RAPID COMMUNICATION

Average position in quantum walks with a U(2) coin*

Li Min(李 敏), Zhang Yong-Sheng(张永生)†, and Guo Guang-Can(郭光灿)

Key Laboratory of Quantum Information, University of Science and Technology of China, Chinese Academy of Sciences, Hefei 230026, *China*

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We investigated discrete-time quantum walks with an arbitary unitary coin. Here we discover that the average position $\langle x\rangle = \max(\langle x\rangle)\sin(\alpha+\gamma)$, while the initial state is $1/\sqrt{2}(|0L\rangle + \mathrm{i}|0R\rangle)$. We verify the result, and obtain some symmetry properties of quantum walks with a U(2) coin with $|0L\rangle$ and $|0R\rangle$ as the initial state.

Keywords: quantum random walk, U(2) coin, average position

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1. Introduction

Quantum walks (QWs) were first introduced in 1993[1] as a generalization of classical random walks. According to the time evolution, QWs can be divided into discrete-time and continuous-time[2] QWs. Recently, both continuous-time[3] and discrete-time[4] QWs are found to be universal for quantum computation. A number of quantum algorithms based on QWs have already been proposed in Refs. [5]– [10]. In addition, QWs in a graph,[11] on a line with a moving boundary,[12] with multiple coins[13] or decoherent coins[14] have also been discussed .

QWs using a SU(2) coin was introduced by Chandrashekar *et al.*,[15] where the standard deviation and measurement entropy properties were discussed. Here we discuss the symmetry and average position properties for the QWs with a U(2) coin.

2. Generalized discrete time quantum walks

In this paper, we discuss within the discrete-time QWs . The total Hilbert space for QWs is given by $\mathscr{H} \equiv \mathscr{H}_{\mathrm{P}} \otimes \mathscr{H}_{\mathrm{C}}$, where $\mathscr{H}_{\mathrm{P}}$ is spanned by the orthonormal states $\{|x\rangle\}$ and $\mathscr{H}_{\mathrm{C}}$ is the two-dimensional coin space spanned by two orthonormal states $|L\rangle$ and $|R\rangle$.

Each step of the QWs can be split into two operations: the evolution of the coin state and the particle movement according to the coin state.

Here for generalization, the coin is evolved by applying a U(2) operation:

$$\boldsymbol{U}_{\alpha,\beta,\gamma,\theta} = \mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha}\cos\beta, & -\mathrm{e}^{-\mathrm{i}\gamma}\sin\beta \\ \mathrm{e}^{\mathrm{i}\gamma}\sin\beta, & \mathrm{e}^{-\mathrm{i}\alpha}\cos\beta \end{pmatrix}. \tag{1}$$

For example, the Hadamard operator can be described as $\mathscr{H} = U_{\pi/2,\pi/4,\pi/2,-\pi/2}$. The particle movement operator is given by

$$S = \mathrm{e}^{\mathrm{i}p\sigma_z} = \sum_x S_x, \tag{2}$$

where p is the momentum operator, and σ_z is the Pauli-z operator,

$$\hat{S}_x = |x+1\rangle\langle x| \otimes |R\rangle\langle R| + |x-1\rangle\langle x| \otimes |L\rangle\langle L|. \tag{3}$$

Then we can obtain the generalized QWs.[15] After t steps of QWs,

$$|\Psi_t\rangle = \left[S(\boldsymbol{I}_{\mathrm{P}} \otimes \boldsymbol{U}_{\alpha,\beta,\gamma,\theta})\right]^t |\Psi_0\rangle, \tag{4}$$

where $|\Psi_0\rangle$ is the initial state of the system.

We can easily obtain the following propositions.

Lemma 1 The quantum walks have the same probability distribution with a U(2) coin or an SU(2) coin which has the same parameters α, β, and γ in the U(2) martrix.

Corollary 1 The average position is the same in quantum walks with a U(2) coin and an SU(2) coin if the two coins have the same parameters α, β, and γ.

According to Corollary 1, if we want to know the average position property in QWs with an arbitrary unitary operator, we only need to study the quantum walk with an SU(2) coin instead. For the rest of this paper, we always use the SU(2) coin which can be written as

$$U^S_{\alpha,\beta,\gamma} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha}\cos\beta, & -\mathrm{e}^{-\mathrm{i}\gamma}\sin\beta \\ \mathrm{e}^{\mathrm{i}\gamma}\sin\beta, & \mathrm{e}^{-\mathrm{i}\alpha}\cos\beta \end{pmatrix}. \tag{5}$$

Based on the analysis in Ref. [16], the state after t steps of QWs with an SU(2) coin

$$|\Psi_t\rangle = \left[S(\boldsymbol{I}_{\mathrm{P}} \otimes \boldsymbol{U}^S_{\alpha,\beta,\gamma})\right]^t |\Psi_0\rangle. \tag{6}$$

The spatial Fourier transformation of the wave function $\Psi(x,t)$ over $\mathscr{Z}$ is given by

$$\tilde{\Psi}(k,t) = \sum_{x=-\infty}^{\infty} \Psi(x,t)\,\mathrm{e}^{\mathrm{i}kx}, \tag{7}$$

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†Corresponding author. E-mail: yshzhang@ustc.edu.cn

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where $k \in [-\pi, \pi]$. We can obtain

$$\tilde{\Psi}(k,t) = (M_k)^t \tilde{\Psi}(k,0), \tag{8}$$

where

$$\begin{aligned} M_k &= \mathrm{e}^{\mathrm{i}k} M_+ + \mathrm{e}^{-\mathrm{i}k} M_- \\ &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(k-\alpha)} \cos\beta, & -\mathrm{e}^{-\mathrm{i}(k+\gamma)} \sin\beta \\ \mathrm{e}^{\mathrm{i}(k+\gamma)} \sin\beta, & \mathrm{e}^{\mathrm{i}(k-\alpha)} \cos\beta \end{pmatrix}. \end{aligned} \tag{9}$$

The eigenvalues of M_k are

$$\begin{cases} \lambda_a = \mathrm{e}^{-\mathrm{i}\omega}, \\ \lambda_b = \mathrm{e}^{\mathrm{i}\omega}, \end{cases} \tag{10}$$

where $\cos\omega = \cos(k-\alpha)\cos\beta$. The eigenstates are

$$\begin{cases} \tilde{\Psi}_k^a &= \dfrac{1}{C_k^a} \begin{pmatrix} P_k \\ Q_k^a \end{pmatrix}, \\ \tilde{\Psi}_k^b &= \dfrac{1}{C_k^b} \begin{pmatrix} P_k \\ Q_k^b \end{pmatrix}, \end{cases} \tag{11}$$

where

$$\begin{cases} P_k = -\mathrm{e}^{-\mathrm{i}(k+\gamma)} \sin\beta, \\ Q_k^a = -\mathrm{i}\sin\omega_k + \mathrm{i}\sin(k-\alpha)\cos\beta, \\ Q_k^b = \mathrm{i}\sin\omega_k + \mathrm{i}\sin(k-\alpha)\cos\beta, \end{cases} \tag{12}$$

$$\begin{aligned} C_k^a &= \sqrt{P_k^* P_k + (Q_k^a)^* Q_k^a} \\ &= \sqrt{2(\sin^2\omega_k - \cos\beta\sin(k-\alpha)\sin\omega_k)}, \end{aligned} \tag{13}$$

$$\begin{aligned} C_k^b &= \sqrt{P_k^* P_k + (Q_k^b)^* Q_k^b} \\ &= \sqrt{2(\sin^2\omega_k + \cos\beta\sin(k-\alpha)\sin\omega_k)}. \end{aligned} \tag{14}$$

3. Average position in quantum walks

Figures 1 and 2 show the average position after 100 steps of QWs with a SU(2) coin in the case of $\beta = \pi/6$, while the initial state is $(1/\sqrt{2})(|0L\rangle + \mathrm{i}|0R\rangle)$. From Fig. 1, we can see that $\langle x\rangle$ only depends on the sum of α and γ, and we can then control the QWs by choosing the coin, which was discussed in Ref. [17]. Figure 2 shows that the actual $\langle x\rangle$ exactly matches the function of $f(\phi) = \max(\langle x\rangle)\sin(\phi)$, then we conject that $\langle x\rangle = G(\beta,t)\sin(\alpha+\gamma)$.

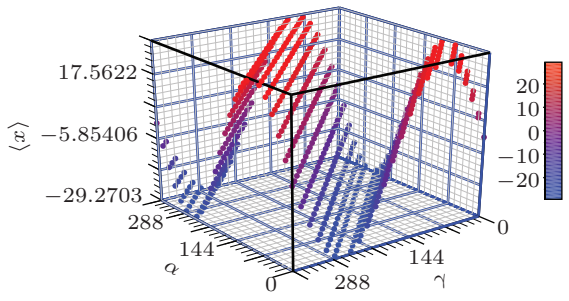


Fig. 1. (color online) Average position of the $\langle x\rangle$ for quantum walks after $t = 100$ steps with a SU(2) coin, where $\beta = \pi/6$.

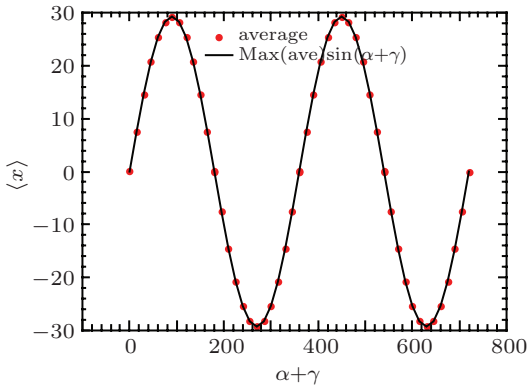


Fig. 2. (color online) The average position with $\alpha+\gamma$ after 100 steps of a quantum walk when $\beta = \pi/6$ (red dot), and $f(\phi) = \max(\langle x\rangle)\sin(\phi)$ (line).

4. Proof in methematics

Theorem 1 The probability distribution for quantum walks with an SU(2) coin is independent of the parameters α and γ, when the initial state is $|0L\rangle$ or $|0R\rangle$. Namely,

$$\begin{cases} P_{|0L\rangle}(\alpha,\beta,\gamma,x,t) = P_{|0L\rangle}(\beta,x,t), \\ P_{|0R\rangle}(\alpha,\beta,\gamma,x,t) = P_{|0R\rangle}(\beta,x,t), \end{cases} \tag{15}$$

for any α and γ.

Proof If the initial state $|\Psi_0\rangle = |0L\rangle$, then

$$\tilde{\Psi}(k,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The probability of $|x\rangle$ is given by

$$\begin{aligned} & P_{|0L\rangle}(\alpha,\beta,\gamma,x,t) \\ &= P_L(\alpha,\beta,\gamma,x,t) + P_R(\alpha,\beta,\gamma,x,t) \\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g_L(k_1,k_2,\alpha,\beta,\gamma,x,t)\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ &\quad + \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g_R(k_1,k_2,\alpha,\beta,\gamma,x,t)\,\mathrm{d}k_1\,\mathrm{d}k_2, \end{aligned} \tag{16}$$

where

$$g_j(k_1,k_2,\alpha,\beta,\gamma,x,t) = \tilde{\Psi}_j^*(k_1,t)\tilde{\Psi}_j(k_2,t)\,\mathrm{e}^{\mathrm{i}(k_1-k_2)x}, \quad (j \in \{L,R\}).$$

If we set $h = k - \alpha$, then we have

$$\begin{aligned} & g_j(k_1,k_2,\alpha,\beta,\gamma,x,t) = g_j(h_1,h_2,\beta,x,t), \\ & g_j(h_1,h_2,\beta,x,t) = g_j(h_1 \pm 2\pi, h_2, \beta, x, t) \\ & = g_j(h_1, h_2 \pm 2\pi, \beta, x, t), \end{aligned}$$

and

$$\begin{aligned} & P_j(\alpha,\beta,\gamma,x,t) \\ &= \frac{1}{4\pi^2}\int_{-\pi-\alpha}^{\pi-\alpha}\int_{-\pi-\alpha}^{\pi-\alpha} g_j(h_1,h_2,\beta,xt)\,\mathrm{d}h_1\,\mathrm{d}h_2 \\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g_j(h_1,h_2,\beta,x,t)\,\mathrm{d}h_1\,\mathrm{d}h_2 \end{aligned}$$

$$= P_j(\beta,x,t). \tag{17}$$

Furthermore, we can know $P_{|0L\rangle}(\alpha,\beta,\gamma,x,t) = P_{|0L\rangle}(\beta,x,t)$. In the same way, we can also obtain $P_{|0R\rangle}(\alpha,\beta,\gamma,x,t) = P_{|0R\rangle}(\beta,x,t)$.

Theorem 2 After t steps, we have

$$\begin{cases} \Psi^{\mathrm{R}}_{|0L\rangle}(x,t) + \Psi^{\mathrm{L}}_{|0R\rangle}(-x,t) \in \mathrm{i}\mathbb{R}, \\ \Psi^{\mathrm{R}}_{|0L\rangle}(x,t) - \Psi^{\mathrm{L}}_{|0R\rangle}(-x,t) \in \mathbb{R}, \end{cases} \tag{18}$$

$$\begin{cases} \Psi^{\mathrm{L}}_{|0L\rangle}(x,t) + \Psi^{\mathrm{R}}_{|0R\rangle}(-x,t) \in \mathbb{R}, \\ \Psi^{\mathrm{L}}_{|0L\rangle}(x,t) - \Psi^{\mathrm{R}}_{|0R\rangle}(-x,t) \in \mathrm{i}\mathbb{R}, \end{cases} \tag{19}$$

where $\Psi^{m}_{|n\rangle}(x,t)$ denotes the coefficient of the $|x\rangle|m\rangle$ state after t steps of quantum walks with the initial state $|n\rangle$.

Proof

$$\begin{aligned} &\Psi^{\mathrm{R}}_{|0L\rangle}(x,t) \pm \Psi^{\mathrm{L}}_{|0R\rangle}(-x,t) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\Psi}^{\mathrm{R}}_{|0L\rangle}(k,t)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \\ &\quad \pm \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\Psi}^{\mathrm{L}}_{|0R\rangle}(k,t)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Q^a_k}{(C^a_k)^2}[P^*_k\,\mathrm{e}^{-\mathrm{i}kx} \mp P_k\,\mathrm{e}^{\mathrm{i}kx}] \\ &\quad \times [\mathrm{e}^{-\mathrm{i}\omega_k t} - \mathrm{e}^{\mathrm{i}\omega_k t}]\,\mathrm{d}k. \end{aligned} \tag{20}$$

As $Q^a_k \in \mathrm{i}\mathbb{R}$, we have

$$\begin{aligned} &\Psi^{\mathrm{R}}_{|0L\rangle}(x,t) + \Psi^{\mathrm{L}}_{|0R\rangle}(-x,t) \in \mathrm{i}\mathbb{R}, \\ &\Psi^{\mathrm{R}}_{|0L\rangle}(x,t) - \Psi^{\mathrm{L}}_{|0R\rangle}(-x,t) \in \mathbb{R}. \end{aligned}$$

Similarly, we can derive Eq. (19).

Corollary 2 The symmetry properties of distribution between quantum walks with a U(2) coin in the initial state $|0L\rangle$ and $|0R\rangle$: for an arbitrary t,

$$P^{\mathrm{R}}_{|0L\rangle}(\beta,x,t) = P^{\mathrm{L}}_{|0R\rangle}(\beta,-x,t),$$

$$P^{\mathrm{L}}_{|0L\rangle}(\beta,x,t) = P^{\mathrm{R}}_{|0R\rangle}(\beta,-x,t).$$

Proof We set $\Psi^{\mathrm{R}}_{|0L\rangle}(x) = C + \mathrm{i}D$, where $C, D \in \mathbb{R}$. From Theorem 2, we can know

$$\begin{aligned} &\Psi^{\mathrm{L}}_{|0R\rangle}(-x) = -C + \mathrm{i}D, \\ &P^{\mathrm{R}}_{|0L\rangle}(\beta,x,t) = P^{\mathrm{L}}_{|0R\rangle}(\beta,-x,t). \end{aligned}$$

Similarly, we can also obtain $P^{\mathrm{L}}_{|0L\rangle}(\beta,x,t) = P^{\mathrm{R}}_{|0R\rangle}(\beta,-x,t)$.

Theorem 3 If the initial state $|\Psi_0\rangle = m|0L\rangle + n|0R\rangle$, where $|m|^2 + |n|^2 = 1$, the probability at state $|xL\rangle$ or $|xR\rangle$ after t steps of a quantum walk is

$$\begin{cases} P^{\mathrm{L}}(x) = |m|^2 P^{\mathrm{L}}_{|0L\rangle} + |n|^2 P^{\mathrm{L}}_{|0R\rangle} \\ \qquad - (\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)} m^* n + \mathrm{e}^{\mathrm{i}(\alpha+\gamma)} m n^*) G^{\mathrm{L}}(\beta,x,t), \\ P^{\mathrm{R}}(x) = |m|^2 P^{\mathrm{R}}_{|0L\rangle} + |n|^2 P^{\mathrm{R}}_{|0R\rangle} \\ \qquad - (\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)} m^* n + \mathrm{e}^{\mathrm{i}(\alpha+\gamma)} m n^*) G^{\mathrm{R}}(\beta,x,t), \end{cases} \tag{21}$$

where G^{L} and G^{R} are independent of α and γ.

Proof The probability at state $|xL\rangle$ after t steps is given by

$$\begin{aligned} &P^{\mathrm{L}}_{m|0L\rangle + n|0R\rangle}(x) \\ &= \frac{1}{4\pi^2}\int\int \tilde{\Psi}^*_L(k_1,t)\tilde{\Psi}_L(k_2,t)\,\mathrm{e}^{\mathrm{i}(k_1-k_2)x}\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ &= |m|^2 P^{\mathrm{L}}_{0L}(\beta,x,t) + |n|^2 P^{\mathrm{L}}_{0R}(\beta,x,t) \\ &\quad - \left(\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)} m^* n \sum_{i=1}^{4} G_i + \mathrm{e}^{\mathrm{i}(\alpha+\gamma)} m n^* \sum_{i=1}^{4} G^*_i\right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} &G_1(\beta,x,t) \\ &= \frac{1}{4\pi^2}\int\int \mathrm{e}^{\mathrm{i}(\omega_{h_1}-\omega_{h_2})t} \frac{1}{(C^b_{h_1}C^b_{h_2})^2} \\ &\quad \times \mathrm{e}^{\mathrm{i}(h_1-h_2)} \sin^3\beta\,\mathrm{e}^{\mathrm{i}(h_1-h_2)x}\,\mathrm{e}^{-\mathrm{i}h_1}(Q^b_{h_2})^*\,\mathrm{d}h_1\,\mathrm{d}h_2 \\ &= \frac{1}{4\pi^2}\int\int \frac{(Q^b_{h_2})^* \sin^3\beta}{(C^b_{h_1}C^b_{h_2})^2}\,\mathrm{i}\sin[(\omega_{h_1}-\omega_{h_2}) \\ &\quad + (h_1-h_2) + (h_1-h_2)x - h_1]\,\mathrm{d}h_1\,\mathrm{d}h_2 \in \mathbb{R}. \end{aligned} \tag{23}$$

Similar to $G_1(\beta,x,t)$, we know $G_i(\beta,x,t) \in \mathbb{R}$, where $i \in \{1,2,3,4\}$. Thus, equation (22) can be written as

$$\begin{aligned} P^{\mathrm{L}}(x) &= |m|^2 P^{\mathrm{L}}_{|0L\rangle} + |n|^2 P^{\mathrm{L}}_{|0R\rangle} \\ &\quad - (\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)} m^* n + \mathrm{e}^{\mathrm{i}(\alpha+\gamma)} m n^*) G^{\mathrm{L}}(\beta,x,t), \end{aligned} \tag{24}$$

where $G^{\mathrm{L}}(\beta,x,t) = \sum_{i=1}^{4} G_i$. In the same way of deriving $P^{\mathrm{L}}(x)$, we can obtain $P^{\mathrm{R}}(x)$ in Eq. (21).

Theorem 4 If the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + \mathrm{i}|0R\rangle)$, the average position after t steps of a quantum walk is given by

$$\langle x\rangle = G(\beta,t)\sin(\alpha+\gamma),$$

where $G(\beta,t)$ only depends on β and t.

Proof From Corollary 2, we can know

$$\begin{cases} \sum_{x=-t}^{t} x[P^{\mathrm{R}}_{|0L\rangle}(\beta,x,t) + P^{\mathrm{L}}_{|0R\rangle}(\beta,x,t)] = 0, \\ \sum_{x=-t}^{t} x[P^{\mathrm{L}}_{|0L\rangle}(\beta,x,t) + P^{\mathrm{R}}_{|0R\rangle}(\beta,x,t)] = 0. \end{cases} \tag{25}$$

Using Eq. (25) and Theorem 3, we can know

$$\begin{aligned} \langle x\rangle &= \sum_{x=-t}^{t} x(P^{\mathrm{L}}(\alpha,\beta,\gamma,x,t) + P^{\mathrm{R}}(\alpha,\beta,\gamma,x,t)) \\ &= G(\beta,t)\sin(\alpha+\gamma), \end{aligned} \tag{26}$$

where $G(\beta,t) = -\sum_{x=-t}^{t} x[G^{\mathrm{L}}(\beta,x,t) + G^{\mathrm{R}}(\beta,x,t)]$ only depends on β and t, instead of α or γ.

5. Conclusions

In this paper, we discussed the properties of the average position in QWs with an arbitrary unitary coin. With a SU(2) coin, if the initial state is $|0L\rangle$ or $|0R\rangle$, the probability distribution is independent of α and γ. Some symmetry properties between different initial states $|0L\rangle$ and $|0R\rangle$ were proven. We derived that $P^{\mathrm{R}}_{|0L\rangle}(\beta,x,t) = P^{\mathrm{L}}_{|0R\rangle}(\beta,-x,t)$ and $P^{\mathrm{L}}_{|0L\rangle}(\beta,x,t) = P^{\mathrm{R}}_{|0R\rangle}(\beta,-x,t)$. If the initial state $|\Psi_0\rangle = 1/\sqrt{2}(|0L\rangle + \mathrm{i}|0R\rangle)$, we know the average $\langle x\rangle = G(\beta,t)\sin(\alpha+\gamma)$. Thus, if we replace the Hadamard operator with an arbitrary unitary operator, the average position is always not equal to 0, unless $\alpha+\beta = n\pi$, $(n \in \mathbb{Z})$.

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