Noise destroys the coexistence of periodic orbits of a piecewise linear map*

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The effects of Gaussian white noise and Gaussian colored noise on the periodic orbits of period-5 (P-5) and period-6 (P-6) in their coexisting domain of a piecewise linear map are investigated numerically. The probability densities of some orbits are calculated. When the noise intensity is $D = 0.0001$, only the orbits of P-5 exist, and the coexisting phenomenon is destroyed. On the other hand, the self-correlation time τ of the colored noise also affects the coexisting phenomenon. When $\tau_c < \tau < \tau_c'$, only the orbits of P-5 appear, and the stability of the orbits of P-5 is enhanced. However, when $\tau > \tau_c'$ (τ_c and τ_c' are critical values), only the orbits of P-6 exist, and the stability of the P-6 orbits is enhanced greatly. When $\tau < \tau_c$, the orbits of P-5 and P-6 coexist, but the stability of the P-5 orbits is enhanced and that of P-6 is weakened with τ increasing.

Keywords: piecewise linear map, noise, periodic orbit

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1. Introduction

In recent years, the investigations of the effect of noise on nonlinear dynamic systems have become an interesting and hot topic. In many cases, the noise plays a positive role and is helpful to control the dynamic system. For example, the noise can induce the transition in a bistable system,[1,2] enhance the stability,[3,4] retrain the growth of the tumor cell,[5–9] and induce the switch between the the TF-A monomer concentration in the gene transcriptional regulatory process.[10] The noise enhances the stability effect via noise recycling in a metastable system.[11] In particular, the stochastic resonance in a single mode laser system[12,13] and the suprathreshold stochastic resonance of a nonlinear multilevel threshold neuronal network system were investigated.[14] Reference [15] showed the effects of an electromagnetic field on intracellular calcium oscillations in a cell with external noise. Generally, noise can induce chaos.[16–19] Reference [17] investigated the effects of fluctuation on the period-doubling bifurcation to chaos, it was observed that the noise can induce an inverse period-doubling cascade in the logistic map, that is, the period of sixteen merged into eight and the period of eight merged into four, and the periodic sequences with high periods of the clean system are replaced by chaoslike sequences when there is noise. Later, Gao *et al.*[18] found that when noise induces the chaos, the complete period-doubling cascade is inhibited, otherwise the cascade is simply masked by the noise. Therefore, only the first few period-doubling bifurcations can be observed in experiments. Thus, it is important to understand in detail how noise affects the dynamic behavior of each periodic orbit which belongs to the periodic sequence in a nonlinear dynamical system. Recently, the logistic map driven by noise was extensively investigated, such as the effect of noise on bifurcation behavior,[20] the estimation of Lyapunov exponents,[21] and the fractal structure.[22]

However, in the previous studies, the effect of noise on periodic orbits was not well understood. In recent years, the effects of Gaussian white noise and Ornstein–Uhlenbeck noise on the periodic orbits of the logistic map were revealed.[23,24] As we know, the logistic map is a continuous map. But in real systems, such as, the models of nerve cells and cardiopathy,[25], relaxation and impact oscillators,[26,27] and DC–DC converters,[28] they are often described by discontinuous maps with gaps. Therefore, it is necessary to explore the effect of noise on discontinuous maps. In our work, we investigate a piecewise linear map with a gap, which is derived from an electronic relaxation oscillator.[26] The piecewise linear map with a gap including the discontinuity and the non-invertibility simultaneously is more complicated, but it presents many interesting phenomena, for example, type V intermittencies,[26,29] multiple Devil's staircases,[30,31] and the coexistence of attractors.[32]

Our main aim is to study the effects of Gaussian white noise and Ornstein–Uhlenbeck noise on the periodic orbits in the coexisting domain of period-5 (P-5) and period-6 (P-6). This study may be enlightening for the further understanding of some aspects of the evolution of chaotic systems in the pres-

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ence of noise. Special attention is paid to finding the effect of noise on the changes of the probability densities of the orbits.

2. Model

A piecewise linear map with both discontinuity and non-invertibility is given by[26,32]

$$x_{n+1} = f_i(x_n) = k_i x_n + b_i \mod 1, \tag{1}$$

where $i = 1,2,3,4$, and

$$\begin{aligned} k_1 &= \frac{y_b - y_A}{x_b - x_A}, \quad b_1 = y_A, & x \in [x_A, x_b];\\ k_2 &= \frac{y_C - y_b}{x_g - x_b}, \quad b_2 = y_b - k_2 x_b, & x \in [x_b, x_g];\\ k_3 &= \text{const.}, \quad b_3 = \text{const.}, & x \in [x_g, x_F];\\ k_4 &= \text{const.}, \quad b_4 = \text{const.}, & x \in [x_F, x_G], \end{aligned} \tag{2}$$

with $y_b = b_1 - \mu$. Here μ is selected as the control parameter. The values of the other parameters are selected as $y_A = 0.0203921$, $y_C = 0.460000$, $y_G = y_A$, $x_A = 0$, $x_b = 0.107663$, $x_g = 0.350000$, $x_F = 0.497121$, $x_G = 1$, $k_3 = 3.07055$, $b_3 = -0.530165$, $k_4 = 0.405507$, and $b_4 = -0.201586$.

The bifurcation diagram for parameter $\mu \in [0.025, 0.065]$ is shown in Fig. 1. It demonstrates that a stable period-5 attractor coexists, in turn, with other types of attractors, i.e.,

$$\begin{aligned} &0.02879 \le \mu < 0.03040: \text{ coexisting with band-11 chaos;}\\ &0.03952 \le \mu < 0.04506: \text{ coexisting with P-6;}\\ &0.04506 \le \mu < 0.05011: \text{ coexisting with band-12 chaos;}\\ &0.05011 \le \mu < 0.05114: \text{ coexisting with band-6 chaos;}\\ &0.05114 \le \mu: \qquad \text{no coexisting.} \end{aligned} \tag{3}$$

Here we only discuss the periodic orbits in the coexisting domain of P-5 and P-6 with control parameter $0.03952 \le \mu < 0.04506$ (the shadow part shown in Fig. 1).

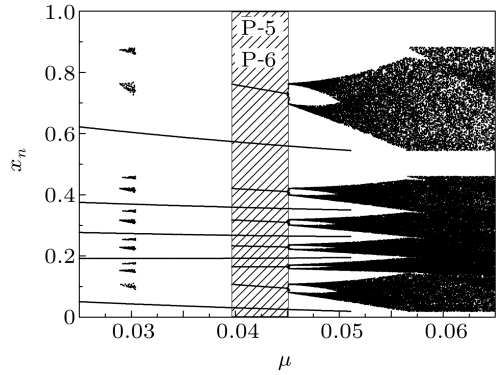


Fig. 1. Bifurcation diagram of the piecewise linear map for parameter $\mu \in [0.025, 0.065]$.

3. The effect of Gaussian white noise

We will investigate the effect of changes in the intensity of noise on the behavior of Eq. (1). The model is given by

$$x_{n+1} = f_i(x_n) = k_i x_n + b_i + \xi_n \mod 1, \tag{4}$$

where ξ_n denotes the Gaussian white noise with zero mean and correlation function $\langle \xi_n \xi_m \rangle = 2D\delta_{m,n}$ for $m,n = 1,2,3,\ldots$, $\langle \ldots \rangle$ denotes an ensemble average over the distribution of noise, δ is the Dirac delta function, and D is the intensity of the noise. Here, we will concentrate our attention on the behavior of the probability density of the periodic orbits induced by noise at the bifurcation parameter $\mu = 0.0425$. Without noise, the stable orbits of P-5 coexist with the orbits of P-6 at $\mu = 0.0425$. In the presence of noise, the orbits will change with the noise intensity.

Figure 2 shows the probability densities of the orbits each as a function of x_{n+1} for different noise intensities ($D = 0$, 0.000001, 0.00001,0.00005, 0.0001, 0.00003). Without noise ($D = 0$), there are six low peaks (six delta functions) in Fig. 1(a). Their heights are equal to 0.03296, and they are located at 0.1, 0.16333, 0.23, 0.31333, 0.41333, and 0.74333 respectively. There are also five high peaks (five delta functions) in Fig. 1(a). Their heights are equal to 0.18275, and they are located at 0.02667, 0.19, 0.26333, 0.35667, and 0.56333 respectively. As the noise intensity D increases, the heights of the low peaks (P-6) and the high peaks (P-5) gradually become lower when $D < 0.0001$. It should be mentioned that P-6 disappears and the coexistence is destroyed when $D = 0.0001$ (Fig. 1(e)). However, when $D > 0.0001$, the coexistence of P-5 and P-6 appears again (Fig. 1(f)). Moreover, with D increasing, the stabilities of the orbits of P-5 and P-6 are still weakened. There is a critical value of $D \approx 0.0001$.

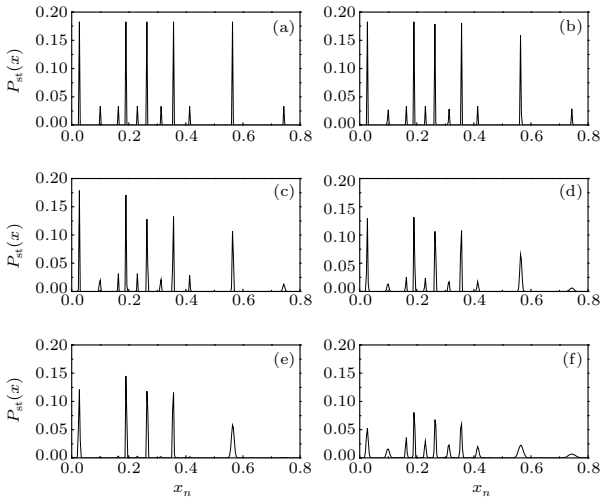


Fig. 2. Normalized probability densities at $\mu = 0.0425$ for different noise intensities: (a) $D = 0.0$, (b) $D = 0.000001$, (c) $D = 0.00001$, (d) $D = 0.00005$, (e) $D = 0.0001$, and (f) $D = 0.0003$. The 4×10^6 iterations, after initial 6×10^6 iterations, are partitioned into 300 bins.

In order to further understand how the noise affects the probability density of each orbit, we choose a series of different noise intensities and study the dynamic behavior of each orbit. Figure 3 shows the probability densities as a function of the noise intensity D. Figure 3(a) shows the effects of the noise on the orbits of P-6. From Fig. 3(a), the effects of the noise on the stabilities of the six orbits of P-6 are consistent. Namely, when D is very small, the heights of the six orbits are

monotonously decreased. As D is increased, the probability densities of the six orbits are monotonously increased. When $D = 0.0001$, the probability densities of the six orbits suddenly become zero, it means that P-6 disappears. It must be emphasized that these orbits change at different rates. We also find that the orbit of 0.16333 is more stable than the others. Figure 3(b) shows the effect of noise on the orbits of P-5. The tendencies of the five orbits of P-5 are consistent. It is found that the orbit of 0.19 is more stable than the others.

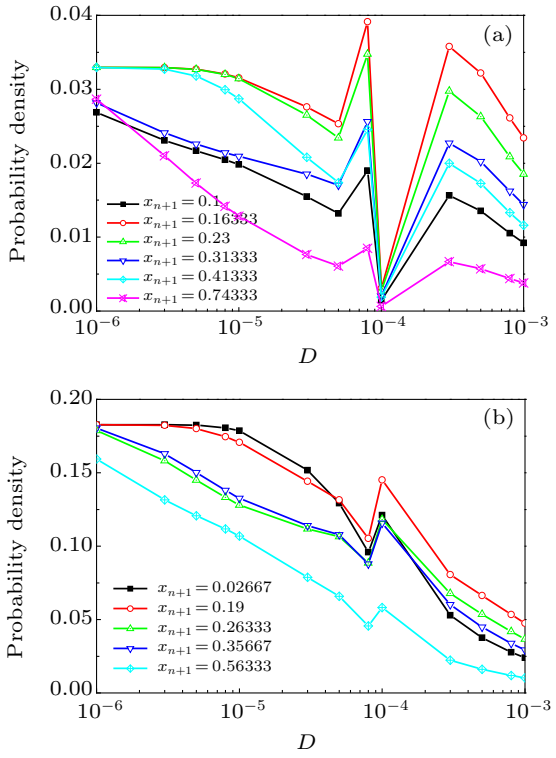


Fig. 3. (color online) Normalized probability densities of orbits x_{n+1} as a function of noise intensity D at $\mu = 0.0425$ for (a) P-6 and (b) P-5.

4. The effect of Ornstein–Uhlenbeck noise

However, more realistic models of physical systems require the consideration of noise sources with finite correlation time (i.e., colored noise). Therefore, in order to further understand how colored noise affects the probability density of each orbit, we introduce an exponentially correlated colored noise γ_n to investigate the dynamic behavior of the periodic orbits of the piecewise linear map at $\mu = 0.0415$. The model is given by

$$x_{n+1} = f_i(x_n) = k_i x_n + b_i + \gamma_n \quad \mod 1, \tag{5}$$

where γ_n denotes the Gaussian colored noise with zero mean and correlation function $\langle \gamma_n \gamma_m \rangle = (\alpha/\tau)\exp[-(m-n)/\tau]$ for $m, n = 1, 2, 3, \ldots$, $\langle \ldots \rangle$ denotes an ensemble average over the distribution of noise, α is the intensity of the noise, and τ is the self-correlation time. When $\tau = 0$, γ_n becomes Gaussian white noise.

Figure 4 shows the probability densities of the orbits as a function of x_n at $\mu = 0.0425$ with $\alpha = 10$, $\tau = 1$, 10, 90, 100, and 1000. Figure 4(a) shows the case without noise, where P-6 and P-5 coexist. Figure 4(b) shows that when α is fixed and τ is very small ($\tau = 1$), the periodic orbits of P-5 and P-6 still coexist. The peaks of the probability densities of the periodic orbits are broadened, and the heights become low compared with those in Fig. 4(a). However, when the correlation time τ reaches a critical value ($\tau_c \approx 10$), the periodic orbits of P-6 disappear (Fig. 4(c)), and only the orbits of P-5 exist. With τ increasing, the stability of the periodic orbits of P-5 is enhanced (Fig. 4(d)). However, it is very interesting to find that the periodic orbits of P-5 disappear suddenly and those of P-6 appear again (Fig. 4(e)) when τ reaches the second critical value $\tau_c' \approx 91.76$. Moreover, the heights of the peaks of the periodic orbits of P-6 are almost equal to those of P-5 without noise as τ increases (Fig. 4(f)). We emphasize that the two critical values τ_c and τ_c' are related to the noise intensity α.

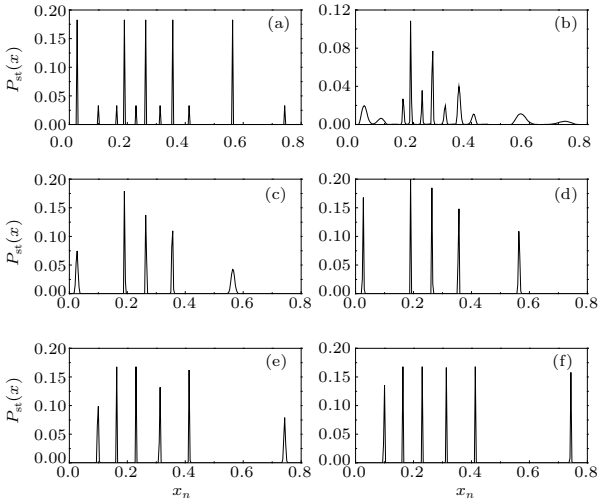


Fig. 4. Normalized probability densities at $\mu = 0.0415$ for (a) without noise; (b) $\alpha = 10$, $\tau = 1$; (c) $\alpha = 10$, $\tau = 10$; (d) $\alpha = 10$, $\tau = 90$; (e) $\alpha = 10$, $\tau = 100$; (f) $\alpha = 10$, $\tau = 1000$. The 4×10^6 iterations, after initial 6×10^6 iterations, are partitioned into 300 bins.

Figure 5 shows how the noise affects the probability density of each orbit. We choose a series of different self-correlation times τ and a fixed noise intensity $\alpha = 10$, and study the dynamic behavior of each orbit. The probability densities are plotted as a function of τ . Figure 5(a) shows that the effects of the self-correlation time τ on the orbits of P-6. From Fig. 5(a), we find that the effects of the self-correlation time τ on the stabilities of the six orbits of P-6 are consistent. When $\tau < \tau_c$, the heights of the six orbits are decreased. When $\tau > \tau_c'$, the probability densities of the six orbits are increased with τ increasing. When $\tau_c < \tau < \tau_c'$, the probability densities of the six orbits suddenly become zero, which means that P-6 disappears. It must be emphasized that these orbits change at different rates. We also find that the orbit of 0.23 is more stable than the others. Figure 5(b) shows that the effects of the self-correlation time τ on the orbits of P-5. The tendencies of

the five orbits of P-5 are consistent. When $\tau < \tau_c$, the probability densities increase. The orbit of 0.19 is more stable than the others.

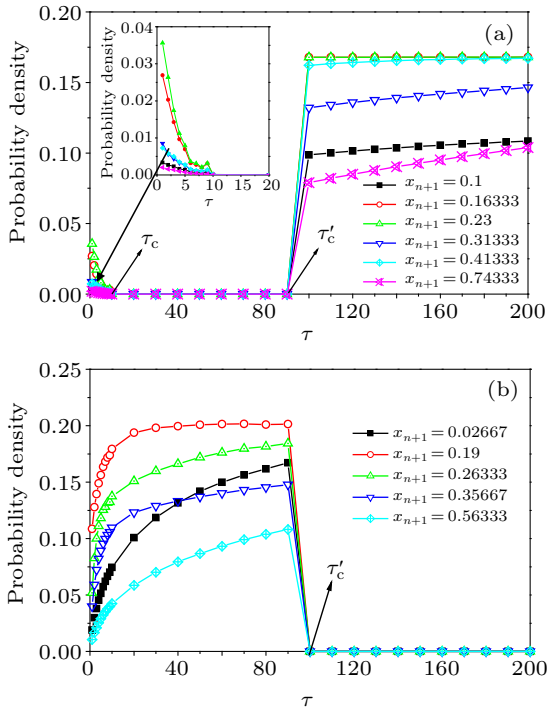


Fig. 5. (color online) Normalized probability densities of orbits x_{n+1} as a function of the self-correlation time τ for (a) P-6 and (b) P-5.

5. Conclusion

In summary, we investigate the effects of Gaussian white noise and Gaussian colored noise on the periodic orbits of P-5 and P-6 in their coexisting domain numerically. The probability densities of some orbits are calculated. When the noise intensity is $D = 0.0001$, only the orbits of P-5 exist, and the coexisting phenomenon is destroyed. On the other hand, the self-correlation time τ of the colored noise also affects the coexisting phenomenon. There exist two critical values τ_c and τ_c'. When $\tau_c < \tau < \tau_c'$, only the orbits of P-5 appear, and the stability of the orbits of P-5 is enhanced. However, when $\tau > \tau_c'$, only the orbits of P-6 exist, and the stability of the orbits of P-6 is greatly enhanced. When $\tau < \tau_c$, the orbits of P-5 and P-6 coexist, but the stability of the orbits of P-5 is enhanced and those of P-6 are weakened with τ increasing. Furthermore, it is reasonable to envisage that we may be able to estimate the noise intensity by measuring the changes in the probability density.

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