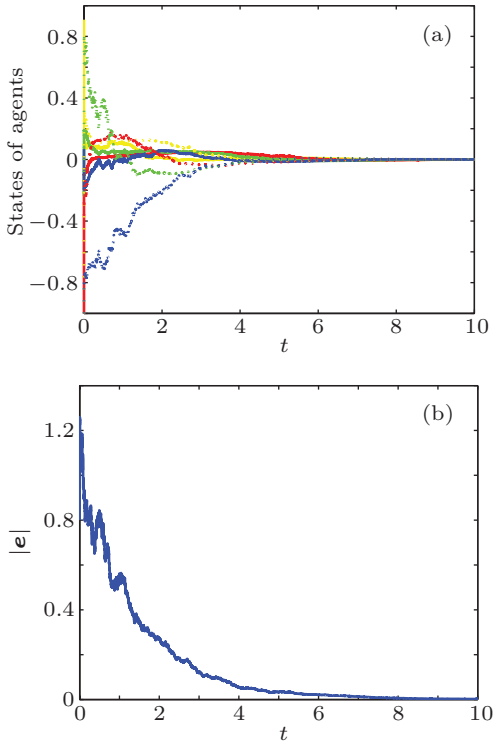



**Fig. 6.** (color online) (a) State trajectories and (b) the group disagreement for a network with switching topologies and time-varying delays.

## 5. Conclusion

The average-consensus problem of noise-perturbed multi-agent systems with fixed and switching topologies as well as communication time delays has been investigated. By virtue of the stability theory for the stochastic differential delay equations, sufficient conditions in terms of LMIs ensuring the successful consensus with probability one have been rigorously established. The numerical simulations are very consistent with the theoretical results. The simulation results show that the speed of convergence in the environment with a large noise intensity is lower than that in the environment with a relatively small noise intensity. In particular, the simulation results show that a network of agents with a switching topology can reach the average consensus even if at every point in time the network topology is insufficiently connected to achieve the consensus.

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