Singularities of noncompact charged objects*

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We formulate a model for noncompact spherical charged objects in the framework of noncommutative field theory. The Einstein–Maxwell field equations are solved with charged anisotropic fluid. We choose matter and charge densities as functions of the two parameters, instead of defining these quantities in terms of the Gaussian distribution function. It is found that the corresponding densities and the Ricci scalar are singular in origin, whereas the metric is nonsingular, indicating a spacelike singularity. The numerical solution of the horizon equation implies that there are two or one or no horizon(s), depending on the mass. We also evaluate the Hawking temperature and find that a black hole with two horizons is evaporated to an extremal black hole with one horizon.

Keywords: noncommutative geometry, electromagnetic field, spacetime singularity

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1. Introduction

Recently, there has been growing interest in studying the nature of spacetime singularity and black holes (BHs) in the context of noncommutative (NC) field theory. This is perhaps due to the fact that some of the problems of BH physics remain unsolved even to this day.[1] For example, a satisfactory description of the final stage of BH evaporation is still an open question. According to BH correspondence principle,[2] one cannot neglect the string effects during the last stage of BH evaporation. This has inspired researchers to adopt the string field approach in various aspects of theoretical physics. Noncommutative field theory is one of the outcomes of the string theory, in which spacetime coordinates become noncommuting operators on a D-brane.[3,4] This motivates researchers to reconsider the older ideas of Snyder.[5]

Zade *et al.*[6] discussed the gravitational collapse of the radiating Dyon field and clarified the status of CCH. Nahed[7] derived the regular charged solutions in Mullör's tetrad theory of gravitation, and Ying *et al.*[8] formulated the double NC form of a spacetime and introduced the complex symmetric theory of gravitation. Meanwhile, Guang[9] investigated the singularities of spherically symmetric, static charged scalar field solutions.

Spacetime noncommutativity can be expressed by the following relation:[10]

$$[x^{\mu},x^{\nu}]=\imath\sigma^{\mu\nu}, \tag{1}$$

where $\sigma^{\mu\nu}$ is an anti-symmetric matrix that determines the spacetime cell discretization as $\hbar$ (Planck's constant) discretizes the phase space. There are several approaches to formulating NC field theory, for instance, one method is based on the cross product, and another on coordinate coherent state formalism. Using the second approach, Smailagic and Spallucci[7] argued that the problems of Lorentz invariance and unitary arising from the cross product can be solved by assuming $\sigma^{\mu\nu}=\sigma\,\mathrm{diag}(\varepsilon_{ij},\varepsilon_{ij},\ldots)$, where σ is a constant with length square dimensions. In addition, the coordinate coherent state modifies the Feynman propagators. It is believed that NC would remove the singularities (divergences) appearing in general relativity (GR). In GR, the metric field is a geometrical structure and the curvature measures its strength. Since noncommutativity is the fundamental property of the metric, it affects the geometry through the field equations, and thus the energy–momentum tensor. In the NC GR, we keep the geometry part of the field equations unchanged, while a small change is introduced in the matter part.

It was pointed out by Doplicer *et al.*[12] that when matter density is extremely large, a BH is formed. According to the Heisenberg uncertainty principle, the measurement of a spacetime separation causes an uncertainty in momentum, i.e., momentum preserves inverse proportionality to the extent of the separation. For a small enough separation, the momentum becomes large and the system leads to BH formation. The behavior of BH in NC field theory has been studied by many people. Banerjee *et al.*[13] examined the behavior of the NC Schwarzschild BH, while Modesto and Nicolini[14] discussed the charged rotating NC BH. Bastos *et al.*[15,16] explored the non-canonical phase space, singularity problem, and BH in the context of NC geometry. Moreover, Bartolami and Zarro[17] investigated the NC correction to pressure, particle numbers, and energy density for fermion gas and radiation. The NC correction to these quantities leads to the fact that NC affects the matter dispersion relation and equation of state. Inspired by the NC correction to BH physics, Oh and Park[18] explored

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the gravitational collapse of shell with a smeared gravitational source in the NC Schwarzschild geometry. We have extended this work for the NC Reissner–Nordström background.[19] Sun *et al.*[20] studied the gravitational collapse of a spherically symmetric star in NC GR using the spacetime quantization approach.

In all these papers, the matter density is taken as the Gaussian distribution function. However, Castro[21] studied the singularity associated with a noncompact matter source by using matter density in a particular form that belongs to the most general two parametric forms of the density distribution. Here we extend this work by studying the nature of singularity associated with noncompact charged matter sources for $0 \leq r \leq \infty$. This paper is organized as follows. In the next section, we formulate the noncompact charged object model. Section 3 is devoted to discussing some of the properties of this model, and in the last section, we conclude our results.

2. The noncompact charged object model

To formulate the noncompact charged objects model, we solve the Einstein–Maxwell field equations with anisotropic fluid and an electromagnetic field. In Ref. [21], instead of choosing density in the form of Gaussian width length[10]

$$\rho = \frac{m_0 \mathrm{e}^{-\frac{r^2}{4\sigma^2}}}{(4\pi\sigma^2)^{3/2}}, \tag{2}$$

they wrote the density as

$$\rho(r,\sigma) = \frac{m_0}{4\pi r^2}\frac{3\sigma^3}{2}\frac{1}{r^4[1+(\sigma/r)^3]^{3/2}}, \tag{3}$$

where σ is the NC parameter, and the constant m_0 corresponds to the total gravitational mass of the system.

The form of matter density ρ given by Eq. (3) belongs to the most general two parameter families of the density distribution function

$$\rho(r,\sigma,k) = \frac{m_0}{4\pi r^2}\frac{k\sigma^k}{2}\frac{1}{r^{1+k}[1+(\sigma/r)^k]^{3/2}}, \quad (k>2). \tag{4}$$

We follow Ref. [21], take the same expression for matter density as expressed by Eq. (3), and use the charge density in the form

$$\rho(r,\sigma) = \frac{q_0}{4\pi r^2}\frac{3\sigma^3}{2}\frac{1}{r^4[1+(\sigma/r)^3]^{3/2}}, \tag{5}$$

instead of

$$\rho_{el} = \frac{q_0 \mathrm{e}^{-\frac{r^2}{4\sigma^2}}}{(4\pi\sigma^2)^{3/2}}, \tag{6}$$

as given in Ref. [22]. Here, constant q_0 corresponds to the total charge of the system.

Taking matter density ρ and electric density ρ_{el} defined by Eqs. (2) and (6), respectively, many authors[10,13,14,22] have derived the neutral and charged rotating NC BH solutions. All these exhibit a de-Sitter core at the origin of the underlying geometry due to the quantum fluctuations. Applying the same strategy used in the above studies, but using a modified definition of the matter density (defined by Eq. (3)), Castro[21] derived an NC solution which has no de-Sitter core and exhibits timelike naked singularity at the origin. The present work is the extension of Ref. [21] to the electromagnetic field case.

We consider a spherically symmetric spacetime in the following form:

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 - \frac{1}{f(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{7}$$

where $f(r)$ is an unknown function to be determined using the field equations.

The matter under consideration is charged anisotropic fluid. The components of the anisotropic fluid energy–momentum tensor are

$$\boldsymbol{T}^{\mu}_{\mu} = (\rho(r), -p_r, -p_\theta, -p_\phi). \tag{8}$$

The energy–momentum tensor of the electromagnetic field is

$$\boldsymbol{T}^{\nu\,(\mathrm{em})}_{\mu} = \frac{1}{4\pi}\left(-\boldsymbol{F}^{\nu\lambda}\boldsymbol{F}_{\mu\lambda} + \frac{1}{4}\delta^{\nu}_{\mu}\boldsymbol{F}_{\pi\lambda}\boldsymbol{F}^{\pi\lambda}\right). \tag{9}$$

The Maxwell equations are given by

$$\boldsymbol{F}_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu}, \quad \boldsymbol{F}^{\mu\nu}{}_{;\nu} = 4\pi\boldsymbol{J}^{\mu}, \tag{10}$$

where $\boldsymbol{F}_{\mu\nu}$ is the Maxwell field tensor, ϕ_μ is the four potential, and $\boldsymbol{J}_\mu$ is the four current. For the static spherically symmetric charge distribution, the four potential and the four current are given in the following form:

$$\phi_\mu = \phi\,\delta^0_\mu, \quad \boldsymbol{J}^{\mu} = \rho_{\mathrm{el}}\delta^{\mu}_0, \tag{11}$$

where ϕ is the electric potential, and ρ_{el} is the electric charge density. Solving the Maxwell field equations, we obtain

$$F_{01} = -\frac{\partial\phi}{\partial r} = \frac{q(r,\sigma)}{r^2}, \tag{12}$$

where

$$q(r,\sigma) = 4\pi\int_0^r r^2\rho_{\mathrm{el}}\,\mathrm{d}r = \frac{q_0}{\sqrt{1+(\sigma/r)^3}}. \tag{13}$$

In order to find the relation between different components of the energy–momentum tensor, we use the conservation of energy–momentum tensor, i.e, $\tilde{\boldsymbol{T}}^{\mu}_{\nu;\mu} = 0$ ($\tilde{\boldsymbol{T}}^{\mu}_{\nu} = \boldsymbol{T}^{\mu}_{\nu} + \boldsymbol{T}^{\mu\,(\mathrm{em})}_{\nu}$). For the Schwarzschild-like nature of the solution, i.e, $g_{00} = (g_{rr})^{-1}$, we make a restriction that $T^0_0 = T^r_r$, which implies that $p_r = -\rho(r)$ (negative radial pressure pointing towards the center $r = 0$) and $p_\theta = p_\phi$. With these conditions, the conservation equation provides the following relation:

$$p_\theta = -\frac{r}{2}\partial_r(\rho + \frac{q^2}{8\pi r^4}) - \rho - \frac{q^2}{4\pi r^4}. \tag{14}$$

The Einstein field equations together with Eqs. (3), (7)–(9), (12), and (14) lead to the following solution:

$$\begin{aligned} \mathrm{d}s^2 &= \left(1-\frac{2m(r,\sigma)}{r}-\frac{Q^2(r,\sigma)}{r}\right)\mathrm{d}t^2 \\ &\quad -\left(1-\frac{2m(r,\sigma)}{r}-\frac{Q^2(r,\sigma)}{r}\right)^{-1}\mathrm{d}r^2 \\ &\quad -r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2), \end{aligned} \tag{15}$$

where

$$m(r,\sigma)=4\pi\int_0^r \rho r^2\,\mathrm{d}r=\frac{m_0}{\sqrt{1+(\sigma/r)^3}}, \tag{16}$$

and

$$\begin{aligned} Q^2(r,\sigma) &= \frac{{q_0}^2}{18\sigma}\left(\sqrt{3}\pi-6\sqrt{3}\arctan\left(\frac{\sigma-2r}{\sqrt{3}\sigma}\right)\right. \\ &\quad \left.+\log\left(\frac{(\sigma^2-\sigma r+r^2)^3}{(\sigma+r)^6}\right)\right). \end{aligned} \tag{17}$$

The solution given by Eq. (15) represents a noncompact charged object.

3. The properties of the model

This section is devoted to exploring the properties of the charged object model. In the limit $\sigma\to 0$, the line element (15) is reduced to the classical Reissner–Nordström solution. We observe that the metric component

$$g_{00}=1-\frac{2m(r,\sigma)}{r}-\frac{Q^2(r,\sigma)}{r} \tag{18}$$

with Eqs. (16) and (17) satisfy the condition $g_{tt}(r=0)=g_{tt}(r=\infty)=1$. This implies that the solution (15) is asymptotically flat and not singular at $r=0$. The corresponding Ricci scalar is

$$R=-\frac{3}{2}\left(\frac{\sigma}{r^3}\right)^3\left(5\sigma^3 m+2r^2{q_0}^2\sqrt{1+\left(\frac{\sigma}{r}\right)^3}-4mr^3\right), \tag{19}$$

which is singular at $r=0$. Moreover, the Ricci scalar can be found directly from the field equations as $R=-T^{\mu}_{\mu}$, where we have used $T^{\mu\,(\mathrm{em})}_{\mu}=0$. Furthermore, using the values of T^{μ}_{μ} along with p_r and p_θ, we obtain

$$R=-\left(4\rho+\frac{r}{\partial_r}(\rho+\frac{q^2}{8\pi r^4})+\frac{q^2}{4\pi r^4}\right). \tag{20}$$

After a straightforward but laborious calculation, we obtain the relation for R as the same as Eq. (19). This verifies the fact that the solution (15) is the valid solution of the Einstein field equations with the charged anisotropic source. Despite the fact that the metric is nonsingular at $r=0$, the Ricci scalar is singular at $r=0$. This is due to the direct dependence of the Ricci scalar on the matter density, which is singular at $r=0$.

When $g_{00}(r)=0$ at the horizon radius $r=r_{\mathrm{H}}$, the metric (15) becomes singular. However, this is a coordinate singularity.

The horizon equation, $g_{00}(r)=0$, for the solution (15) indicates that it is not possible to find an exact solution of the horizon equation for r. Thus, with the arbitrary choice of parameters, we evaluate the values of r for $g_{00}(r)=0$, as shown in Fig. 1(a). This shows that for $m_0=0.44$, one horizon exists when the mass is smaller than this value, and no horizon or two horizons exist for mass larger than $m_0=0.44$. Hence, the case $m_0<0.44$ corresponds to the existence of naked singularity, as no horizons exist to hide the singularity at $r=0$. This implies that the minimum mass for the existence of BH in this framework is $m_0=0.44$. Since matter density and the Ricci scalar are divergent at $r=0$, while metric components are finite at this value of r, a spacelike singularity exists. Consequently, we find that the solution represents a spacelike naked singularity, extremal BH and a BH with inner and outer horizons depending on the mass.

The Hawking temperature for the BH is

$$T_{\mathrm{H}}=\left(\frac{1}{4\pi}\frac{\mathrm{d}g_{00}}{\mathrm{d}r}\right)_{r=r_{\mathrm{H}}}. \tag{21}$$

The behavior of T_{H} (after eliminating $m=m(r,q_0,\sigma)$ from $\mathrm{d}g_{00}/\mathrm{d}r$ and $g_{00}(r)=0$) is shown in Fig. 1(b). As r_{H} decreases, the temperature increases and reaches the maximum value at $r_{\mathrm{H}}\simeq 2$, and then suddenly drops to zero at $r_{\mathrm{H}}\simeq 1.1$, corresponding to the radius of the extremal BH. Since a further decrease in r_{H} results in a negative temperature, which is not acceptable, no naked singularity exists at $r_{\mathrm{H}}=0$ in this case. Consequently, the singularity at $r=0$ is covered by extremal horizons. Hence, the possibility that the mass is less than the critical mass is neglected in our discussion.

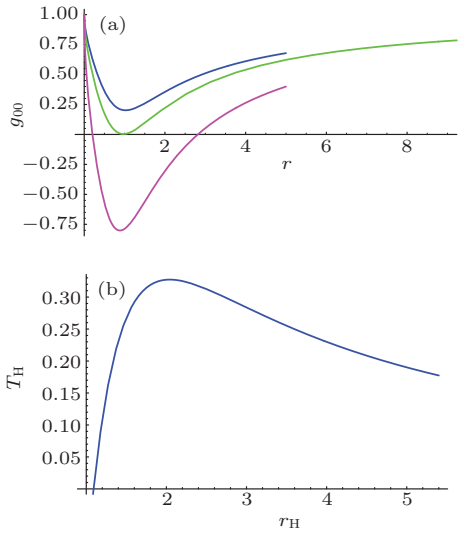


Fig. 1. (color online) (a) The value of g_{00} versus r with $q_0=1$ and $\sigma=1$ for two horizons ($m_0=1$, pink curve), extremal BH ($m_0=0.44$, green curve), and no horizon ($m_0=0.3$, blue curve). The intercepts on the horizontal axis label the radius of the event horizons. (b) The value of T_{H} versus r_{H} with $T_{\mathrm{H}}=0$ and $r_{\mathrm{H}}\backsimeq 1.1$. Namely, for the extremal BH, $T_{\mathrm{H}}\backsimeq 0.32$ corresponds to mass $m\backsimeq 0.6$.

The radial null geodesics from the singularity at $r = 0$ reach an observer in finite (naked singularity case) and infinite (one and two horizon cases) time. The equation of the radial null geodesics is

$$\mathrm{d}s^2 = \left(1 - \frac{2m(r,\sigma)}{r} - \frac{Q^2(r,\sigma)}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m(r,\sigma)}{r} - \frac{Q^2(r,\sigma)}{r}\right)^{-1}\mathrm{d}r^2 = 0, \quad (22)$$

which can be written as

$$t = \int_0^r \left[1 - \frac{2m(r,\sigma)}{r} - \frac{Q^2(r,\sigma)}{r}\right]^{-1}\mathrm{d}r. \quad (23)$$

For cases of one and two horizon(s), i.e., $m_0 = 0.44$ and $m_0 > 0.44$, the graph shows that the integral diverges. In both cases, we have $t = \infty$ for $r \neq \infty$. This implies that the light signals from singularity take an infinite time to reach the observer at $r \neq 0$.

4. Conclusion

In this paper, we formulated a new static, spherically symmetric charged solution of the Einstein–Maxwell field equations in the framework of NC geometry. This describes the final stage of the collapsing charged non-compact object. Our work extends the work of Castro[21] to the charge case, and his results can be recovered by taking the charged parameter $Q = 0$. The present solution is reduced to the classical RN solution by taking the NC parameter $\sigma \to 0$. It is found that the solution is asymptotically flat and not singular at the origin, while the Ricci scalar and matter density are singular. Hence, spacelike singularity appears in this case. The horizon analysis implies that for a suitable set of initial data and $m_0 < 0.44$, $m_0 = 0.44$, and $m_0 > 0.44$, there is the existence of no horizon, a single horizon, or two horizons, respectively. However, the thermodynamical results imply that the final stage of the object is extremal BH containing a singularity at $r = 0$.

In Ref. [21], the horizon equation is cubic polynomial, which admits three possible exact solutions that correspond to the cases of two horizons, one horizon, and no horizon. It was concluded on the basis of Hawking temperature (with the assumption that the given matter configuration has less mass than the critical mass) that a timelike naked singularity exists. In our case, we conclude that the spacelike singularities associated with the noncompact charged object are covered by an extremal BH horizon. There is curvature singularity at $r = 0$, while there is de-Sitter geometry around the origin[10] instead of curvature singularity.

References

[1] Padmanabhan T 2005 *Phys. Rep.* **406** 49
[2] Susskind L 1993 *Phys. Rev. Lett.* **71** 2367
[3] Witten E 1996 *Nucl. Phys. B* **460** 335
[4] Siberg N and Witten E 1999 *JHEP* **1999** 032
[5] Snyder S H 1947 *Phys. Rev.* **71** 38
[6] Zade S S, Patil K D and Mohod A D 2008 *Chin. Phys. Lett.* **25** 854
[7] Nashed G G L 2007 *Chin. Phys. Lett.* **24** 3059
[8] Doplicer S, Fredenhagen K and Robert J E 1995 *Commun. Math. Phys.* **172** 187
[9] Guang C 2001 *Chin. Phys.* **10** 787
[10] Nicolimi P, Smailagic A and Spallucci E 2006 *Phys. Lett. B* **632** 547
[11] Smailagic A and Spallucci E 2003 *J. Phys. A* **36** 467
[12] Doplicer S, Fredenhagen K and Robert J E 1995 *Commun. Math. Phys.* **172** 187
[13] Banerjee R, Majhi R B and Modak S K 2009 *Class. Quantum Grav.* **26** 085010
[14] Modesto L and Nicolini P 2010 *Phys. Rev. D* **82** 104035
[15] Bastos C, Bertolami O, Dias N C and Prata J N 2009 *Phys. Rev. D* **80** 124038
[16] Bastos C, Bertolami O, Dias N C and Prata J N 2011 *Phys. Rev. D* **84** 024005
[17] Bertolami O and Zaro C D A 2010 *Phys. Rev. D* **81** 025005
[18] Oh J J and Park C 2010 *JHEP* **1003** 86
[19] Sharif M and Abbas G 2012 *J. Phys. Soc. Jpn.* **81** 044002
[20] Sun W, Wang D, Xie N Q, Zhang R B and Zhang X 2010 *Eur. Phys. J. C* **69** 271
[21] Castro C 2008 *Phys. Lett. B* **665** 384
[22] Ansoldi S, Nicolimi P, Smailagic A and Spallucci E 2007 *Phys. Lett. B* **645** 261