Adaptive control of bifurcation and chaos in a time-delayed system*

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In this paper, the stabilization of a continuous time-delayed system is considered. To control the bifurcation and chaos in a time-delayed system, a parameter perturbation control and a hybrid control are proposed. Then, to ensure the asymptotic stability of the system in the presence of unexpected system parameter changes, the adaptive control idea is introduced, i.e., the perturbation control parameter and the hybrid control parameter are automatically tuned according to the adaptation laws, respectively. The adaptation algorithms are constructed based on the Lyapunov–Krasovskii stability theorem. The adaptive parameter perturbation control and the adaptive hybrid control methods improve the corresponding constant control methods. They have the advantages of increased stability, adaptability to the changes of the system parameters, control cost saving, and simplicity. Numerical simulations for a well-known chaotic time-delayed system are performed to demonstrate the feasibility and superiority of the proposed control methods. A comparison of the two adaptive control methods is also made in an experimental study.

Keywords: delay, parameter perturbation control, hybrid control, adaptive control

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1. Introduction

A time delay exists in a wide variety of physical systems such as population dynamics, communication networks, electronic oscillators, lasers, biophysics, nuclear reactors, etc.[1,2] Time delay is often a source of instability and gives rise to undesired oscillations which degrade the performance of the systems. It is well known that even a first-order time-delayed system may behave chaotically, since the delay causes the system to be infinite-dimensional. A well-studied chaotic time-delayed system is the Mackey–Glass system, as a mathematical model for the generation of white blood cells.[3] In the last decades, it has been realized that bifurcation and chaos control by suppressing bifurcation and chaos can be useful when they are harmful or undesirable. Hence, the task of controlling oscillations in time-delayed systems using the chaos and bifurcation control strategies is of great importance in a diverse range of applications such as inertial two-neuron systems, oncolytic virus dynamics, small-world networks, cellular neural networks, and so on.[4–10]

From the viewpoint of the control strategy, the developed methods can be classified into two main categories: feedback control methods[11–13] in which feedback strategies are applied to control bifurcation and chaos, and non-feedback control methods[14–16] in which bifurcation and chaos are suppressed by applying weak periodic perturbations or parameter perturbations. Based on these works, a hybrid control strategy[17,18] is proposed, in which the parameter perturbation and the state feedback are combined to control the period-doubling bifurcation and chaos. Theoretical analysis and numerical simulations show that the hybrid control strategy has some evident advantages over other existing methods. However, in recent studies two problems have emerged, which need further investigation.

First, in Refs. [19]–[23], the non-feedback control method has been proposed to control the bifurcation and chaos for some particular systems, such as the Lorenz system, Hénon map, and so on. Hybrid control has been proposed in Refs. [24]–[26] to control the Hopf bifurcations in a predator–prey system and internet congestion models. However, few people consider general time-delayed systems, and the extension of the parameter perturbation control and the hybrid control to general time-delayed systems need further investigation.

Secondly, in the above control schemes, the control parameters such as the perturbation control parameter and the hybrid control parameter have to be exactly selected to stabilize a fixed point. The control strategy is valid only when the parameters of the system are fixed, and the stabilization may be destroyed by the effects of system parameter perturbations. In such cases, an adaptive technique is required to adjust the control parameters. To stabilize a chaotic time-delayed system, the adaptive feedback control technique has

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been proposed in Ref. [27], where the analytical result of the adaptive law to adjust the feedback gain is obtained. However, the adaptive parameter perturbation control method and adaptive hybrid control strategy have not been extensively used for chaotic time-delayed systems yet.

In this paper, to solve the above two problems, the parameter perturbation control method and the hybrid control strategy are presented for the stabilization of chaotic time-delayed systems. Then, to cope with the unexpected changes of the system parameters, the idea of adaptive control is incorporated into the above control methods. The main contributions of this paper can be summarized as follows.

i) We consider the stabilization of a class of chaotic time-delayed systems. To control the Hopf bifurcations and stabilize the unstable equilibrium points of a time-delayed system, parameter perturbation control and a hybrid control strategy are proposed. Some analytical results for adjusting the perturbation control parameter and the hybrid control parameter have been obtained. We can see that by choosing appropriate values of the control parameters, the undesirable Hopf bifurcations can be postponed and the fixed point can be stabilized. These two control methods may be applicable to some real time-delayed systems due to their good performance and easy implementation. Hence, research about the above control methods for general time-delayed systems not only extends the existing results theoretically, but may also provide resource managers with some new ideas in reality.

ii) To ensure the asymptotic stability of the closed-loop system in the presence of parameter oscillations, the adaptive control idea is introduced to combine with the parameter perturbation control and hybrid control, respectively. Based on the Lyapunov–Krasovskii (L–K) stability theorem, the adaptation algorithms for tuning the perturbation control parameter and the hybrid control parameter are constructed. Theoretical analysis and numerical simulations show that the unstable fixed point of the system can always be stabilized by the proposed control methods when the system parameters change. The adaptive control techniques improve the corresponding constant controllers, and they have the advantages of increased stability, adaptability to the changes of the system parameters, minimized consumed power or time needed to reach the control goal, and thereby control cost saving, simplicity, and easy implementation.

Furthermore, there is one technique detail that needs to be mentioned. Notice that the adjustment of the perturbation control parameter and the hybrid control parameter requires knowledge of the fixed point of the system. However, for many real systems, the exact value of the fixed point is difficult to measure. In order to solve this problem, an estimation mechanism of capturing the fixed point of the system is provided. A straightforward idea is the usage of a low-pass filter as the fixed point estimator. Using this approach, the unknown fixed point can be replaced by the output of the filter as its estimated value. Numerical simulations are carried out to verify the effectiveness of the proposed method.

The rest of the paper is organized as follows. In Section 2, parameter perturbation control and hybrid control are presented. In Section 3, adaptive parameter perturbation control and adaptive hybrid control are proposed. In Section 4, numerical simulations are performed to verify the feasibility and effectiveness of the proposed methods. The conclusions and discussion are presented in Section 5.

2. Parameter perturbation control and hybrid control

In this paper, we consider the problem of bifurcation and chaos control for a class of time-delayed system described by the following first-order delayed differential equation:

$$\dot{x}(t) = f(x(t)) + g(x(t-\tau), p), \tag{1}$$

where $x(t) \in \mathbf{R}$ is the state of the system at time t, $\tau > 0$ is the delay, f and g are nonlinear functions, and $p \in \mathbf{R}$ is the parameter of the system. We consider the following assumptions.

(i) Model (1) has an equilibrium point x^*.

(ii) The functions f and g are continuously differentiable, and satisfy

$$\frac{\partial f(x)}{\partial x}|_{x=x^*} \leq 0, \quad \frac{\partial g(x,p)}{\partial x}|_{x=x^*} < 0, \quad \frac{\partial^2 g(x,p)}{\partial x \partial p}|_{x=x^*} < 0.$$

(iii) Model (1) is supplemented with the initial condition of the form $x_0(\theta) = \phi(\theta)$ $(\theta \in [-\tau, 0])$, where $\phi(\cdot) \in C([-\tau, 0], \mathbf{R})$ is a continuous real-valued function in $[-\tau, 0]$.

2.1. The estimation algorithm

In most cases we cannot conveniently measure the fixed point of the system. Under this condition, the fixed point may be replaced by its estimated value which is calculated for the whole time using an estimation algorithm. The unknown fixed point can be attained using a low-pass filter in which the filtered state estimates the fixed point. Supposing the state variable $x(t)$ of the system is available for the whole time, we consider a low-pass filter of the form

$$H_{\mathrm{f}}(D) = \frac{1}{\varepsilon D + 1},$$

where $D = \mathrm{d}/\mathrm{d}t$ is the differential operator, and ε is the time constant which determines the pass band of $H_{\mathrm{f}}(D)$. Then, the fixed point x^* can be replaced by

$$x_{\mathrm{f}}(t) = H_{\mathrm{f}}(D)x(t),$$

where $x_{\mathrm{f}}(t)$ is the output of the filter and gives the unknown fixed point x^*. If $x(t)$ oscillates around x^*, then the output $x_{\mathrm{f}}(t)$ converges to x^*. Hence, in the control strategies proposed in this paper, the fixed point x^* can always be replaced by x_{f}.

2.2. Parameter perturbation control

Consider the controlled system in the following form:

$$\dot{x}(t) = f(x(t)) + g(x(t-\tau), p+\delta(k)), \tag{2}$$

where $\delta(k)$ is the parameter perturbation control. Under this control, the fixed point of the closed-loop system (2) may change. For simplicity, we still denote it by x^*. We consider the aforementioned assumptions (i)–(iii) still hold for the closed-loop system (2). The linearized equation of system (2) around x^* can be described as follows:

$$\dot{x}(t) = a(x(t)-x^*) + (b+h(k))(x(t-\tau)-x^*), \tag{3}$$

where

$$a = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*} \le 0,$$
$$b = b(p) = \left.\frac{\partial g(x,p)}{\partial x}\right|_{x=x^*,\delta(k)=0} < 0,$$

and $h(k)$ is the perturbation control parameter. Then the characteristic equation of system (3) is written as follows:

$$Q(\lambda,\tau) = \lambda - a - (b+h(k))\mathrm{e}^{-\lambda\tau} = 0. \tag{4}$$

Inserting purely imaginary roots $\lambda = \pm \mathrm{i}\omega$ with $(\omega > 0)$ into Eq. (4) gives

$$a + (b+h(k))\cos(\omega\tau) = 0, \quad \omega + (b+h(k))\sin(\omega\tau) = 0. \tag{5}$$

Therefore, if one can find b and $h(k)$ such that $0 > a > b+h(k)$ and $(b+h(k))^2 = a^2+\omega^2$, where ω is the solution of $\omega/a - \tan(\omega\tau) = 0$, then there exist critical values of $p = p_n$ satisfying Eq. (5) such that $b = b(p_n)$, and

$$\omega = \omega_n \in \left[\frac{(4n+1)\pi}{2\tau}, \frac{(4n+2)\pi}{2\tau}\right], \quad (n = 0,1,\ldots).$$

Differentiating Eq. (4) with respect to p and applying the implicit function theory, we obtain

$$\frac{\mathrm{d}\lambda}{\mathrm{d}p} = \frac{b'(p)e^{-\lambda\tau}}{1+(b+h(k))\tau e^{-\lambda\tau}} = \frac{b'(p)(\lambda-a)/(b+h(k))}{1+(\lambda-a)\tau},$$

where

$$b'(p) = \left.\frac{\partial^2 g(x,p)}{\partial x \partial p}\right|_{x=x^*,\delta(k)=0} < 0.$$

Therefore, for $\lambda = \pm \mathrm{i}\omega$, we have

$$\mathrm{Re}\left\{\frac{\mathrm{d}\lambda}{\mathrm{d}p}\right\} = \frac{(-a+a^2\tau+\omega^2\tau)\cdot b'(p)/(b+h(k))}{(1-a\tau)^2+(\omega\tau)^2} > 0.$$

Thus, system (2) exhibits a sequence of Hopf bifurcations at $p = p_n$ $(n = 0,1,\ldots)$.

Next, we determine whether the characteristic equation (4) has any roots with positive real parts for $p = p_n$. Let $\lambda = \beta + \mathrm{i}\omega$ be a root of Eq. (4) with positive β and ω. Thus, for $p = p_n$, we obtain

$$\beta - a - (b+h(k))\,\mathrm{e}^{-\tau\beta}\cos(\omega\tau) = 0,$$
$$\omega + (b+h(k))\,\mathrm{e}^{-\tau\beta}\sin(\omega\tau) = 0. \tag{6}$$

Since $b+h(k) < a < 0$, then the second equation implies that $\omega\tau < (4n+1)\pi/2$ $(n = 0,1,\ldots)$, and the first one implies that

$$\frac{(4m+1)\pi}{2} < \omega\tau < \frac{(4m+3)\pi}{2}, \quad (m = 0,1,\ldots,n-1).$$

Therefore, system (2) may have roots with a positive real part except for $n = 0$, and the characteristic equation (4) has at least $2(n+1)$ roots with positive real parts for $p_n < p < p_{n+1}$. As a result, we have $p = p_0$ as the critical value for Hopf bifurcation occurrence.

The above analysis can be summarized in the following theorem.

Theorem 2.1 Suppose assumptions (i)–(iii) hold true for the closed-loop system (2), $0 > a > b+h(k)$, where

$$a = \frac{\partial f(x)}{\partial x}|_{x=x^*}, \quad b = b(p) = \left.\frac{\partial g(x,p)}{\partial x}\right|_{x=x^*,\delta(k)=0},$$

and $h(k)$ is the perturbation control parameter in the linearized equation (3). A Hopf bifurcation occurs at the equilibrium x^* of the closed-loop system (2) when the parameter p passes through the critical value p_0, where p_0 satisfies $(b(p_0)+h(k))^2 = a^2+\omega_0^2$, and $\omega_0 \in [\pi/2\tau, \pi/\tau)$ is the solution of $\omega/a - \tan(\omega\tau) = 0$.

Theorem 2.1 indicates that by choosing appropriate values of $h(k)$, the Hopf bifurcation can be postponed using the proposed parameter perturbation control method.

In Refs. [17] and [18], a hybrid control strategy was firstly proposed to control the period-doubling bifurcations and the resulting chaos for nonlinear dynamic systems. The advantages of the hybrid control strategy over other methods have been demonstrated in detail. From the implementation point of view, the combination of state feedback and parameter perturbation may mean the resource managers have more choices according to their real needs. Hence, the hybrid control method may have wide applications in reality. However, the systematic study of this method for general time-delayed systems has been seldom seen. Hence, we will consider this problem in the next section.

2.3. Hybrid control

The hybrid control for system (1) can be described as follows:

$$\dot{x}(t) = \alpha[f(x(t)) + g(x(t-\tau),p)] - (1-\alpha)(x(t)-x^*), \tag{7}$$

where $\alpha \in (0,1)$ is the control parameter. The closed-loop system (7) has the same fixed points as the original system (1).

The linearized equation of system (7) around the fixed point x^* has the characteristic equation as follows:

$$Q(\lambda,\tau)=\lambda-\alpha a+1-\alpha-\alpha be^{-\lambda\tau}=0, \tag{8}$$

where

$$a=\left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*}\leq 0,\quad b=b(p)=\left.\frac{\partial g(x,p)}{\partial x}\right|_{x=x^*}<0.$$

Inserting purely imaginary roots $\lambda=\pm i\omega$ ($\omega>0$) into Eq. (8) gives

$$\begin{aligned}&-\alpha a+1-\alpha-\alpha bcos(\omega\tau)=0,\\&\omega+\alpha bsin(\omega\tau)=0.\end{aligned} \tag{9}$$

Therefore, if one can find b and α such that $0<-\alpha a+1-\alpha<-\alpha b$, $(\alpha b)^2=(-\alpha a+1-\alpha)^2+\omega^2$, where ω is the solution of $\omega/(-\alpha a+1-\alpha)+\tan(\omega\tau)=0$, then there are critical values $p=p_n$ satisfying Eq. (9) such that $b=b(p_n)$, and

$$\omega=\omega_n\in\left[\frac{(4n+1)\pi}{2\tau},\frac{(4n+2)\pi}{2\tau}\right],\quad (n=0,1,\ldots).$$

Differentiating Eq. (8) with respective to p and applying the implicit function theory, we obtain

$$\frac{d\lambda}{dp}=\frac{\alpha b'(p)e^{-\lambda\tau}}{1+\alpha b(p)\tau e^{-\lambda\tau}}=\frac{b'(p)(\lambda-\alpha a+1-\alpha)/b(p)}{1+\tau(\lambda-\alpha a+1-\alpha)},$$

where

$$b'(p)=\left.\frac{\partial^2 g(x,p)}{\partial x\partial p}\right|_{x=x^*}<0.$$

Therefore, for $\lambda=\pm i\omega$, we have

$$\mathrm{Re}\left\{\frac{d\lambda}{dp}\right\}=\frac{\{\tau\omega^2+(-\alpha a+1-\alpha)[1+\tau(-\alpha a+1-\alpha)]\}b'(p)/b(p)}{[1+\tau(-\alpha a+1-\alpha)]^2+(\tau\omega)^2}>0.$$

Thus, system (7) exhibits a sequence of Hopf bifurcations at $p=p_n$ ($n=0,1,\ldots$).

Next, we determine whether the characteristic equation (8) has any roots with a positive real part for $p=p_n$. Let $\lambda=\beta+i\omega$ be a root of Eq. (8) with positive β and ω. Thus, for $p=p_n$, we obtain

$$\begin{aligned}&\beta-\alpha a+1-\alpha-\alpha be^{-\tau\beta}cos(\omega\tau)=0,\\&\omega+\alpha be^{-\tau\beta}sin(\omega\tau)=0.\end{aligned} \tag{10}$$

Since $b<0$, then the second equation implies that $\omega\tau<(4n+1)\pi/2$ ($n=0,1,\ldots$), and the first one implies that $(4m+1)\pi/2<\omega\tau<(4m+3)\pi/2$ ($m=0,1,\ldots,n-1$). Therefore, system (7) may have roots with positive real parts except for $n=0$, the characteristic equation (8) has at least $2(n+1)$ roots with positive real parts for $p_n<p<p_{n+1}$. As a result, we have $p=p_0$ as the critical value for Hopf bifurcation occurrence.

The above analysis is summarized into the following theorem.

Theorem 2.2 Suppose assumptions (i)–(iii) hold, $\alpha(-a+b-1)+1<0$, where

$$a=\left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*},\quad b=b(p)=\left.\frac{\partial g(x,p)}{\partial x}\right|_{x=x^*}.$$

A Hopf bifurcation occurs at the equilibrium x^* of the closed-loop system (7) when the parameter p passes through the critical value p_0, such that $(\alpha b(p_0))^2=(-\alpha a+1-\alpha)^2+\omega_0^2$, where ω_0 is the solution of $\omega=-(-\alpha a+1-\alpha)\tan(\omega\tau)$, for $\omega_0\in[\pi/2\tau,\pi/\tau)$.

Theorem 2.2 shows that by choosing appropriate values of α, the Hopf bifurcation can be postponed using the proposed hybrid control method.

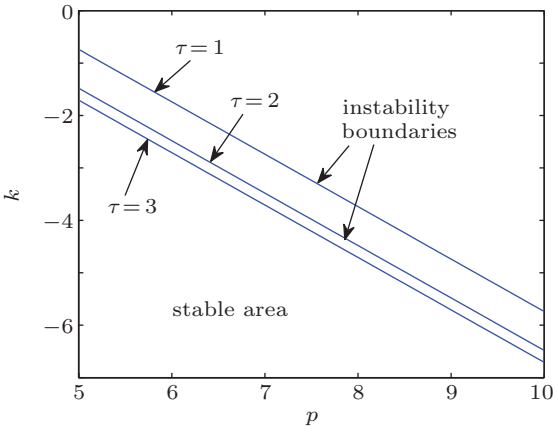


Fig. 1. (color online) The stable area of the closed-loop system (2) in the (p,k) plane.

We now study the effect of parameter perturbation control and hybrid control in a time-delayed system. Using the bifurcation diagram in the plane of the system parameter p and the perturbation control parameter k, we can see how they influence the stability of the system. Figure 1 shows the stable areas in the (p,k) plane, where $a=-1$, $b(p)=2-p$, $\delta(k)=k$, and $h(k)=-k$. Outside the stable areas, the fixed point is unstable and oscillations may occur, i.e., there is a critical value at which the Hopf bifurcation occurs. The lines indicate the instability boundaries for different values of τ. In addition, we can see that when the system parameter p varies, a transition from stable area to unstable area will be possible, and Hopf bifurcation may occur. Figure 2 shows the stable areas in the

(p,α) plane, where $a=-1$, and $b(p)=2-p$. Due to fluctuations in p, a transition from stable to unstable can also be seen. Therefore, in order to ensure the asymptotic stability of the closed-loop system in the presence of parameter oscillations, adaptive control strategies are required for adjusting the perturbation control parameter k and hybrid control parameter α.

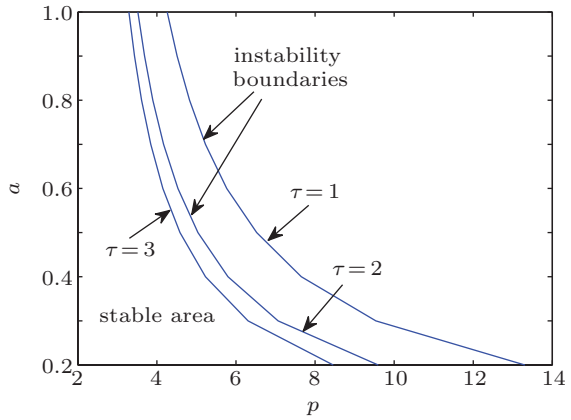


Fig. 2. (color online) The stable area of the closed-loop system (7) in the (p,α) plane.

3. Adaptive parameter perturbation control and adaptive hybrid control

In this section, we propose the adaptive parameter perturbation control and adaptive hybrid control to stabilize the equilibrium point of the chaotic time-delayed system (1) in the presence of unexpected system parameter perturbations.

3.1. Adaptive parameter perturbation control

In this section, we propose the adaptive parameter perturbation control scheme to stabilize the fixed point x^* of a time-delayed system (1). The proposed adaptive control scheme is depicted in Fig. 3. The scheme has three components: the main parameter perturbation controller, which stabilizes the fixed point x^* of the system; the model estimator, which estimates the fixed point of the system; the adaptive module including an adaptation algorithm, which tunes the perturbation control parameter according to an adaptation law.

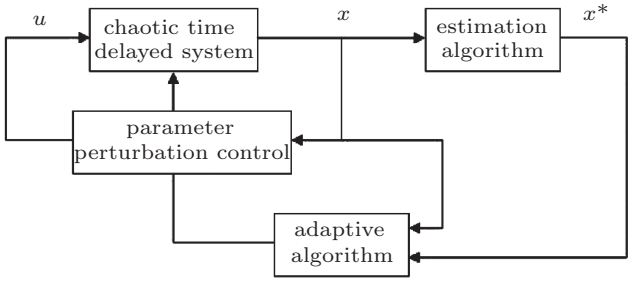


Fig. 3. The adaptive parameter perturbation control method with an estimator block.

Assume that the controlled closed-loop system and the linearized equation have the form of Eqs. (2) and (3), respectively. To obtain an adaptation algorithm for the perturbation control parameter, analysis of the stability of the closed-loop system around the equilibrium point x^* is required. To construct an adaptation mechanism for adjusting the perturbation control parameter, such that $x(t)$ converges to x^*, we consider two cases: $a<0$ and $a=0$.

Case I $a<0$. According to the standard procedure of the L–K method, let us choose the following L–K functional:

$$V(t)=\frac{1}{2}(x(t)-x^*)^2+\frac{1}{2\gamma}(b(p)+h(k))^2, \tag{11}$$

where $\gamma>0$ is the adaptation gain.

It can be easily seen that this L–K functional satisfies the L–K theorem. In addition, the time derivative of V is

$$\begin{aligned}\dot{V}(t) &= (x(t)-x^*)\dot{x}(t)+\frac{1}{\gamma}(b(p)+h(k))\frac{\mathrm{d}h(k)}{\mathrm{d}t}\\ &= (x(t)-x^*)[a(x(t)-x^*)+(b(p)+h(k))\\ &\quad\times(x(t-\tau)-x^*)]+\frac{1}{\gamma}(b(p)+h(k))\frac{\mathrm{d}h(k)}{\mathrm{d}t}.\end{aligned} \tag{12}$$

If the parameter $h(k)$ is rewritten as

$$\frac{\mathrm{d}h(k)}{\mathrm{d}t}=-\gamma(x(t)-x^*)(x(t-\tau)-x^*), \tag{13}$$

then we obtain

$$\dot{V}(t)=a(x(t)-x^*)^2<0, \tag{14}$$

which implies that system (2) is asymptotically stable at x^*.

The above analysis can be summarized into the following theorem.

Theorem 3.1 The closed-loop system (2) is asymptotically stable at a fixed point x^* for $a<0$ if the perturbation control parameter $h(k)$ in the linearized equation (3) is tuned by Eq. (13).

Note that the value of x^* in Eq. (13) can be replaced by its estimation, i.e., the output $x_{\mathrm{f}}(t)$ of the low-pass filter. Thus, the adaptation law can be described as

$$\frac{\mathrm{d}h(k)}{\mathrm{d}t}=-\gamma(x(t)-x_{\mathrm{f}}(t))(x(t-\tau)-x_{\mathrm{f}}(t)). \tag{15}$$

Case II $a=0$. If we choose the same L–K functional as that in Eq. (11), then the time derivative of V is

$$\begin{aligned}\dot{V}(t) &= (x(t)-x^*)\dot{x}(t)+\frac{1}{\gamma}(b(p)+h(k))\frac{\mathrm{d}h(k)}{\mathrm{d}t}\\ &= (x(t)-x^*)(b(p)+h(k))(x(t-\tau)-x^*)\\ &\quad+\frac{1}{\gamma}(b(p)+h(k))\frac{\mathrm{d}h(k)}{\mathrm{d}t}\\ &= (x(t)-x^*)(b(p)+h(k))(x(t-\tau)-x(t))\\ &\quad+(b(p)+h(k))(x(t)-x^*)^2\\ &\quad+\frac{1}{\gamma}(b(p)+h(k))\frac{\mathrm{d}h(k)}{\mathrm{d}t}.\end{aligned} \tag{16}$$

If the parameter $h(k)$ is renewed as

$$\frac{\mathrm{d}h(k)}{\mathrm{d}t} = -\gamma(x(t)-x^*)(x(t-\tau)-x(t)), \tag{17}$$

and similar to the constant parameter perturbation control, if the parameters satisfy $b(p)+h(k)<0$, then we obtain

$$\dot{V}(t) = (b(p)+h(k))(x(t)-x^*)^2 < 0, \tag{18}$$

which implies that system (2) is asymptotically stable at x^*.

Summarizing the above analysis, we obtain the following theorem.

Theorem 3.2 For $a=0$, if the parameters satisfy $b(p)+h(k)<0$, then the closed-loop system (2) is asymptotically stable at the fixed point x^* when the perturbation control parameter $h(k)$ in the linearized equation (3) is tuned by Eq. (17).

The value of x^* in Eq. (17) can also be replaced by its estimated value, i.e., the output $x_{\mathrm{f}}(t)$ of the low-pass filter. Thus, the adaptation law becomes

$$\frac{\mathrm{d}h(k)}{\mathrm{d}t} = -\gamma(x(t)-x_{\mathrm{f}}(t))(x(t-\tau)-x(t)). \tag{19}$$

In the usual constant parameter perturbation control, the control parameter needs to be maximal when system parameter oscillations exist, which means a kind of waste in practice. However, the control parameter in the proposed adaptive method is automatically adapted to the nearest suitable value, and therefore the method is more efficient in control cost saving than the constant parameter perturbation control scheme.

3.2. Adaptive hybrid control

In this section, we propose the adaptive hybrid control scheme in order to stabilize the fixed point x^* of time-delayed system (1). The proposed adaptive control scheme is depicted in Fig. 4. The scheme has three components: the main hybrid controller, which stabilizes the fixed point x^* of the system; the model estimator, which estimates the fixed point of the system; the adaptive module including an adaptation algorithm, which tunes the hybrid control parameter according to an adaptation law.

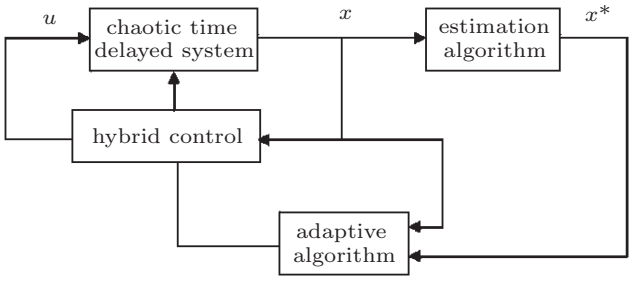


Fig. 4. The adaptive hybrid control method with an estimator block.

Assume that the closed-loop system has the form of Eq. (7). To obtain an adaptation algorithm for the hybrid control parameter α, we will analyze the stability of the closed-loop system (7) around the equilibrium point x^*. We will construct an adaptation mechanism for adjusting the hybrid control parameter α, such that $x(t)$ converges to x^*. The linearized controlled delayed differential equation has the following form:

$$\begin{aligned}\dot{x}(t) &= \alpha[a(x(t)-x^*)+b(p)(x(t-\tau)-x^*)] \\ &\quad -(1-\alpha)(x(t)-x^*),\end{aligned} \tag{20}$$

where

$$a = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*} \le 0, \quad b = b(p) = \left.\frac{\partial g(x,p)}{\partial x}\right|_{x=x^*} < 0.$$

According to the standard procedure of the L–K method, let us choose the following L–K functional:

$$V(t) = \frac{1}{2}(x(t)-x^*)^2 - \frac{1}{2\gamma}b(p)\alpha^2, \tag{21}$$

where $\gamma>0$ is the adaptation gain.

It is easily seen that this L–K functional satisfies the L–K theorem. In addition, the time derivative of V can be given by

$$\begin{aligned}\dot{V}(t) &= (x(t)-x^*)\dot{x}(t) - \frac{1}{\gamma}b(p)\alpha\dot{\alpha}(t) \\ &= (x(t)-x^*)[(\alpha a+\alpha-1)(x(t)-x^*) \\ &\quad +\alpha b(p)(x(t-\tau)-x^*)] - \frac{1}{\gamma}b(p)\alpha\dot{\alpha}(t).\end{aligned} \tag{22}$$

If the parameter α is renewed as

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \gamma(x(t)-x^*)(x(t-\tau)-x^*), \tag{23}$$

then we obtain

$$\dot{V}(t) = (\alpha a+\alpha-1)(x(t)-x^*)^2 < 0, \tag{24}$$

which implies that system (7) is asymptotically stable at the fixed point x^*.

Summarizing the above analysis, we obtain the following theorem.

Theorem 3.3 The closed-loop system (7) is asymptotically stable at the fixed point x^* if the hybrid control parameter α is tuned by Eq. (23).

The value of x^* in Eq. (23) can still be replaced by its estimated value, i.e., the output $x_{\mathrm{f}}(t)$ of the low-pass filter. Thus, the adaptation law becomes

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \gamma(x(t)-x_{\mathrm{f}}(t))(x(t-\tau)-x_{\mathrm{f}}(t)). \tag{25}$$

The hybrid control parameter in the proposed adaptive method is automatically adapted to the nearest suitable value. Therefore, the method is also more efficient in the control cost saving than the constant hybrid control scheme.

4. Simulations

Now, we present the stabilization of the equilibrium point for the time-delayed logistic system to evaluate the performance of the proposed control methods. Consider the following model:

$$\dot{x}(t) = -x(t) + px(t-\tau)[1-x(t-\tau)], \tag{26}$$

where $\tau > 0$ is the delay, and p is the parameter of the system. This model has been studied in Ref. [28], and it has been shown that this model can exhibit chaos in a range of p. This model has one equilibrium point $x^* = 1 - 1/p$ which has been proven to be stable. We will study the behavior of model (26) near x^*.

The characteristic equation of the linearized model is $\lambda + 1 + (p-2)\,\mathrm{e}^{-\lambda\tau} = 0$. One can determine the critical value for Hopf bifurcation occurrence $p_0 = 2 + \sqrt{1+\omega_0^2}$, where $\omega_0 = -\tan(\omega_0\tau)$, and $\pi/2 \le \omega_0\tau < \pi$. Assuming $\tau = 3$, we have $\omega_0 \approx 0.8186$, and $p_0 \approx 3.2923$. Figure 5 shows the waveform plots for the open-loop system (26) for $p = 2.5$, $p = 3.6$, and $p = 4.3$. It is seen that the model can be stable, periodic, and chaotic for different values of p.

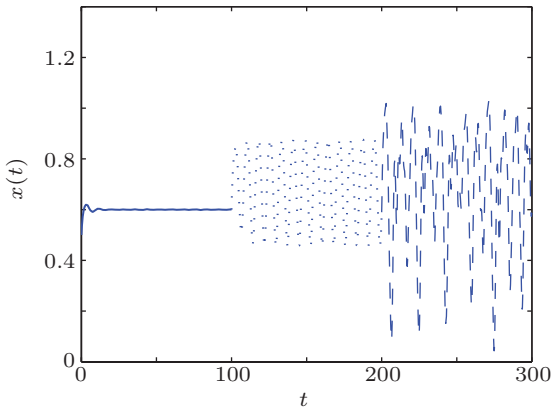


Fig. 5. (color online) The dynamic behavior of system (26) for $p = 2.5$, $p = 3.6$, and $p = 4.3$.

By applying the parameter perturbation control method to model (26), the Hopf bifurcation can be postponed, and the first periodic limit cycle appears for a larger value of p. Choosing $\delta(k) = k = -2$, we can obtain the critical value for Hopf bifurcation occurrence $p_0 \approx 5.2923$. Figure 6 shows the waveform plots for $p = 5.001$, $p = 5.6$, and $p = 6.4$ which represent the stable, periodic, and chaotic cases.

From Fig. 6 we can see how a sudden change in the parameter p can have important effect on the stability of the system and can limit the success of the parameter perturbation control method. When a sudden change in parameter p occurs, the system may show Hopf bifurcation or even chaotic behavior. Therefore, the success of the control method depends not only on the control parameter $\delta(k)$, but also on the specific value of parameter p.

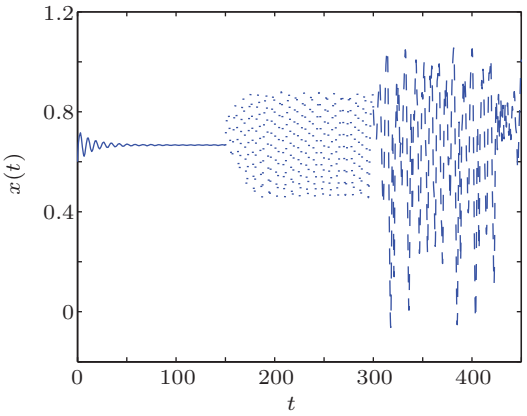


Fig. 6. (color online) The stabilization of system (26) by applying the parameter perturbation control for $p = 5.001$, $p = 5.6$, and $p = 6.4$.

Applying the hybrid control method to model (26), the Hopf bifurcation can also be postponed. Choosing $\alpha = 0.6$, the first periodic limit cycle appears for a larger value of p. It occurs in the critical value $p_0 \approx 4.1539$. Figure 7 shows the results for $p = 3.7$, $p = 4.8$, and $p = 6$ which represent the stable, periodic, and chaotic cases. It can also be seen that a sudden change in parameter p can have important effect on the stability of the system, and can limit the success of the hybrid control method.

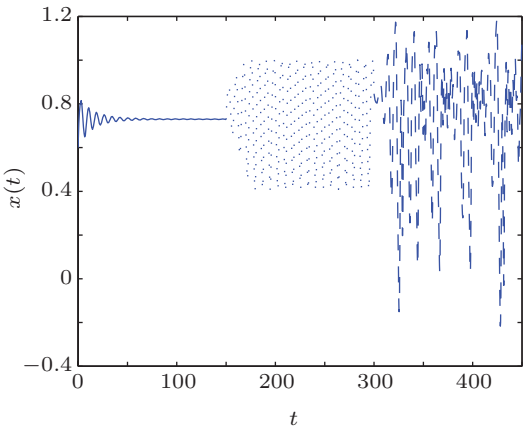


Fig. 7. (color online) The stabilization of system (26) by applying the hybrid control for $p = 3.7$, $p = 4.8$, and $p = 6$.

Now, we consider the adaptive parameter perturbation control effect. Choosing $\delta(k) = k$, we can obtain $h(k) = -k$. On the basis of the theoretical analysis results in Theorem 3.1, the adaptation law is $\dot{k}(t) = \gamma(x(t) - x_{\mathrm{f}}(t))(x(t-\tau) - x_{\mathrm{f}}(t))$. Waveform plots and the parameter convergence for the closed-loop system are shown in Fig. 8. The adaptation gain is chosen as $\gamma = 6.6$, and the pass band is $\varepsilon = 1$. The initial condition for adaptation is assumed to be $k = -2$. It can be seen that the adaptive parameter perturbation control method successfully stabilizes the fixed point of the system and does not depend on the change in parameter p. Parameter k is renewed through the adaptive algorithm in order to overcome the effect of parame-

ter perturbations. Once the stabilization is achieved, k stops to be renewed, and then k converges to a suitable value.

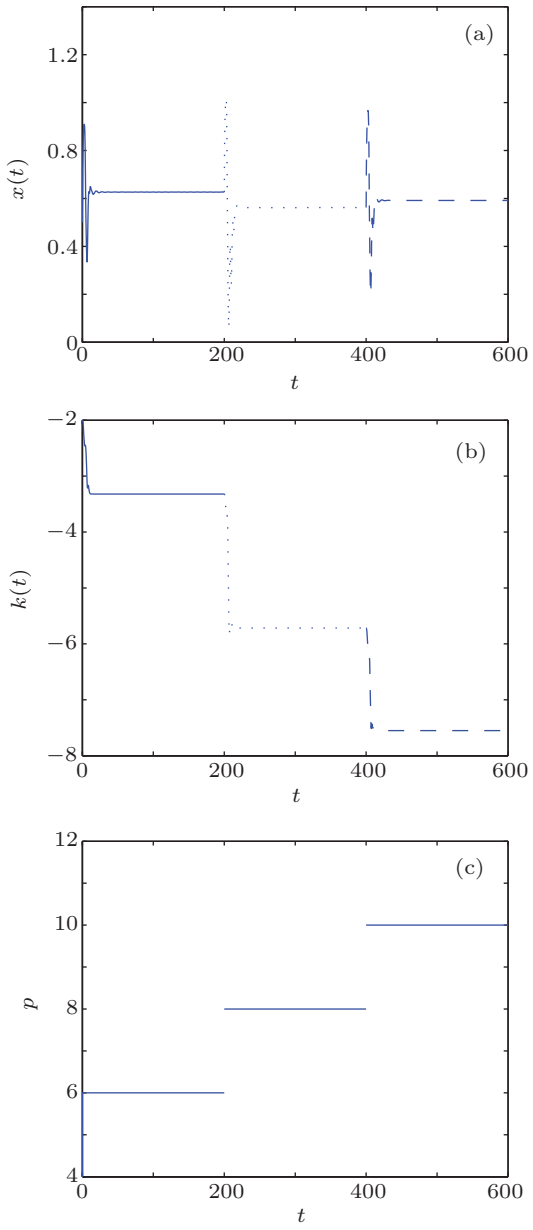


Fig. 8. (color online) The stabilization of system (26) by applying the adaptive parameter perturbation control. (a) Time series of $x(t)$. (b) Convergence of $k(t)$. (c) Values of p in (t,p) plane.

Figure 9 shows the control results of the adaptive hybrid control method. On the basis of the theoretical analysis results in Theorem 3.3, the adaptation law is $\dot{\alpha}(t) = \gamma(x(t)-x_{\rm f}(t))(x(t-\tau)-x_{\rm f}(t))$. Waveform plots and the parameter convergence for the closed-loop system are shown in Fig. 9. The adaptation gain is chosen as $\gamma = 2$, and the pass band is $\varepsilon = 1$. The initial condition for adaptation is assumed to be $\alpha = 0.5$. It can be seen that the adaptive hybrid control method successfully stabilizes the fixed point of the system and does not depend on the change in parameter p. Parameter α is renewed through the adaptive algorithm in order to overcome the effect of system parameter perturbations. Once the stabilization is achieved, α stops to be renewed, and converges to a suitable value.

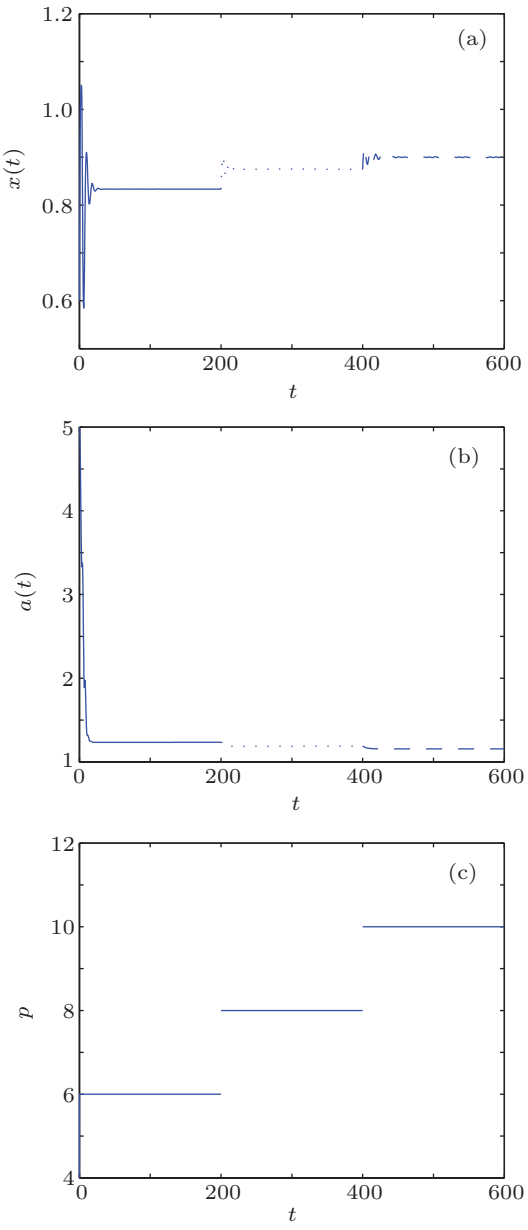


Fig. 9. (color online) The stabilization of system (26) by applying the adaptive hybrid control. (a) Time series of $x(t)$. (b) Convergence of $\alpha(t)$. (c) Values of p in (t,p) plane.

Now, we compare the two adaptive control methods: adaptive parameter perturbation control and adaptive hybrid control, in the following three aspects.

(i) From the analytical expressions (2) and (7) of the two adaptive control methods, we can see that the original equilibrium point may be changed under the adaptive parameter perturbation control. For model (26), the original fixed point is $x^* = 1 - 1/p$. While the fixed point becomes $x^* = 1 - 1/(p+k)$ under the adaptive parameter perturbation control, it is easily seen that the equilibrium point remains unchanged under the adaptive hybrid control.

Sometimes changing the fixed point may be undesirable.

Under this condition, adaptive hybrid control may be a better choice. This also explains why we continue to study the adaptive hybrid control after proposing the adaptive parameter perturbation control.

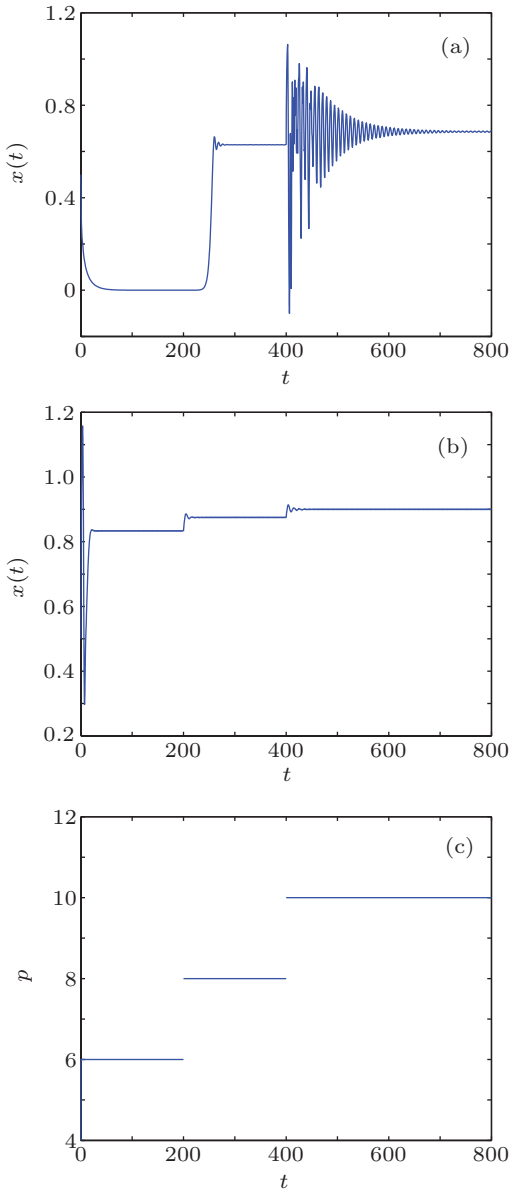


Fig. 10. (color online) The control effects of system (26) by applying the two adaptive control methods. (a) Adaptive parameter perturbation control. (b) Adaptive hybrid control. (c) Values of p in (t,p) plane.

(ii) We will compare the control effects of model (26) under the two adaptive control schemes. From the analytical expressions (2) and (7) of the two adaptive control methods, we can see that their working mechanisms are different. In the adaptive parameter perturbation control, parameter p is directly controlled by adding $\delta(k)=k$ on it. Therefore, when parameter p is large, a large adaptation gain γ in Eq. (15) is needed to pull it back to the stable range. In the adaptive hybrid control, when the control parameter α is reduced, the nonlinearity of the system is reduced, and then the oscillations can be controlled. As $\alpha\in(0,1)$, to adjust α properly, a smaller adaptation gain γ in Eq. (25) is needed. In other words, for the same value of γ, adaptive hybrid control may have better performance, especially for large values of parameter p. Figure 10 verifies the above analysis. In Fig. 10, we always choose $\gamma=1$ in Eqs. (15) and (25), we can see good performance of the adaptive hybrid control, and that large oscillations occur under the adaptive parameter perturbation control for $p=10$.

Figure 11 illustrates the dependence of the largest real parts of the eigenvalues upon the parameter p for $\gamma=1$. We can see that when model (26) is controlled by the adaptive parameter perturbation control or the adaptive hybrid control, perturbations to parameter p cannot destroy the stability of the fixed point. However, when parameter p approaches 10, the largest real parts of the eigenvalues under the adaptive parameter perturbation control are obviously larger than those under the adaptive hybrid control. Hence, when we choose the same adaptation gain γ, the adaptive hybrid controlled system has a better stability. This verifies, on the other hand, that when the parameter p becomes large, for the same value of γ, adaptive hybrid control may have better control performance due to its working mechanism.

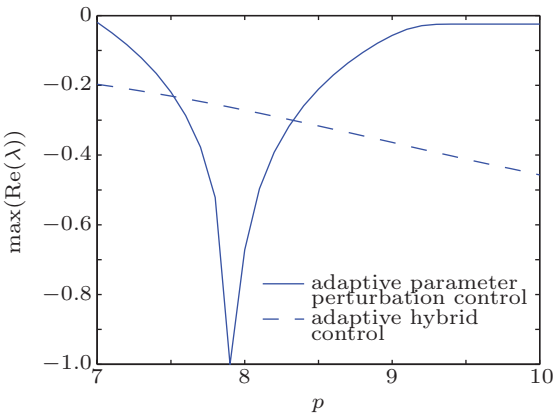


Fig. 11. (color online) The largest real parts of the eigenvalues of the adaptive controlled system.

(iii) Through a simple test for model (26), we will compare the adaptability to the system parameter changes of the two adaptive control strategies. Consider a system with a simple periodic perturbation $0.1\sin(10t)$ on the first item coefficient as follows:

$$\dot{x}(t)=-(1+0.1\sin(10t))x(t)+px(t-\tau)[1-x(t-\tau)]. \quad (27)$$

Applying the adaptive parameter perturbation control and the adaptive hybrid control to model (27), respectively, we have

$$\begin{aligned}\dot{x}(t) &= -(1+0.1\sin(10t))x(t)+(p+k)x(t-\tau)\\ &\quad\times[1-x(t-\tau)], \quad (28)\\ \dot{x}(t) &= \alpha[-(1+0.1\sin(10t))x(t)+px(t-\tau)(1-x(t-\tau))]\\ &\quad-(1-\alpha)(x(t)-x_{\rm f}(t)), \quad (29)\end{aligned}$$

where the adaptation laws for $h(k) = -\delta(k) = -k$ and α are defined by Eq. (15) and (25), respectively.

From the analytical expressions of systems (28) and (29), we can see that with the same adaptation gain γ, the adaptive hybrid control may have better adaptability to the system parameter changes. In system (28), the system parameter perturbation $0.1\sin(10t)$ is controlled by adding directly $\delta(k) = k$ on p. When p becomes large, a large adaptation gain γ in Eq. (15) is needed. In system (29), when the control parameter α is reduced, the system parameter perturbation $0.1\sin(10t)$ is obviously weakened. As $\alpha \in (0,1)$, to adjust α properly, a smaller adaptation gain γ in Eq. (25) is needed. In other words, when we choose the same value of γ, parameter perturbation $0.1\sin(10t)$ can be better suppressed under the adaptive hybrid control, especially for a large p value. Figure 12 illustrates this clearly, in which the adaptation gains γ in Eqs. (15) and (25) are all chosen to be 1.

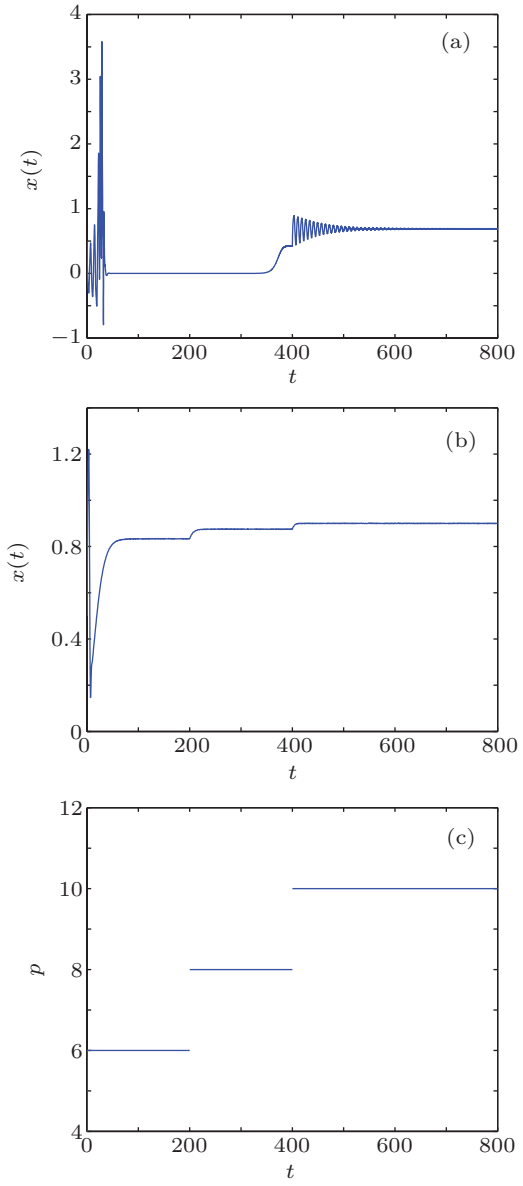


Fig. 12. (color online) The adaptability to the system parameter changes of system (26) by applying the two adaptive control methods. (a) Adaptive parameter perturbation control. (b) Adaptive hybrid control. (c) Values of p in (t, p) plane.

In the above test, a small parameter perturbation is added to the first item coefficient of model (26). When it is added to the second item coefficient, we obtain a similar result. Here, we neglect it for simplicity.

5. Conclusions and discussion

In this paper, a parameter perturbation control method and a hybrid control method have been proposed for stabilizing an unstable fixed point for a class of first-order time-delayed systems. Using an analytical approach, we have proved that the proposed control methods can postpone the onset of Hopf bifurcation by choosing appropriate control parameters. However, we notice that in the presence of system parameter perturbations, the control effect may be destroyed, and the stability of the closed-loop system may be lost. Therefore, adaptive schemes are proposed for tuning the perturbation control parameter and the hybrid control parameter when the parameter of the system varies. The adaptation algorithms are derived based on the Lyapunov–Krasovskii stability theorem. The adaptive control techniques have the advantages of increased stability, adaptability to the changes of the system parameter, high efficiency in control cost saving, and simplicity. Simulations with a typical chaotic time-delayed model demonstrate the effectiveness and correctness of the proposed control methods.

Comparisons between the two adaptive control strategies are made in detail. Summarizing the results from the analysis, we find that adaptive parameter perturbation control may change the fixed point of the original uncontrolled system. For the same adaptation gain, adaptive hybrid control may have a better performance. In other words, to obtain the same control result, a larger adaptation gain in adaptive parameter perturbation control is needed. However, there is one point that should be mentioned: the above comparisons between the two adaptive controllers are only from the theoretical point of view. In the real world, whether the adjustment of the adaptation gain in the adaptive parameter perturbation control or the adjustment of the adaptation gain in the adaptive hybrid control is easier to implement is determined according to the real situation. Hence, the choices of the two adaptive controllers are all possible, and we cannot predict arbitrarily which of the two adaptive control methods is better implemented in the real world. We wish to choose a suitable adaptive control scheme according to the real situation.

In the endeavor of using the adaptive parameter perturbation control method and adaptive hybrid control method for bifurcation and chaos control, there remain some problems to be studied in future research.

(i) For discrete systems, the problem of stabilization of unstable periodic orbits via the adaptive parameter perturbation control method and adaptive hybrid control method has not been addressed at all.

(ii) Using the above adaptive control strategies to control the bifurcation and chaos in some areas, such as complex network models, is a promising topic due to their good performance, cost saving, and easy implementation.

References

[1] Dugard L and Verriest E I 1998 *Stability and Control of Time-Delay Systems* (London: Springer)
[2] Gu K, Kharitonov V L and Chen J 2003 *Stability of Time-Delay Systems* (Boston: Birkhäuser)
[3] Mackey M C and Glass L 1977 *Science* **197** 287
[4] Ye Z Y, Yang G and Deng C B 2011 *Chin. Phys. B* **20** 010207
[5] Zhao H Y, Yu X H and Wang L 2012 *Int. J. Bifurcat. Chaos* **22** 1250036
[6] Wang S L, Wang S L and Song X Y 2012 *Nonlinear Dyn.* **67** 629
[7] Zhao H Y and Xie W 2011 *Nonlinear Dyn.* **63** 345
[8] Balasubramaniam P, Kalpana M and Rakkiyappan R 2012 *Chin. Phys. B* **21** 048402
[9] Yang J H and Liu X B 2012 *Acta Phys. Sin.* **61** 010505 (in Chinese)
[10] Gong D W, Zhang H G and Wang Z S 2012 *Chin. Phys. B* **21** 030204
[11] Yu P, Yuan Y and Xu J 2002 *Commun. Nonlinear Sci. Numer. Simulat.* **7** 69
[12] Chen G and Yu X 1999 *IEEE Trans. Circ. Sys.* I **46** 767
[13] Luo X S, Fang J Q, Kong L J and Weng J Q 2000 *Acta Phys. Sin.* **49** 1423 (in Chinese)
[14] Peng J H, Tang J S, Yu D J, Hai W H and Yan J R 2003 *Chin. Phys.* **12** 17
[15] Yang L, Liu Z and Mao J 2000 *Phys. Rev. Lett.* **84** 67
[16] Yang L, Liu Z and Chen G 2002 *Int. J. Bifurcat. Chaos* **12** 1411
[17] Luo X S, Chen G R, Wang B H and Fang J Q 2003 *Chaos Soliton. Fract.* **18** 775
[18] Liu Z R and Chung K W 2005 *Int. J. Bifurcat. Chaos* **15** 3895
[19] Jimenez-Triana A, Tang W K S and Chen G R 2010 *IEEE Trans. Circ. Sys. II* **57** 305
[20] Wu Z M, Xie J Y, Fang Y Y and Xu Z Y 2007 *Chaos Soliton. Fract.* **32** 104
[21] Kavitha A and Uma G 2010 *J. Electr. Eng. Technol.* **5** 171
[22] Zhou Y F, Tse C K, Qiu S S and Chen J N 2005 *Chin. Phys.* **14** 61
[23] Wang T S, Wang X Y and Wang M J 2011 *Commun. Nonlinear Sci. Numer. Simulat.* **16** 3367
[24] Zhang L P, Wang H N and Xu M 2011 *Acta Phys. Sin.* **60** 010506 (in Chinese)
[25] Ding D W, Zhu J and Luo X S 2008 *Chin. Phys. B* **17** 105
[26] Liu F, Guan Z H and Wang H 2008 *Chin. Phys. B* **17** 2405
[27] Rezaie B and Motlagh M R J 2011 *Nonlinear Dyn.* **64** 167
[28] Jiang M, Shena Y, Jian J and Liao X 2006 *Phys. Lett. A* **350** 221