Quantum mechanical version of the classical Liouville theorem*

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In terms of the coherent state evolution in phase space, we present a quantum mechanical version of the classical Liouville theorem. The evolution of the coherent state from $|z\rangle$ to $|sz-rz^*\rangle$ corresponds to the motion from a point $z(q,p)$ to another point $sz-rz^*$ with $|s|^2-|r|^2=1$. The evolution is governed by the so-called Fresnel operator $U(s,r)$ that was recently proposed in quantum optics theory, which classically corresponds to the matrix optics law and the optical Fresnel transformation, and obeys group product rules. In other words, we can recapitulate the Liouville theorem in the context of quantum mechanics by virtue of coherent state evolution in phase space, which seems to be a combination of quantum statistics and quantum optics.

Keywords: Fresnel operator, Liouville theorem, quantum mechanical version

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1. Introduction

In classical statistics mechanics, the dynamical variables p_i and q_i with $i=1,2,\ldots,n$ may be regarded as the coordinates in a $2n$-dimensional space called the phase space. Every set of these variables (p_i,q_i) is thus represented by a point, and the quantity

$$\mathrm{d}s=\prod_{i=1}^{n}\mathrm{d}p_i\mathrm{d}q_i \tag{1}$$

represents the volume element in the phase space. A mechanical system found at point $P(p_i,q_i)$ at time t will follow a definite phase orbit so that at a later time t', it will be found at another point $P'\left(p'_k,q'_k\right)$ with the set $\left(p'_k,q'_k\right)$ obtained from the initial set (p_i,q_i) by forward integrating Hamilton's equations.[1] Since the new values are determined by the initial values in this fashion, we can write

$$p'_k=p'_k(p_i,q_i),\quad q'_k=q'_k(p_i,q_i),\quad k=1,2,\ldots,n. \tag{2}$$

Equation (2) can be considered as a transformation of the set (p_i,q_i) corresponding to the volume element $\mathrm{d}s$ to the set $\left(p'_k,q'_k\right)$ and the volume element $\mathrm{d}s'$, i.e., geometrically as a mapping of the phase space at time t to that at time t', as described in Fig. 1 (without loss of generality, taking the two-dimensional case as an example). As described in classical statistics mechanics, during the process of the above evolution, the magnitude of the volume element is kept unchanged, i.e.,

$$\mathrm{d}s'=\mathrm{d}s. \tag{3}$$

In classical mechanics, this character is recapitulated as the classical Liouville theorem, i.e., the magnitude of the volume element in the phase space does not change in the course of its motion along the phase orbits.

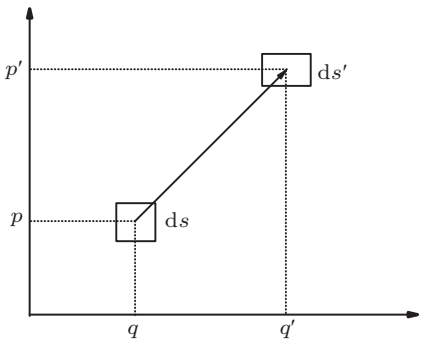


Fig. 1. Motion according to classical Liouville theorem.

In this paper, we will present a quantum mechanical version of the classical Liouville theorem in terms of the coherent state[2–5] evolution in the phase space. The evolution of the coherent state from $|z\rangle$ to $|sz-rz^*\rangle$ corresponds to the motion from a point $z(q,p)$ to another point $sz-rz^*$ with $|s|^2-|r|^2=1$. The evolution is governed by the Fresnel operator $U(s,r)$, that has recently been proposed[6–10] in quantum optics theory, classically corresponds to the matrix optics law and the optical Fresnel transformation, and obeys the group product rules. In Section 2, we will introduce the motion of the coherent state in the phase space and the corresponding Fresnel operator. In Section 3, according to the fact that the product of two Fresnel operators is still a Fresnel operator, we

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give the quantum mechanical version of the classical Liouville theorem, and finally, we provide a discussion.

2. The coherent state motion in phase space and the Fresnel operator

In quantum mechanics, the coherent state's coordinate X and momentum P satisfy the minimum uncertainty relationship[11]

$$\left\langle (\Delta X)^2 \right\rangle \left\langle (\Delta P)^2 \right\rangle = \frac{1}{4}, \tag{4}$$

where the reduced Plank constant $\hbar$ is taken as the unit. Let

$$\Delta x = \sqrt{\left\langle (\Delta X)^2 \right\rangle}, \ \Delta p = \sqrt{\left\langle (\Delta P)^2 \right\rangle}, \tag{5}$$

then

$$\Delta x \Delta p = \frac{1}{2}, \tag{6}$$

which corresponds to a small round with area $1/2$ in the phase space, and the center of the round represents the quantum state. In the phase space, the representation point of a coherent state is $z = (q + \mathrm{i}p)/\sqrt{2}$, so $\left|\begin{pmatrix} q \\ p \end{pmatrix}\right\rangle$ can represent a coherent state. Let $\begin{pmatrix} q \\ p \end{pmatrix}$ be transformed to $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$, then the corresponding coherent state is transformed from $\left|\begin{pmatrix} q \\ p \end{pmatrix}\right\rangle$ to $\left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}\right\rangle$. According to the symplectic transformation, the real parameters should satisfy the relation

$$AD - BC = 1. \tag{7}$$

By introducing

$$\begin{aligned} s &= \frac{1}{2}\left[A + D - \mathrm{i}B + \mathrm{i}C\right], \\ r &= \frac{1}{2}\left(D - A - \mathrm{i}B - \mathrm{i}C\right), \end{aligned} \tag{8}$$

then the relation $AD - BC = 1$ becomes the relation $|s|^2 - |r|^2 = 1$ that should be satisfied by the complex parameters s and r. Then the coherent state $\left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}\right\rangle$ can be rewritten as

$$\begin{aligned} &\left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}\right\rangle \\ &= \left|\begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix}\begin{pmatrix} z \\ z^* \end{pmatrix}\right\rangle \equiv |sz - rz^*\rangle \\ &= \exp\left[-\frac{|sz - rz^*|^2}{2} + (sz - rz^*)a^\dagger\right]|0\rangle, \end{aligned} \tag{9}$$

where $|0\rangle$ is the vacuum state. From the above equations, corresponding to the evolution of the coherent state, we can construct the ket–bra integration

$$U = \sqrt{s}\int \frac{\mathrm{d}^2 z}{\pi} |sz - rz^*\rangle \langle z|. \tag{10}$$

As will be explained later, U is a motion causing phase point movement from z to $sz - rz^*$. Using the integration within an ordered product of operators (IWOP),[12–15] we can get

$$\begin{aligned} U(s,r) &= \sqrt{s}\int \frac{\mathrm{d}^2 z}{\pi} :\exp\left[-|s|^2|z|^2 + sza^\dagger + z^*\left(a - ra^\dagger\right)\right. \\ &\quad \left. + \frac{r^* s}{2}z^2 + \frac{rs^*}{2}z^{*2} - a^\dagger a\right]: \\ &= \frac{1}{\sqrt{s^*}}\exp\left[-\frac{r}{2s^*}a^{\dagger 2}\right] \\ &\quad \times :\exp\left[\left(\frac{1}{s^*} - 1\right)a^\dagger a\right]: \exp\left[\frac{r^*}{2s^*}a^2\right], \end{aligned} \tag{11}$$

where $: \ :$ denotes the normal ordering, a and $a^\dagger$ are the corresponding annihilation and creation operators, and $U(s,r)$ is called the Fresnel transformation. In Fresnel's age, there was no quantum mechanics, so let us explain why it is called as the Fresnel operator. Using the overlap of the coordinate eigenstate and the coherent state

$$\langle x|z\rangle = \pi^{-\frac{1}{4}}\exp\left(-\frac{x^2}{2} + \sqrt{2}xz - \frac{z^2}{2} - \frac{|z|^2}{2}\right), \tag{12}$$

and the completeness of the coherent states, we can calculate the matrix of $U(s,r)$ in the coordinate states

$$\begin{aligned} &\langle x_2|U(s,r)|x_1\rangle \\ &= \int \frac{\mathrm{d}^2 z}{\pi}\langle x_2|z\rangle\langle z|U(s,r)\int \frac{\mathrm{d}^2 z'}{\pi}|z'\rangle\langle z'|x_1\rangle \\ &= \frac{1}{\sqrt{s^*}}\int \frac{\mathrm{d}^2 z}{\pi}\int \frac{\mathrm{d}^2 z'}{\pi}\exp\left(-\frac{x_1^2 + x_2^2}{2} + \sqrt{2}x_2 z + \sqrt{2}x_1 z'\right. \\ &\quad \left. - |z|^2 - |z'|^2 - \frac{r}{2s^*}z^{*2} + \frac{r^*}{2s^*}z'^2 + \frac{1}{s^*}z^* z' - \frac{z^2}{2} - \frac{z'^{*2}}{2}\right) \\ &= \frac{1}{\sqrt{s^* - s + r^* - r}}\exp\left\{\frac{1}{s^* - s + r^* - r}\right. \\ &\quad \times \left[2x_1 x_2 + (r^* - s)x_1^2 - \frac{x_2^2}{s^* - r}\right] \\ &\quad \left. - \frac{rx_2^2}{s^* - r} - 2\frac{x_1^2 + x_2^2}{2}\right\}. \end{aligned} \tag{13}$$

From Eq. (8), we know

$$\begin{aligned} A &= \frac{1}{2}(s - r + s^* - r^*), \quad B = \frac{\mathrm{i}}{2}(s + r - s^* - r^*), \\ C &= -\frac{\mathrm{i}}{2}(s - r - s^* + r^*), \quad D = \frac{1}{2}(s + r + s^* + r^*), \end{aligned} \tag{14}$$

which satisfy $AD - BC = 1$, so equation (13) is the integration kernel of the generalized Fresnel transformation

$$\begin{aligned} &\langle x_2|U(s,r)|x_1\rangle \\ &= \frac{1}{\sqrt{2\pi \mathrm{i}B}}\exp\left[\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_1 x_2 + Dx_2^2\right)\right]. \end{aligned} \tag{15}$$

So $\langle x_2|U(s,r)|x_1\rangle$ is the quantum correspondence of the Fresnel operator in classical optics, and we name $U(s,r)$ the Fresnel operator.[6–8] The motion of a coherent state in the phase space corresponds to the Fresnel transformation. This is wonderful!

$$g(x_2) = \frac{1}{\sqrt{2\pi \mathrm{i}B}}\int \mathrm{d}x_1 f(x_1) \times \exp\left[\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_1x_2 + Dx_2^2\right)\right] \tag{16}$$

is simply the Collins diffraction integration formula in classical optics, where $f(x_1)$ and $g(x_2)$ are the input and the output optical fields, respectively.

Further, we explain the above motion from z to $sz - rz^*$ that arose from $U(s,r)$ in the phase space. From Eq. (11), we can get

$$U^\dagger a U = sa - ra^\dagger, \qquad U^\dagger a^\dagger U = s^* a^\dagger - r^* a, \tag{17}$$

so according to the relations $X = (a + a^\dagger)/\sqrt{2}$ and $P = (a - a^\dagger)/(\sqrt{2}\mathrm{i})$, we have

$$U^\dagger X U = U^\dagger\left(\frac{a+a^\dagger}{\sqrt{2}}\right)U = \frac{1}{\sqrt{2}}\left(sa - ra^\dagger + s^* a^\dagger - r^* a\right), \tag{18}$$

$$U^\dagger P U = U^\dagger\left(\frac{a-a^\dagger}{\mathrm{i}\sqrt{2}}\right)U = \frac{1}{\mathrm{i}\sqrt{2}}\left(sa - ra^\dagger - s^* a^\dagger + r^* a\right). \tag{19}$$

Then in state $U(s,r)|z\rangle$, the expectation values of X and P are

$$\langle z|U^\dagger X U|z\rangle = \frac{1}{\sqrt{2}}\left[(s - r^*)z - (r - s^*)z^*\right], \tag{20}$$

$$\langle z|U^\dagger P U|z\rangle = \frac{1}{\mathrm{i}\sqrt{2}}\left[(s + r^*)z - (r + s^*)z^*\right]. \tag{21}$$

We can get the uncertainty relation for X and P in the state of $U|z\rangle$ (the coherent state after the motion arose from the Fresnel operator) as

$$\left\langle(\Delta X)^2\right\rangle\Big|_{U|z\rangle} = \langle X^2\rangle - \langle X\rangle^2 = \frac{1}{2}, \qquad \left\langle(\Delta P)^2\right\rangle\Big|_{U|z\rangle} = \langle P^2\rangle - \langle P\rangle^2 = \frac{1}{2}, \tag{22}$$

i.e.,

$$\left\langle(\Delta X)^2\right\rangle\Big|_{U|z\rangle}\left\langle(\Delta P)^2\right\rangle\Big|_{U|z\rangle} = \frac{1}{4}, \tag{23}$$

which is equal to the case for the coherent state $|z\rangle$ in Eq. (6). From Eq. (23), we can see that for the moving in the phase space, the phase volume is a constant.

3. Quantum mechanical version of classical Liouville theorem

The product of two Fresnel operators is still a Fresnel operator, i.e.,

$$U(s,r)U'(s',r') = U''(s'',r''), \tag{24}$$

and this can be seen as the mapping of the classical Liouville theorem in quantum optics. It is named the quantum Liouville theorem. According to Eq. (10), we should prove the following product rules (noticing $s's'^* - r'r'^* = 1$):

$$\sqrt{ss'}\int\frac{\mathrm{d}^2z\mathrm{d}^2z'}{\pi^2}\left|\begin{pmatrix} s & -r \\ -r^* & s^*\end{pmatrix}\begin{pmatrix} z \\ z^*\end{pmatrix}\right\rangle \times\left\langle z\right|\begin{pmatrix} s' & -r' \\ -r'^* & s'^*\end{pmatrix}\begin{pmatrix} z' \\ z'^*\end{pmatrix}\Big\rangle\langle z'| = \sqrt{s''}\int\frac{\mathrm{d}^2z}{\pi}\left|\begin{pmatrix} s'' & -r'' \\ -r''^* & s''^*\end{pmatrix}\begin{pmatrix} z \\ z^*\end{pmatrix}\right\rangle\langle z|. \tag{25}$$

We can see that the classical transformation automatically satisfies the product rules

$$\begin{pmatrix} s'' & -r'' \\ -r''^* & s''^*\end{pmatrix} = \begin{pmatrix} s & -r \\ -r^* & s^*\end{pmatrix}\begin{pmatrix} s' & -r' \\ -r'^* & s'^*\end{pmatrix}, \tag{26}$$

i.e., $s'' = ss' + rr'^*$ and $r'' = rs'^* + r's$, and assures the relation

$$s''s''^* - r''r''^* = 1. \tag{27}$$

If we can prove Eq. (25), then this shows that the product rules of the matrix can map the product rules of the squeezed operator represented by the coherent ket–bra projection operator. If we use the group language, the coherent ket–bra projection operator representation realizes the product rules of the symplectic group. This property is very important as it ensures that the result of many successive squeezing transformations is still a squeezing transformation. The last squeezing transformation parameter is many successive transformations of the product of the transformation group.

Now using the IWOP method and the formula

$$\mathrm{e}^{\lambda a^2}\mathrm{e}^{\sigma a^{\dagger 2}} = \frac{1}{\sqrt{1-4\lambda\sigma}}\exp\left(\frac{\sigma a^{\dagger 2}}{1-4\lambda\sigma}\right) \times\exp\left[-a^\dagger a\ln(1-4\lambda\sigma)\right]\exp\left(\frac{\lambda a^2}{1-4\lambda\sigma}\right), \tag{28}$$

we can get

$$\begin{aligned} &U(s,r)U(s',r') \\ &= \frac{1}{\sqrt{s^*s'^*}}\exp\left(-\frac{r}{2s^*}a^{\dagger 2}\right)\exp\left[a^\dagger a\ln(s^*)^{-1}\right]\exp\left(\frac{r^*}{2s^*}a^2\right) \\ &\quad\times\exp\left(-\frac{r'}{2s'^*}a^{\dagger 2}\right)\exp\left[a^\dagger a\ln(s'^*)^{-1}\right]\exp\left(\frac{r'^*}{2s'^*}a^2\right) \\ &= \frac{1}{\sqrt{s^*s'^* + r^*r'}}\exp\left[-\frac{rs'^* + r's}{2(s^*s'^* + r^*r')}a^{\dagger 2}\right] \\ &\quad\times\exp\left[a^\dagger a\ln(s^*s'^* + r^*r')^{-1}\right] \\ &\quad\times\exp\left[-\frac{r'^*(s^*s'^* + r^*r') + r^*}{2s'^*(s^*s'^* + r^*r')}a^2\right]. \end{aligned} \tag{29}$$

From Eq. (26), we can get

$$\frac{r'^{*}\left(s^{*}s'^{*}+r^{*}r'\right)}{s'^{*}}=r^{*}s'+r'^{*}s^{*}=r''^{*}, \tag{30}$$

so

$$\begin{aligned} & U\left(s,r\right)U\left(s',r'\right) \\ = & \frac{1}{\sqrt{s''^{*}}}\exp\left(-\frac{r''}{2s''^{*}}a^{\dagger 2}\right) \\ & \times :\exp\left[\left(\frac{1}{s''^{*}}-1\right)a^{\dagger}a\right]:\exp\left(\frac{r''^{*}}{2s''^{*}}a^{2}\right) \\ = & U\left(s'',r''\right). \end{aligned} \tag{31}$$

The product rules are thus proved.

In the phase space, if a coherent state undergoes many successive evolutions, every evolution is governed by the Fresnel transformation, i.e., $|z\rangle$ to $U\left(s,r\right)|z\rangle$, then to $U\left(s,r\right)U\left(s',r'\right)|z\rangle$, and so on. Equation (24) assures us that during the process of the evolutions, the uncertainty relation for X and P in every state is the same as given by Eq. (23), i.e., for the moving in the phase space, the phase volume is a constant. So equation (24) is named the quantum Liouville theorem, the counterpart of the classical Liouville theorem in quantum optics.

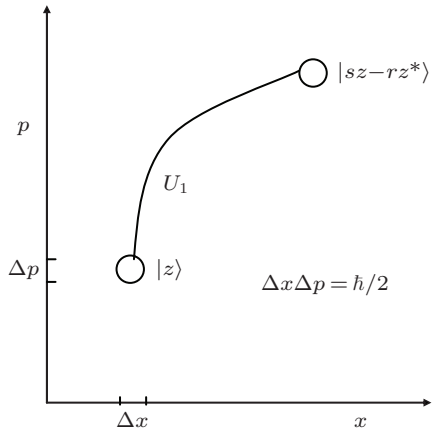


Fig. 2. Motion according to the quantum Liouville theorem.

Since the discovery of the Fresnel operator $U\left(s,r\right)$, it has been widely used in quantum optics. For example, in Ref. [16], Fan *et al.* introduced $U\left(s,r\right)|x\rangle\equiv|x\rangle_{s,r}$ as the intermediate coordinate–momentum representation, where $|x\rangle$ is the coordinate representation, and found that the project operator $|x\rangle_{s,r\ s,r}\langle x|$ is just the Radon transformation of the Wigner operator, based on which excellent applications in the tomography theorem in quantum optics have been obtained.[17,18] So from the above excellent performances of the Fresnel operator $U\left(s,r\right)$, we can see that the Fresnel operator $U\left(s,r\right)$ can be considered as a bridge combining quantum statistics and quantum optics. In future, we can probe more inherent relations between them.

4. Conclusion

In a word, in this paper we recapitulate the Liouville theorem in the context of quantum mechanics by virtue of the coherent state evolution in the phase space, which seems to be a combination of quantum statistics and quantum optics.

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