Chaotic dynamic behavior analysis and control for a financial risk system*

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According to the risk management process of financial markets, a financial risk dynamic system is constructed in this paper. Through analyzing the basic dynamic properties, we obtain the conditions for stability and bifurcation of the system based on Hopf bifurcation theory of nonlinear dynamic systems. In order to make the system's chaos disappear, we select the feedback gain matrix to design a class of chaotic controller. Numerical simulations are performed to reveal the change process of financial market risk. It is shown that, when the parameter of risk transmission rate changes, the system gradually comes into chaos from the asymptotically stable state through bifurcation. The controller can then control the chaos effectively.

Keywords: chaos attractor, Hopf bifurcation, financial risk, chaos feedback control

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1. Introduction

Chaos,[1,2] as a special complex dynamic phenomenon, has been studied for about four decades. It has now been applied to various disciplines of natural science and social science, and forms a new subject or a new research field.[3–5] Chaotic economics is a new physical economics theory. It is able to reveal ordered structures and regularity hidden in the seemingly random economic phenomena, and provides a way of understanding the complexity of economic phenomena as some of their own internal structures or purposive behaviors, rather than external or accidental behaviors. Initially, the related work is mainly concentrated in the business cycle[6–8] and the volatility of security price.[9–11] Based on empirical applications and the development of the chaos theory, it is gradually extended to more economic areas, such as monetary aggregate,[12,13] duopoly game,[14,15] and the relationship between R&D spending and production growth.[16,17]

In the economic field, financial risk is one of the most important problems which nations and all economic entities are concerned about. Financial risk means the possibilities of suffering losses caused by uncertain changes of endogenous factors in financial activities or investment activities, which display erratic fluctuations. Chaos supports an endogenous explanation of the complexity observed in an economic time series. According to chaos theory, risk behaviors of different scale systems have self-similarity. The origin of risk lies in the strange attractor, while the key of risk management lies in the control of the chaotic attractor, which decides the formulation of risk control measures.[18,19]

Since Burlando[20] introduced the structure and terms of chaos theory to risk management, some scholars have tried applying the chaos theory to study financial market risks. Zhou[21] thought that financial crisis is a kind of expression of chaos characteristics. She used the Duffing–Holmes model to discuss the production conditions of financial risk and give some concrete measures in risk control. In recent years, chaotic economics has obtained extensive attention and has been raised to an alternative scientific methodology to understand the complex dynamics of the real financial market.[22–24]

In this paper, according to Ref. [25], an improved financial risk system is constructed. Through dynamic behavior analysis, it reveals the procedure from stability to chaos for the improved financial risk system, and describes the change of financial market risk using the complex dynamics of the system. Besides, it is an important mission to design the control system of chaos, because chaos disappearance is of great significance to guard against financial system risks. Inspired by some recent work on designing controllers,[26–28] we applied the positive feedback gain matrix method[29,30] to construct the controlled system which makes the system's chaos disappear quickly. At the same time, the asymptotic stability of the controlled system is proved. This means that the financial market risk can be controlled effectively under some conditions.

2. Model construction and analysis

Based on Ref. [25], a cross-product item is added to the first equation, and one has the following improved model de-

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scribing the risk management process in the financial market:

$$\begin{cases} R_i = mU_{i-1} - n(R_{i-1} - dU_{i-1}V_{i-1}), \\ U_i = rR_{i-1} - R_{i-1}V_{i-1}, \\ V_i = R_{i-1}U_{i-1} + kV_{i-1}, \end{cases} \tag{1}$$

where R_i, U_i, and V_i represent occurrence value risk, analysis value risk, and control value risk in the current market, respectively; $U_{i-1}V_{i-1}$ stands for the value risk preliminarily controlled by the analysis risk; $R_{i-1}V_{i-1}$ is the value risk preliminarily controlled by the occurrence risk; $R_{i-1}U_{i-1}$ represents the value risk preliminarily analyzed to the occurrence risk; m is the analysis risk efficiency; n is the risk satisfaction rate; d is the risk control efficiency; r is the transmission rate of previous risk; k stands for the distortion coefficient of risk control; $m,n,d,r \in R^+$, $k \in R$. Because risk in financial markets always exists as the market occurs, it is obvious that the three state variables R_i, U_i, and V_i are nonnegative.

Different from the system in Ref. [25], the model (1) reflects the influence of controlling the previous risk analysis value on current risk. This can be closer to the actual situation of risk management process in the financial market. Actually, as shown in the first equation of system (1), the occurrence of current risk is decided by the previous risk analysis result, the judgment on previous risk, and the previous risk control to risk analysis.

Next, we will investigate the dynamic properties of the financial risk system (1). Let $m = \delta$, $n+1 = \delta$, $nd = 1$, and $k = 1-b$, then the discrete system (1) can be transformed into the following continuous form:

$$\begin{cases} R' = \delta(U - R) + UV, \\ U' = rR - U - RV, \\ V' = RU - bV. \end{cases} \tag{2}$$

2.1. Stability analysis of equilibrium

Let the right hand of system (2) be zero, by solving the system of equations, we can obtain five equilibriums of system (2): one zero equilibrium ($E_0(0,0,0)$), four nonzero equilibriums ($E_{1,2}(\pm R_1^*, \pm U_1^*, V_1^*)$, $E_{3,4}(\pm R_2^*, \pm U_2^*, V_2^*)$), where

$$R_{1,2}^* = \sqrt{\frac{b(r-\delta \pm \sqrt{\Delta})}{r+\delta \mp \sqrt{\Delta}}},$$
$$U_{1,2}^* = \frac{1}{2}\sqrt{b}\sqrt{r^2 - (\delta \mp \sqrt{\Delta})^2},$$
$$V_{1,2}^* = \frac{1}{2}(r-\delta \pm \sqrt{\Delta}), \Delta = (r+\delta)^2 - 4\delta.$$

As stated above, in the financial market, the three state variables R_i, U_i, and V_i are nonnegative, so we are only interested in the positive equilibriums. From the calculated results, we can obtain the following conclusions.

(i) $E_1(R_1^*, U_1^*, V_1^*)$ is a positive equilibrium ($R_1^* > 0, U_1^* > 0, V_1^* > 0$) if and only if $b > 0$, $r - \delta + \sqrt{\Delta} > 0$, and $\Delta \geq 0$.

(ii) $E_3(R_1^{**}, U_1^{**}, V_1^{**})$ is a positive equilibrium if and only if $b > 0$, $r - \delta - \sqrt{\Delta} > 0$, and $\Delta \geq 0$.

Obviously, when the conditions in situation (ii) are satisfied, both $E_3(R_2^*, U_2^*, V_2^*)$ and $E_1(R_1^*, U_1^*, V_1^*)$ are the positive equilibriums. Note that $E_2(-R_1^*, -U_1^*, V_1^*)$ and $E_4(-R_2^*, -U_2^*, V_2^*)$ are not the positive equilibriums in any case.

Next, we will discuss the stability at the equilibriums E_0 and E_1. The stability at E_3 can be discussed similarly. By linearizing system (2), the Jacobin matrix at any equilibrium $\overline{E}(\overline{R}, \overline{U}, \overline{V})$ is given by

$$\boldsymbol{J} = \begin{bmatrix} -\delta & \delta + \overline{V} & \overline{U} \\ r - \overline{V} & -1 & -\overline{R} \\ \overline{U} & \overline{R} & -b \end{bmatrix}.$$

The characteristic polynomial of matrix $\boldsymbol{J}$ is $f(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$.

For zero equilibrium $E_0(0,0,0)$, the corresponding eigenvalues are

$$\lambda_1 = -b, \lambda_{2,3} = -\frac{\delta+1}{2} \pm \frac{1}{2}\sqrt{(\delta+1)^2 + 4\delta(r-1)}.$$

Thus, if $r < 1$, then the equilibrium E_0 is locally asymptotically stable; if $r > 1$, then the equilibrium E_0 is an unstable saddle point.

At the zero equilibrium, the system is locally asymptotically stable when the risk transmission rate r is less than one, which means that the financial market risk tends to be zero under a low risk transmission rate. However, this situation is not supposed to occur, because the financial risk cannot be completely eliminated in the real financial market. When the risk transmission rate r is greater than one, $E_0(0,0,0)$ is unstable, which means that the financial market risk is not determinate at this moment.

For nonzero equilibrium $E_1(R_1^*, U_1^*, V_1^*)$, the parameters of the characteristic polynomial are

$$\begin{cases} p_1 = \delta + b + 1, \\ p_2 = (R_1^*)^2 + (V_1^*)^2 - (U_1^*)^2 + b\delta + \delta \\ \quad + b + \delta V_1^* - \delta r - rV_1^*, \\ p_3 = \delta(R_1^*)^2 - (U_1^*)^2 + b(V_1^*)^2 + 2R_1^*U_1^*V_1^* + b\delta \\ \quad + \delta R_1^*U_1^* + \delta bV_1^* - rR_1^*U_1^* - \delta br - brV_1^*. \end{cases} \tag{3}$$

According to the Rorth–Hurwitz criterion, if the parameters satisfy the following conditions:

$$p_1 > 0, \ \ p_3 > 0, \ \ p_1p_2 - p_3 > 0, \tag{4}$$

then the equilibrium E_1 is locally asymptotically stable.

The above results show that $E_0(0,0,0)$ is unstable when the transmission rate of risk r is greater than one. However, in the meantime, the equilibrium E_1 is asymptotically stable within a certain scope of parameters.

When the system is in a stable state, the market risk is in a controllable range, and the market itself can adjust the risk

within a certain scope that will not influence normal economic activities. Otherwise, the market risk reaches an uncontrollable extent. In this case, efficient measures should be taken to control risk.

2.2. Bifurcation analysis

Bifurcation is a way in which the system goes into chaos and complexity from stability. If the risk system is stable, the trend of the system is predictable; if the risk system is unstable, then no matter how good the prediction model fits the historical data, the prediction of the trend is always unreliable. When the stable risk system goes through the bifurcation node, the stable trend of the risk system will change with the occurrence of the period-doubling bifurcation, and with the stable risk cycle becoming shorter and shorter, the risk system will gradually lose its stability and go into chaos.

In the following, we will study the Hope bifurcation at the equilibrium E_1. Suppose the characteristic equation $f(\lambda)=0$ has pure imaginary roots $\lambda_{2,3}=\pm \mathrm{i}\omega$ for $\omega>0$. Since $\lambda_1+\lambda_2+\lambda_3=-(\delta+b+1)$, we have $\lambda_1=-(\delta+b+1)$. Substituting λ_1 into $f(\lambda)$, it follows

$$0=f(-\delta-b-1)=p_2\lambda_1+p_3. \tag{5}$$

Note that $p_3=-\lambda_1\lambda_2\lambda_3=-\lambda_1\omega^2$, we have $\omega^2=p_2$. Hence, two pure imaginary roots are $\lambda_{2,3}=\pm \mathrm{i}\sqrt{p_2}$. Besides, substituting p_3 and p_2 into Eq. (5), it follows

$$r_{\mathrm{h}}=\frac{2R_1^*U_1^*V_1^*+\delta R_1^*U_1^*-(b+1)(R_1^*)^2-(\delta+1)(V_1^*)^2+(\delta+b)(U_1^*)^2}{R_1^*U_1^*-\delta^2-\delta-\delta V_1^*-V_1^*}$$
$$-\frac{b(\delta+1)(\delta+b+1)+\delta(\delta+1)+\delta(\delta+1)V_1^*}{R_1^*U_1^*-\delta^2-\delta-\delta V_1^*-V_1^*}.$$

If $r=r_h>1$, then Hopf bifurcation occurs at equilibrium E_1.

Combined with the actual background of the model, the occurrence of Hopf bifurcation implies that the management system of market risk is in a transition from a controllable stable state to other multiple states. The market risk continuously changes after bifurcation occurs, and the management system may remain stable in some states. However, the cycle becomes shorter and shorter, and ultimately cannot maintain a stable state.

3. Chaos control of the financial risk system

The goal of chaos control is to make the system's trajectory tend to be a limited cycle or an equilibrium, so that the system is stable. In this section, the feedback gain matrix method[31] is applied to control chaos. In addition, the asymptotic stability of the controlled system is also proved by the stability theory of the nonlinear dynamic system.

According to the financial risk systems (2), a new controlled system is constructed as follows:

$$\begin{cases} R'=\delta(U-R)+UV-k_1(R-R_1^*), \\ U'=rR-U-RV-k_2(U-U_1^*), \\ V'=RU-bV-k_3(V-V_1^*), \end{cases} \tag{6}$$

where $E_1(R_1^*,U_1^*,V_1^*)$ is the equilibrium of system (2), k_1, k_2, and k_3 are the positive numbers which stand for the positive feedback gain, $U_1=-k_1(R-R_1^*)$, $U_2=-k_2(U-U_1^*)$, and $U_3=-k_3(V-V_1^*)$ are the external control input. Obviously, $E_1(R_1^*,U_1^*,V_1^*)$ is also an equilibrium of system (6). Let $c=r-\delta+\sqrt{\Delta}$ and $d=r+\delta-\sqrt{\Delta}$, according to the preceding results, we have

$$E_1(R_1^*,U_1^*,V_1^*)=\left(\sqrt{\frac{bc}{d}},\frac{1}{2}\sqrt{bcd},\frac{1}{2}c\right).$$

After simple derivation, we can obtain the following conclusions.

There exist three nonnegative numbers k_1, k_2, and k_3 such that the controlled system (6) is asymptotically stable at equilibrium E_1. In fact, let $\boldsymbol{J}$ be the Jacobin matrix of system (6) at equilibrium E_1. The characteristic polynomial of the matrix $\boldsymbol{J}$ is $f(\lambda)=\lambda^3+G_1\lambda^2+G_2\lambda+G_3$, where

$$\begin{aligned} G_1&=p_1+k_1+k_2+k_3, \\ G_2&=p_2+(1+b)k_1+(\delta+b)k_2+(\delta+1)k_3 \\ &\quad +k_1k_2+k_1k_3+k_2k_3, \\ G_3&=p_3+\left(\frac{bc}{d}+b\right)k_1+\left(\delta b-\frac{1}{4}bcd\right)k_2 \\ &\quad +\left(\frac{1}{2}c\delta-r\delta+\frac{1}{4}c^2-\frac{1}{2}cr+\delta\right)k_3+bk_1k_2 \\ &\quad +k_1k_3+\delta k_2k_3+k_1k_2k_3. \end{aligned}$$

The parameters p_1, p_2, and p_3 are defined by Eq. (3). It is obvious that $G_1>0$ for any nonnegative numbers k_1, k_2, and k_3. Next, we will choose parameters k_1, k_2, and k_3 such that they satisfy $G_1>0$ and $G_1G_2-G_3>0$. In order to simplify the treatment, we choose $k_3=0$ below. Thus, $G_3>0$ when $k_1>cd/4$ and $k_2>-p_3/(\delta b)$. Since

$$\begin{aligned} G_1G_2-G_3 &> p_1p_2-p_3+(k_1+k_2)p_2 \\ &\quad +(k_1+k_2)[(1+b)k_1+(\delta+b)k_2+k_1k_2] \\ &\quad -\frac{bc}{d}k_1+\frac{1}{4}bcdk_2, \end{aligned}$$

it follows $G_1G_2-G_3>0$ when

$$k_1>-p_2/(1+b),$$

$$k_2 > \max\left\{\frac{4(p_3 - p_1 p_2)}{bcd}, \frac{bc}{d(\delta + b)}\right\}.$$

Hence, when the parameters k_1 and k_2 are large enough, the Rorth–Hurwitz criterion is satisfied. Therefore, the controlled system (6) is asymptotically stable at equilibrium E_1.

4. Numerical simulation

4.1. Chaotic attractors in the financial risk system

With given parameters $\delta = 10$, $b = 8/3$, and $r = 28$, and chosen initial point $(R_0, U_0, V_0) = (20, 20, 20)$, the trajectories of system (2) are shown in Fig. 1. It is obvious that system (2) has a chaotic attractor.

The Lyapunov exponents of system (2) are 2.2140, -0.2722, and -15.4845. Furthermore, the Lyapunov exponent dimension of the system is 2.1254. The above results again show that there exists a chaotic attractor in the financial system.

4.2. Sensitivity on initial conditions

In system (2), with given parameters $\delta = 10$, $b = 8/3$, $r = 28$, and chosen initial points $x_{00} = (20, 20, 20)$ and $x_{01} = (20.0001, 20, 20)$, the trajectories of system (2) are shown in Fig. 2. Obviously, in the beginning, two trajectories are inseparable, but after a period of iterations, they separate quickly. It shows that the financial risk arises from small to large with a change in the initial value, and that the financial risk system has typical sensitive dependence on the initial conditions.

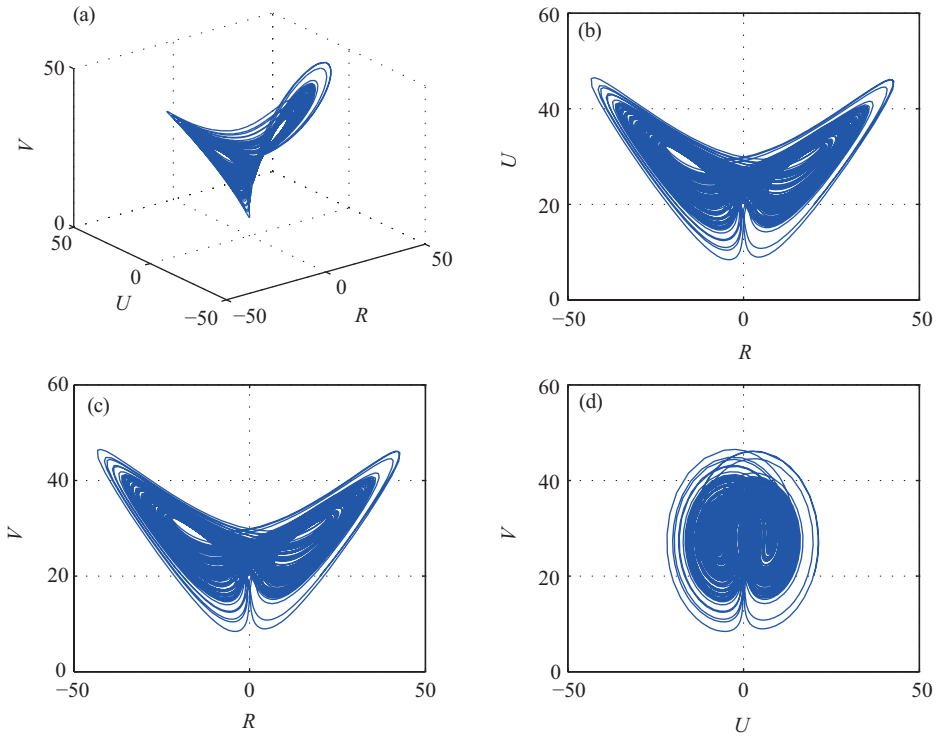


Fig. 1. (color online) Phase diagram of system (2). (a) Risk model; (b) R–U curve; (c) R–V curve; (d) U–V curve.

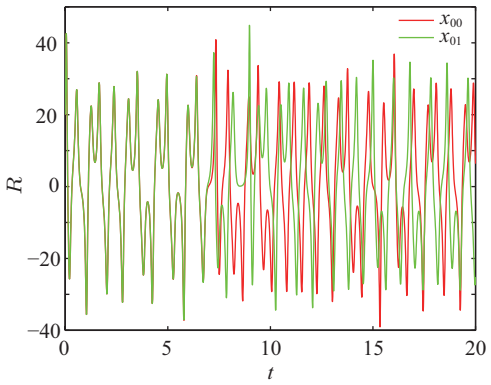


Fig. 2. (color online) Sensitive dependence on the initial conditions.

4.3. Influence of the risk transmission rate r

In order to investigate the influence of the risk transmission rate r on the stability of the financial system, we keep the parameters $\delta = m = 10$ and $b = 1 - k = 8/3$ unchanged, and vary r.

When the risk transmission rate r varies in interval [6,10], the bifurcation diagram of the current risk value of occurrence R is shown in Fig. 3. The figure reveals the evolving procedure of the risk R from stability to chaos as parameter r increases.

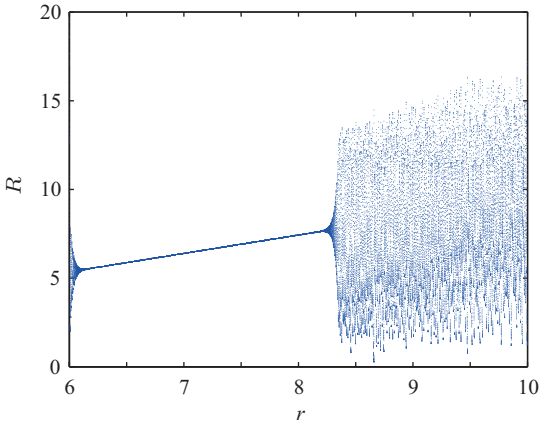


Fig. 3. (color online) Bifurcation diagram of R.

When $r = 8$, we have $E_1 = (5.8744, 3.3711, 7.4261)$, and the eigenvalues of the Jacobin matrix at E_1 are $\lambda_1 = -13.4536$ and $\lambda_{2,3} = -0.1065 \pm 7.0427\mathrm{i}$. Hence, the system is asymptotically stable.

When $r = 9$, we obtain $E_1 = (6.4524, 3.4957, 8.4582)$, and the eigenvalues are $\lambda_1 = -13.6884$ and $\lambda_{2,3} = 0.0109 \pm 7.6840\mathrm{i}$. The corresponding Lyaponov-exponents are: 0.9733, -0.9740, and -13.7319. The system is therefore chaotic.

Numerical simulation shows that system (2) is asymptotically stable when $r < 8$, the Hopf bifurcation occurs at the critical point located in the interval $(8, 9)$, and the system has structural changes and goes into the chaotic state when $r > 9$.

4.4. Effect of controller

In order to reveal the effect of chaos control, some numerical simulations are given below. At equilibrium E_1, keep the parameters $\delta = 10$ and $b = 8/3$ unchanged, and vary r.

Set $k_1 = 0$, $k_2 = 2$, and $k_3 = 0$. When $r = 9$, we have $E_1 = (6.4524, 3.4957, 8.4582)$, and the corresponding eigenvalues are $\lambda_1 = -13.9425$ and $\lambda_{2,3} = -0.8621 \pm 7.7005\mathrm{i}$. It implies that the system is asymptotically stable at E_1. If $r = 19$, it follows that $E_1 = (11.9373, 4.1664, 18.6510)$, and the eigenvalues are $\lambda_1 = -15.7063$ and $\lambda_{2,3} = 0.0198 \pm 13.4323\mathrm{i}$. It shows that the equilibrium E_1 is unstable.

However, if we let $k_1 = 0$, $k_2 = 2$, and $k_3 = 2$, and keep $r = 19$, then the corresponding eigenvalues are $\lambda_1 = -17.0111$ and $\lambda_{2,3} = -0.3278 \pm 18.1355\mathrm{i}$. Therefore, the system is still asymptotically stable at $E_1 = (16.7059, 4.4272, 27.7350)$. Even when $r = 28$, the system continues to be stable (see Fig. 4, the initial point is $(R_0, U_0, V_0) = (20, 20, 20)$). Therefore, by adding the positive feedback controller, the risk can be controlled to change from the unstable chaotic state to the stable state.

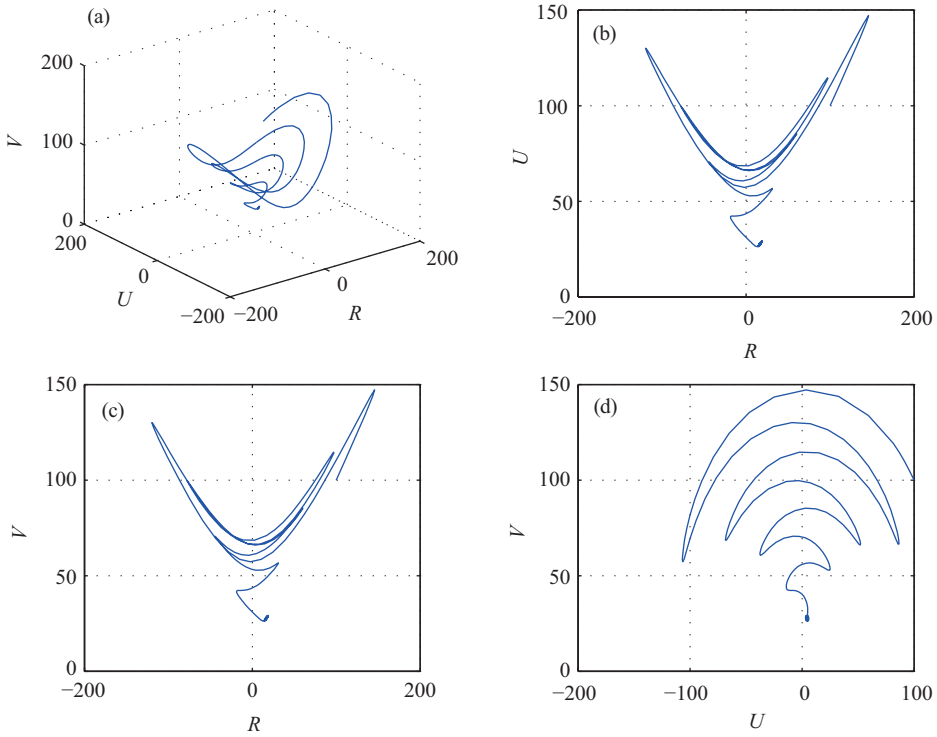


Fig. 4. (color online) Phase diagram of a controlled system. (a) Risk model; (b) R–U curve; (c) R–V curve; (d) U–V curve.

5. Conclusions and discussion

In this paper, an improved financial risk system is constructed based on the risk management process, which is closer to the real financial market. Through a series of dynamic behavior analysis, the changing process of the financial market risk is described. Also, the controlled system of a financial risk system is constructed by applying the positive feedback gain matrix method. It can make the system's chaos disappear, and

ensure that the financial risk management system is in a stable state. We believe that the model is of significance in guiding the macro-financial risk management.

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