Maximum confidence measurements via probabilistic quantum cloning*

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(Received 9 June 2012; revised manuscript received 30 July 2012)

Probabilistic quantum cloning (PQC) cannot copy a set of linearly dependent quantum states. In this paper, we show that if incorrect copies are allowed to be produced, linearly dependent quantum states may also be cloned by the PQC. By exploiting this kind of PQC to clone a special set of three linearly dependent quantum states, we derive the upper bound of the maximum confidence measure of a set. An explicit transformation of the maximum confidence measure is presented.

Keywords: probabilistic quantum cloning, quantum state discrimination, maximum confidence measurement

PACS: 03.67.−a, 03.65.−w **DOI:** 10.1088/1674-1056/22/3/030312

1. Introduction

Quantum state discrimination[1–3] lies at the heart of quantum cryptography.[4,5] Restrained by quantum mechanics, we are unable to discriminate nonorthogonal quantum states deterministically. In order to discriminate given quantum states, one method to detect them is to cause an error probability that we wish to minimize through measurement. This strategy is called minimal error discrimination (MED), and was first introduced by Helstrom via quantum hypothesis testing and is hence called the Helstrom theorem.[6] Another method is known as optimal unambiguous discrimination (OUD). In the second strategy, we do not want any error in measurement, and thus inconclusive measurement results will inevitably occur. The pioneering study on the OUD is performed under the IDP limit[7–9] with equal prior probabilities, and under the JS limit[10] with unequal prior probabilities. Many recent studies have been reported on MED,[11–18] and many interesting results have also been presented.[19–28]

Croke *et al.* considered a combination of MED and OUD, i.e., maximum confidence measurements (MCMs).[29] In the process of discriminating quantum states, both an error and an inconclusive result may be induced. If suitable outcomes are obtained, then the probability of the confidence measurements is maximal. If an inconclusive result occurs, the discrimination is then invalid. For a special set of linearly dependent quantum states, an MCM experiment was reported,[30] and the maximum confidence in the measurement result is 2/3. Subsequently, the MCM was generalized to two mixed quantum states[31] with an experimental realization,[32] and to symmetric qudit states.[33] Generally, state discrimination is analyzed by using positive-operator valued measurement (POVM). One of the main tasks of state discrimination is to find suitable POVM elements and then implement them in experiments. In Ref. [34], the authors used the no-signaling condition[35] to rederive the bound of the MCM.

Probabilistic quantum cloning (PQC)[36] has an intimate relation to state discrimination. PQC cannot copy a set of linearly dependent quantum states, only linearly independent quantum states. If incorrect copies are allowed to be produced, however, linearly dependent quantum states may also be cloned by PQC. In this paper, we show the existence of this kind of PQC. We first present a formalism of $1 \to M$ PQC. If the measured result is inconclusive, then the PQC fails. If the PQC is successful, we can measure M copies and learn part of the information about the input states. In the limit $M \to \infty$, the measurement of the collapsed states can tell us their complete information. For a special set of three linearly dependent quantum states, we first design optimal $1 \to M$ PQC. Based on optimal $1 \to M$ PQC as $M \to \infty$, we present a formulation of MCM with a maximal measurement confidence probability, which is coincident with a previous contribution.[29] In the meantime, an explicit unitary transformation of MCM is derived, which will facilitate the realization of MCM. In Refs. [29] and [33], the authors took two steps to complete the POVM, i.e., the measurement of the MCM. In the first step, if an inconclusive result is obtained, the measurement fails. If the measurement is successful, then the second step is needed to gain further information about the input state from the collapsed state. In our paper, the MCM derived from the PQC has two distinct advantages. The first is that the interpretation

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11074002, 61073048, and 11104057), the Natural Science Foundation of the Education Department of Anhui Province, China (Grant Nos. KJ2010ZD08 and KJ2012A245), and the Postgraduate Program of Huainan Normal University of China.

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of MCM derived from the PQC is very clear and simple, and the second is that the realization of MCM based on an explicit unitary transformation only requires one step.

The paper is organized as follows. In Section 2, we briefly review some of the contributions, including MCM and PQC. In Section 3, we first present a formulation of $1 \to M$ PQC, and then derive the optimal PQC. Based on optimal PQC, we give a formulation of MCM. For a special set of three linearly dependent quantum states, an explicit unitary transformation is presented.

2. The MCM and PQC

In this section, we briefly review some of the contributions, i.e., MCM and PQC. In Ref. [29], the authors ascribed the discrimination of three linearly dependent quantum states to the MCM, namely

$$\begin{aligned} |\psi_0\rangle &= \cos\theta|0\rangle + \sin\theta|1\rangle, \\ |\psi_1\rangle &= \cos\theta|0\rangle + \mathrm{e}^{2\pi\mathrm{i}/3}\sin\theta|1\rangle, \\ |\psi_2\rangle &= \cos\theta|0\rangle + \sin\theta\,\mathrm{e}^{4\pi\mathrm{i}/3}|1\rangle, \end{aligned} \tag{1}$$

where $\theta \in [0, \pi/4]$, with a prior probability $p_i = 1/3$ $(i = 0,1,2)$. In order to distinguish them, the MCM allows the concurrence of an error and an inconclusive result. The POVM elements are defined as $\Pi_i = (3\cos^2\theta)^{-1}|\phi_0\rangle\langle\phi_0|$ (i=0,1,2), where

$$\begin{aligned} |\phi_0\rangle &= \sin\theta|0\rangle + \cos\theta|1\rangle, \\ |\phi_1\rangle &= \sin\theta|0\rangle + \mathrm{e}^{2\pi\mathrm{i}/3}\cos\theta|1\rangle, \\ |\phi_2\rangle &= \sin\theta|0\rangle + \mathrm{e}^{4\pi\mathrm{i}/3}\cos\theta|1\rangle, \end{aligned} \tag{2}$$

and the POVM element corresponding to the inconclusive outcome is $\Pi_? = (1-\tan^2\theta)|0\rangle\langle 0|$. If the POVM element Π_i is measured to produce outcome ω_i $(i = 0,1,2)$, the input state is not deterministic: $\rho_i = |\psi_i\rangle\langle\psi_i|$ $(i = 0,1,2)$, since an error may be produced. The maximum probability of MCM is

$$P_i^{(\mathrm{MCM})}(\rho_i|\omega_i) = \frac{2}{3}. \tag{3}$$

For the POVM element $\Pi_?$, the inconclusive probability is given by

$$P(?) = \cos 2\theta. \tag{4}$$

In order to implement the POVM elements Π_i, two steps are needed.[29,33] In the first step, the outcomes may be $\omega_{\mathrm{succ}}^{(1)}$ and ω_{fail}. If the outcome ω_{fail} is obtained, no further measurement is made, and the result may be considered to be inconclusive. If the outcome $\omega_{\mathrm{succ}}^{(1)}$ is obtained, the second step is performed on the collapsed state of the outcome $\omega_{\mathrm{succ}}^{(1)}$ to gain knowledge of the input state. In Section 3, we will derive an explicit unitary transformation of MCM. Our result shows that one step is sufficient for the implementation of MCM.

For the PQC, a certain set $\{|\zeta_m\rangle\}_{m=0}^{d-1}$ constructed in terms of linearly independent quantum state $|\zeta_m\rangle$ is used to obtain clones that are perfect with nonzero probability. Optimal $1 \to 2$ PQC in d-dimensional space is given by[36]

$$\begin{aligned} U(|\zeta_m\rangle|\Sigma\rangle|p_0\rangle) &= \sqrt{\gamma_m}|\zeta_m\rangle^{\otimes 2}|p_0\rangle + \sum_{n=1}^{d} c_{mn}|\Phi_{\mathrm{AB}}^{(n)}\rangle|p_n\rangle, \\ &\quad (m = 0,1,\ldots,d-1), \end{aligned} \tag{5}$$

where $|p_0\rangle$, $|p_1\rangle,\ldots,|p_d\rangle$ are orthonormal states of the probe P, and $|\Phi_{\mathrm{AB}}^{(n)}\rangle$ are the normalized states of the composite system AB. After the evolution, the probability of obtaining two perfect clones $|\zeta_m\rangle^{\otimes 2}$ is γ_m if $|p_0\rangle$ is detected. If $|p_j\rangle$ is detected, the PQC then fails. In the case of two dimensions, optimal probability γ_0 and γ_1 are given by[36]

$$\frac{\gamma_0 + \gamma_1}{2} \le \frac{1 - |\langle\zeta_0|\zeta_1\rangle|}{1 - |\langle\zeta_0|\zeta_1\rangle|^2}. \tag{6}$$

The equality holds if and only if $\gamma_0 = \gamma_1$. For the PQC, only linearly independent quantum states can be cloned. If incorrect clones are allowed to occur, however, we show that linearly dependent quantum states may also be cloned. In the next section, we will use this result to derive the upper bound of MCM.

3. The MCM via the PQC

In this section, we first design the formulation of $1 \to M$ PQC to clone three linearly dependent quantum states given by Eq. (1). If the PQC is successful, the measurement of M clones will give complete information about the collapsed states, i.e., the quantum hypothesis testing. Based on the PQC, we then present the MCM, along with an explicit unitary transformation for linearly dependent quantum states $|\psi_i\rangle$ $(i = 0,1,2)$ given by Eq. (1).

We design $1 \to M$ PQC as follows:

$$\begin{aligned} |\psi_0\rangle|0\rangle^{\otimes(M-1)}|0\rangle_{\mathrm{d}} &\to p_{00}|\psi_0\rangle^{\otimes M}|0\rangle_{\mathrm{d}} + p_{01}|\psi_1\rangle^{\otimes M}|1\rangle_{\mathrm{d}} \\ &\quad + p_{02}|\psi_2\rangle^{\otimes M}|2\rangle_{\mathrm{d}} + p_{03}|0\rangle^{\otimes M}|3\rangle_{\mathrm{d}} \\ &= |\Psi_0\rangle, \\ |\psi_1\rangle|0\rangle^{\otimes(M-1)}|0\rangle_{\mathrm{d}} &\to p_{10}|\psi_0\rangle^{\otimes M}|0\rangle_{\mathrm{d}} + p_{11}|\psi_1\rangle^{\otimes M}|0\rangle_{\mathrm{d}} \\ &\quad + p_{12}|\psi_2\rangle^{\otimes M}|0\rangle_{\mathrm{d}} + p_{13}|0\rangle^{\otimes M}|3\rangle_{\mathrm{d}} \\ &= |\Psi_1\rangle, \\ |\psi_2\rangle|0\rangle^{\otimes(M-1)}|0\rangle_{\mathrm{d}} &\to p_{20}|\psi_0\rangle^{\otimes M}|0\rangle_{\mathrm{d}} + p_{21}|\psi_1\rangle^{\otimes M}|1\rangle_{\mathrm{d}} \\ &\quad + p_{22}|\psi_2\rangle^{\otimes M}|2\rangle_{\mathrm{d}} + p_{23}|0\rangle^{\otimes M}|3\rangle_{\mathrm{d}} \\ &= |\Psi_2\rangle, \end{aligned} \tag{7}$$

where the coefficients p_{ik} $(i = 0,1,2;\ k = 0,1,2,3)$ are complex members, and $|i\rangle_{\mathrm{d}}$ is the detected qubit. If $|3\rangle_{\mathrm{d}}$ is detected, then the PQC fails, which gives an inconclusive probability $|p_{i3}|^2$ for measuring the input state $|\psi_i\rangle$. If $|i\rangle_{\mathrm{d}}$ is detected, we can measure the collapsed states $|\psi_i\rangle^{\otimes M}$, and infer that the input state may be $|\psi_i\rangle$. However, the PQC can produce incorrect copies. From the first one of the set of equations (7), for example, we measure the collapsed state $|\psi_0\rangle^{\otimes M}$

with success probability $|p_{00}|^2$ and assume the initial state to be $|\psi_0\rangle$. This is a correct result. However, we measure the collapsed state $|\psi_1\rangle^{\otimes M}$ with a success probability $|p_{01}|^2$, and naturally assume the initial state to $|\psi_1\rangle$. This is an error. Thus, when the input state is $|\psi_0\rangle$, the confidence probability of MCM is

$$p_0^{(\mathrm{MCM})} = |p_{00}|^2 \Big/ \sum_{j=0}^{2} |p_{0j}|^2,$$

which should be maximized. Therefore, in the first one of the set of equations (7), the collapsed states $|\psi_1\rangle^{\otimes M}$ and $|\psi_2\rangle^{\otimes M}$ are incorrect clones. This is because that three input states are linearly dependent. In order to evaluate the bound, $P_i^{(\mathrm{MCM})}$, we should maximize the probability

$$p_i^{(\mathrm{MCM})} = |p_{ii}|^2 / \sum_{j=0}^{2} |p_{ij}|^2,$$

i.e.,

$$P_i^{(\mathrm{MCM})} = \mathrm{Max}[p_i^{(\mathrm{MCM})}]. \tag{8}$$

From Eq. (7), we can give some constraints. The unitarity of the transformation gives

$$\sum_{k=0}^{3} |p_{ik}|^2 = 1, \ (i = 0,1,2), \tag{9}$$

their inner products yield

$$\langle \Psi_i | \Psi_j \rangle = \langle \psi_i | \psi_j \rangle, \ (i,j = 0,1,2), \tag{10}$$

and the linear dependence requires

$$\begin{aligned} & U(|\psi_0\rangle + \mathrm{e}^{2\pi \mathrm{i}3}|\psi_1\rangle + \mathrm{e}^{4\pi \mathrm{i}/3}|\psi_2\rangle) \\ = \ & |\Psi_0\rangle + \mathrm{e}^{2\pi \mathrm{i}/3}|\Psi_1\rangle + \mathrm{e}^{4\pi \mathrm{i}/3}|\Psi_2\rangle \\ = \ & 0. \end{aligned} \tag{11}$$

We assume that a prior probability is equiprobable, and the probabilities of MCM and an inconclusive result should satisfy

$$p_0^{(\mathrm{MCM})} = p_1^{(\mathrm{MCM})} = p_2^{(\mathrm{MCM})}, \quad |p_{03}| = |p_{13}| = |p_{23}|. \tag{12}$$

Our task is to maximize the value of Eq. (8) under conditions given by Eqs. (9)–(12). After some derivation, we obtain the coefficients, and then the PQC is defined as

$$\begin{aligned}
|\psi_0\rangle|0\rangle^{\otimes(M-1)}|0\rangle_{\mathrm{d}} &\to \sqrt{2\sin^2\theta}\left(\sqrt{\frac{2}{3}}|\psi_0\rangle^{\otimes M}|0\rangle_{\mathrm{d}} + \sqrt{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\pi/3}|\psi_1\rangle^{\otimes M}|1\rangle_{\mathrm{d}} + \sqrt{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\pi/3}|\psi_2\rangle^{\otimes M}|2\rangle_{\mathrm{d}}\right) + \sqrt{\cos 2\theta}|0\rangle^{\otimes M}|3\rangle_{\mathrm{d}}, \\
|\psi_1\rangle|0\rangle^{\otimes(M-1)}|0\rangle_{\mathrm{d}} &\to \sqrt{2\sin^2\theta}\left(\sqrt{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\pi/3}|\psi_0\rangle^{\otimes M}|0\rangle_{\mathrm{d}} + \sqrt{\frac{2}{3}}|\psi_1\rangle^{\otimes M}|1\rangle_{\mathrm{d}} + \sqrt{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\pi/3}|\psi_2\rangle^{\otimes M}|2\rangle_{\mathrm{d}}\right) + \sqrt{\cos 2\theta}|0\rangle^{\otimes M}|3\rangle_{\mathrm{d}}, \\
|\psi_2\rangle|0\rangle^{\otimes(M-1)}|0\rangle_{\mathrm{d}} &\to \sqrt{2\sin^2\theta}\left(\sqrt{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\pi/3}|\psi_0\rangle^{\otimes M}|0\rangle_{\mathrm{d}} + \sqrt{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\pi/3}|\psi_1\rangle^{\otimes M}|1\rangle_{\mathrm{d}} + \sqrt{\frac{2}{3}}|\psi_2\rangle^{\otimes M}|2\rangle_{\mathrm{d}}\right) + \sqrt{\cos 2\theta}|0\rangle^{\otimes M}|3\rangle_{\mathrm{d}}.
\end{aligned} \tag{13}$$

From Eq. (13), one can observe that the upper bound of MCM is $P_i^{(\mathrm{MCM})} = 2/3$, and the probability of an inconclusive is $P(?) = \cos 2\theta$, which is coincident with Ref. [29].

Based on Eq. (13), we can obtain a formulation of MCM as

$$\begin{aligned}
|\psi_0\rangle|0\rangle_{\mathrm{d}} &\to \sqrt{2\sin^2\theta}\left(\sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\pi/3}|01\rangle + \sqrt{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\pi/3}|10\rangle\right) + \sqrt{\cos 2\theta}|11\rangle, \\
|\psi_1\rangle|0\rangle_{\mathrm{d}} &\to \sqrt{2\sin^2\theta}\left(\sqrt{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\pi/3}|00\rangle + \sqrt{\frac{2}{3}}|01\rangle + \sqrt{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\pi/3}|10\rangle\right) + \sqrt{\cos 2\theta}|11\rangle, \\
|\psi_2\rangle|0\rangle_{\mathrm{d}} &\to \sqrt{2\sin^2\theta}\left(\sqrt{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\pi/3}|00\rangle + \sqrt{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\pi/3}|01\rangle + \sqrt{\frac{2}{3}}|10\rangle\right) + \sqrt{\cos 2\theta}|11\rangle.
\end{aligned} \tag{14}$$

Note that this transformation satisfies the normalization condition, the inner product condition, and the linear dependence condition. The interpretation of Eq. (14) is very clear and simple. After applying this transformation on the input state $|\psi_i\rangle$ with an ancillary state$|0\rangle_{\mathrm{d}}$, we measure four computational states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. If the outcome is $|11\rangle$ with the probability $P(?) = \cos 2\theta$, then the MCM fails. If the outcome is $|00\rangle$, we can infer that the input state is $|\psi_0\rangle$ with the MCM probability $P_0^{(\mathrm{MCM})} = 2/3$. The cases are the same for the outcomes $|01\rangle$ and $|10\rangle$.

Inserting the explicit states $|\psi_i\rangle$ $(i = 0,1,2)$ into Eq. (14), it is very easy to obtain an explicit transformation

$$|0\rangle|0\rangle_{\mathrm{d}} \to \frac{\tan\theta}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) + \sqrt{1-\tan^2\theta}|11\rangle,$$

$$|1\rangle|0\rangle_{\mathrm{d}} \to \frac{1}{\sqrt{3}}(|00\rangle + \mathrm{e}^{4\pi \mathrm{i}/3}|01\rangle + \mathrm{e}^{2\pi \mathrm{i}/3}|10\rangle). \tag{15}$$

This is a unitary transformation. From the input to the output, the implementation of MCM needs only one step.

We should point out that the condition $M \to \infty$ is important. Only by measuring the infinite identical copies of a quantum state can the complete information can be obtained. The role of $1 \to M$ PQC is quantum hypothesis testing as $M \to \infty$. In our derivation of PQC, we did not prove the optimality, the upper bound of MCM can demonstrate it.

4. Conclusion

In this paper, we show that linearly dependent quantum states may be cloned by PQC, and present a formulation of PQC. By exploiting PQC as the quantum hypothesis testing, we give a formulation of MCM for a special set of three linearly dependent quantum states. The upper bound coincides with existing results. We give an explicit unitary transformation of a set.

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