# New approach for deriving the exact time evolution of the density operator for a diffusive anharmonic oscillator and its Wigner distribution function*

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Using thermal entangled state representation, we solve the master equation of a diffusive anharmonic oscillator (AHO) to obtain the exact time evolution formula for the density operator in the infinitive operator-sum representation. We present a new evolution formula of the Wigner function (WF) for any initial state of the diffusive AHO by converting the WF calculation into an overlap between two pure states in an enlarged Fock space. It is found that this formula is very convenient in investigating the WF's evolution of any known initial state. As applications, this formula is used to obtain the evolution of the WF for a coherent state and the evolution of the photon-number distribution of diffusive AHOs.

**Keywords:** diffusive anharmonic oscillator, thermal entangled state representation, infinitive operator-sum representation, Wigner function

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## 1. Introduction

As one of most important nonlinear models, the anharmonic oscillator (AHO)[1–3] is defined with the relationship between force and displacement, which is not linear but depends on the amplitude of the displacement. In the last few decades, the AHO has become a widely studied model in the literature, and intensive investigations have been carried out with the AHO because of its role in the modeling of quantum field theory and its usefulness in atomic and molecular physics.[4–14] One example is the derivation of the approximate analytical formulas describing the squeezing, the principal squeezing, and the quasiprobability distribution $Q$-function for the AHO model by Tanaś *et al.*[4] For a system of coupled AHOs, Chung and Chew[5] derived a basis set for describing the quantum states of the system by performing the Bogoliubov transformation on the bosonic operators of each oscillator. In addition, Jentschura *et al.*[6] presented a unified treatment of AHOs of arbitrary even and odd degrees in terms of a dispersion relation and found a series of explicit new results, including the general behavior of the large-order perturbation theory for arbitrary levels of odd AHOs. On the other hand, these studies adopted many different effective approaches to treating AHO models. For example, the state-dependent diagonalization method was developed to find the energy eigenvalues and eigenstates of a quantum AHO.[7] Green's function and equation of motion approach was used to address whether the moment sum rule can help to obtain a result closer to the exact solution.[8] The effective operator methods were also introduced to analyze the one-dimensional AHO in both the strong and the weak coupling limits.[9]

Recently, the diffusive AHO, which is defined as the AHO coupled to a thermal bath of oscillators in equilibrium, has drawn much attention.[12–14] Under the conditions of small nonlinearity strength and weak coupling to the reservoir degrees of freedom, the master equation describing the diffusive AHO in the interaction picture reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) &= -\mathrm{i}g\left[(a^{\dagger}a)^2,\rho(t)\right]+k(\bar{n}+1)\\ &\quad\times\left[2a\rho(t)a^{\dagger}-a^{\dagger}a\rho(t)-\rho(t)a^{\dagger}a\right]\\ &\quad+k\bar{n}\left[2a^{\dagger}\rho(t)a-aa^{\dagger}\rho(t)-\rho(t)aa^{\dagger}\right],\end{aligned}\quad(1)$$

where the density operator $\rho(t)$ represents the state of the system at time $t$, $a$ ($a^{\dagger}$) is the annihilation (creation) operator, $\bar{n}$ is the average number of thermal photons in mode $\omega$ of the reservoir ($\omega$ is the natural frequency of the oscillator), $k$ is the damping constant, and $g$ is a nonlinearity constant which can be related to the nonlinear susceptibility $\chi^{(3)}$ of the medium. In contrast to the previous work, here we shall use the newly-introduced thermal entangled state representations (TESRs) to treat the diffusive AHO and obtain the exact time evolution of the density operator and the Wigner function (WF). Our motivations for choosing this topic come from two facts. First, to the best of our knowledge, an investigation of the exact time evolution of the density operator for a diffusive AHO has not

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been reported due to the complicated calculations involved, not to speak of the evolution of the WF. On the other hand, the master equation describing the diffusive AHO in Eq. (1) is a more general one in the sense that many important master equations in quantum optics can be treated as Eq. (1) with particular values of $\bar{n}$ and $k$. For instance, in the diffusive limit, i.e., $k \to 0$, $\bar{n} \to \infty$, while keeping the product $\kappa = k\bar{n}$ finite, the master equation (1) becomes[12]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\mathrm{i}g\left[(a^{\dagger}a)^2, \rho(t)\right] + 2\kappa[a\rho(t)a^{\dagger} + a^{\dagger}\rho(t)a - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a - \rho(t)]. \quad (2)$$

Using the Baker–Campbell–Hausdorff formulas[15] to expand the evolution operator, Peixoto de Faria [12] solved the above master equation (2) and obtained the general solutions for the classical and quantum dynamics of the diffusive AHOs. On the other hand, when $\bar{n} \to 0$ and $k$ is kept finite, equation (1) becomes the master equation for a dissipative Kerr medium[13]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\mathrm{i}g\left[(a^{\dagger}a)^2, \rho(t)\right] + k\left[2a\rho(t)a^{\dagger} - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a\right]. \quad (3)$$

Therefore, it is of interest to derive the exact time evolution of the density operator and the WF for the diffusive AHO by virtue of the TESRs.

This paper is organized as follows. In Section 2, we briefly review two newly-introduced TESRs and their properties. In Section 3, we use the TESRs to derive the exact time evolution of density operator $\rho(t)$ with its infinitive operator-sum representation as the solution of the master equation (1). In Section 4, we present a new evolution formula of WF for any initial state of the diffusive AHO by expressing the WF for $\rho(t)$ as an overlap between two pure states in a two-mode enlarged Fock space. In addition, the analytical evolution of the WF for the initial coherent state and the evolution of the photon-number distribution (PND) of the diffusive AHO are also presented by using this formula. Finally, we draw some conclusions in Section 5.

## 2. Two thermal entangled state representations

In order to obtain the general solution of the above master equation (1), we now introduce a new kind of two-mode TESR in an enlarged Fock space[16–18]

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^{\dagger} - \eta^{*}\tilde{a}^{\dagger} + a^{\dagger}\tilde{a}^{\dagger}\right)|0\tilde{0}\rangle = D(\eta)|\eta = 0\rangle, \quad (4)$$

where $|\eta = 0\rangle = \exp(a^{\dagger}\tilde{a}^{\dagger})|0,\tilde{0}\rangle = \sum_{n=0}^{\infty}|n,\tilde{n}\rangle$, $D(\eta) = \exp(\eta a^{\dagger} - \eta^{*}a)$ is the displacement operator ($\eta$ is complex), $\tilde{a}^{\dagger}$ is a fictitious mode accompanying the real photon creation operator $a^{\dagger}$, $|\tilde{0}\rangle$ is annihilated by $\tilde{a}$, $[\tilde{a}, \tilde{a}^{\dagger}] = 1$, and $[\tilde{a}, a^{\dagger}] = 0$. The $|\eta\rangle$ state is the common eigenvector of $(a - \tilde{a}^{\dagger})$ and $(a^{\dagger} - \tilde{a})$ due to the commutative relation $[(a - \tilde{a}^{\dagger}), (a^{\dagger} - \tilde{a})] = 0$, so it yields the following eigen equations:

$$(a - \tilde{a}^{\dagger})|\eta\rangle = \eta|\eta\rangle, \quad (a^{\dagger} - \tilde{a})|\eta\rangle = \eta^{*}|\eta\rangle. \quad (5)$$

Using the technique of integration within an ordered product of operators and the normally ordered form of vacuum projector $|0,\tilde{0}\rangle\langle 0,\tilde{0}| =: \exp(-a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}):$, where $::$ denotes the normal ordering, we can prove that the state $|\eta\rangle$ is complete and orthogonal, i.e.,

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1, \quad \langle\eta'|\eta\rangle = \pi\delta(\eta - \eta')\delta(\eta^{*} - \eta'^{*}). \quad (6)$$

Further, from Eq. (4), it is easy to see that the state $|\eta = 0\rangle$ possesses the properties

$$a|\eta = 0\rangle = \tilde{a}^{\dagger}|\eta = 0\rangle, \quad a^{\dagger}|\eta = 0\rangle = \tilde{a}|\eta = 0\rangle, \quad a^{\dagger}a|\eta = 0\rangle = \tilde{a}^{\dagger}\tilde{a}|\eta = 0\rangle. \quad (7)$$

Similarly, to obtain Eq. (4), we introduce another new kind of two-mode thermal entangled state $|\xi\rangle$ conjugated to $|\eta\rangle$, which is defined as

$$|\xi\rangle = \exp\left(-\frac{1}{2}|\xi|^2 + \xi a^{\dagger} - \xi^{*}\tilde{a}^{\dagger} + a^{\dagger}\tilde{a}^{\dagger}\right)|0\tilde{0}\rangle = D(\xi)|\xi = 0\rangle = (-1)^{a^{\dagger}a}|\eta = -\xi\rangle. \quad (8)$$

This also possesses the completeness relation and the orthogonal property

$$\int \frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi| = 1, \quad \langle\xi'|\xi\rangle = \pi\delta(\xi - \xi')\delta(\xi^{*} - \xi'^{*}). \quad (9)$$

## 3. The evolution of the density operator of diffusive AHO

In this section, we show how an initial state $\rho_0$ evolves based on the newly introduced state (4). Operating both sides of Eq. (1) on the state $|\eta = 0\rangle \equiv |I\rangle$, and denoting $|\rho(t)\rangle = \rho(t)|I\rangle$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)\rangle &= \{-\mathrm{i}g\left[(a^{\dagger}a)^2, \rho(t)\right] + k(\bar{n}+1) \\ &\quad \times [2a\rho(t)a^{\dagger} - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a] \\ &\quad + k\bar{n}\left[2a^{\dagger}\rho(t)a - aa^{\dagger}\rho(t) - \rho(t)aa^{\dagger}\right]\}|I\rangle \\ &= \{-\mathrm{i}g\left[(a^{\dagger}a)^2 - (\tilde{a}^{\dagger}\tilde{a})^2\right] \\ &\quad + k(\bar{n}+1)(2a\tilde{a} - a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}) \\ &\quad + k\bar{n}(2a^{\dagger}\tilde{a}^{\dagger} - aa^{\dagger} - \tilde{a}\tilde{a}^{\dagger})\}|\rho(t)\rangle, \end{aligned} \quad (10)$$

so its formal solution is

$$\begin{aligned}|\rho(t)\rangle &= \exp\{-\mathrm{i}gt[(a^{\dagger}a)^2 - (\tilde{a}^{\dagger}\tilde{a})^2] \\ &\quad + k(\bar{n}+1)t[2a\tilde{a} - a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}] \\ &\quad + k\bar{n}t[2a^{\dagger}\tilde{a}^{\dagger} - aa^{\dagger} - \tilde{a}\tilde{a}^{\dagger}]\}|\rho_0\rangle, \end{aligned} \quad (11)$$

where $|\rho_0\rangle \equiv \rho_0|I\rangle$, and $\rho_0$ is the initial density operator. By introducing the following operators:

$$\begin{aligned} K_+ &= a^\dagger \tilde{a}^\dagger, \quad K_- = a\tilde{a}, \\ K_z &= \frac{1}{2}(a^\dagger a + \tilde{a}^\dagger \tilde{a} + 1), \quad K_0 = a^\dagger a - \tilde{a}^\dagger \tilde{a}, \end{aligned} \tag{12}$$

where $K_+, K_-$, and $K_z$ yield the following SU(1,1) Lie algebra $[K_-, K_+] = 2K_z$ and $[K_z, K_\pm] = \pm K_\pm$, and $K_0$ is the Casimir operator, which is commutative with generators $K_\pm$ and $K_z$ of SU(1,1). We can rewrite Eq. (11) as

$$\begin{aligned} |\rho(t)\rangle &= \exp\{-\mathrm{i}gt[K_0(2K_z-1)] + k(\bar{n}+1)t \\ &\quad \times (2K_- - 2K_z + 1) + k\bar{n}t(2K_+ - 2K_z + 1)\}|\rho_0\rangle \\ &= \exp[\mathrm{i}gtK_0 + k(2\bar{n}+1)t] \\ &\quad \times \exp(\gamma_+ K_+ + \gamma_z K_z + \gamma_- K_-)|\rho_0\rangle, \end{aligned} \tag{13}$$

where $\gamma_+ = 2k\bar{n}t$, $\gamma_- = 2k(\bar{n}+1)t$, and $\gamma_z = -2t[\mathrm{i}gK_0 + k(2\bar{n}+1)]$. By using the disentangling theorem for SU(1,1) Lie algebra,[19] we have

$$\begin{aligned} |\rho(t)\rangle &= \exp[\mathrm{i}gtK_0 + k(2\bar{n}+1)t]\exp(\Gamma_+ K_+) \\ &\quad \times \exp[(2\ln\sqrt{\Gamma_z})K_z]\exp(\Gamma_- K_-)|\rho_0\rangle, \end{aligned} \tag{14}$$

where $\Gamma_\pm = (2\gamma_\pm \sinh\phi)/(2\phi\cosh\phi - \gamma_z \sinh\phi)$, $\sqrt{\Gamma_z} = 2\phi/(2\phi\cosh\phi - \gamma_z\sinh\phi)$, and $\phi^2 = \gamma_z^2/4 - \gamma_+\gamma_-$. In order to deprive the state $|I\rangle$ from Eq. (14), substituting the completeness relation of the Fock states in the enlarged space $\sum_{m,n=0}^{\infty}|m,\tilde{n}\rangle\langle m,\tilde{n}| = 1$ into Eq. (14) and noticing $a^{\dagger j}|n\rangle = \sqrt{(n+j)!/n!}|n+j\rangle$, we arrive at

$$\begin{aligned} |\rho(t)\rangle &= \sum_{j,m,n=0}^{\infty} \frac{\Gamma_-^j(m,n)}{j!}\exp[\mathrm{i}gtK_0 + k(2\bar{n}+1)t] \\ &\quad \times \sqrt{\Gamma_z^{m+n+1}(m,n)}\sum_{i=0}^{\infty}\frac{\Gamma_+^i}{i!}a^{\dagger i}\tilde{a}^{\dagger i}|m,\tilde{n}\rangle\langle m,\tilde{n}|a^j\rho_0 a^{\dagger j}|I\rangle \\ &= \sum_{i,j,m,n=0}^{\infty}\sqrt{\frac{(m+i)!(n+i)!}{m!n!}\Gamma_z^{m+n+1}(m,n)} \\ &\quad \times \frac{\Gamma_+^i(m,n)\Gamma_-^j(m,n)}{i!j!}\exp[\mathrm{i}g(m-n)t + k(2\bar{n}+1)t] \\ &\quad \times |m+i,\tilde{n}+i\rangle\langle m,\tilde{n}|a^j\rho_0 a^{\dagger j}|I\rangle, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Gamma_+(m,n) &= 2k\bar{n}t\Omega(m,n)\sinh\phi(m,n), \\ \sqrt{\Gamma_z(m,n)} &= \Omega(m,n)\phi(m,n), \\ \Gamma_-(m,n) &= 2kt(\bar{n}+1)\Omega(m,n)\sinh\phi(m,n), \\ \Omega(m,n) &= [\phi(m,n)\cosh\phi(m,n) + t[\mathrm{i}g(m-n) \\ &\quad + k(2\bar{n}+1)]\sinh\phi(m,n)]^{-1}, \\ \phi(m,n) &= t[-g^2(m-n)^2 \\ &\quad + \mathrm{i}2gk(2\bar{n}+1)(m-n) + k^2]^{1/2}. \end{aligned} \tag{16}$$

Using the identity

$$\langle\tilde{n}|I\rangle = |n\rangle, \tag{17}$$

we have

$$\langle m,\tilde{n}|a^j\rho_0 a^{\dagger j}|I\rangle = \langle m|a^j\rho_0 a^{\dagger j}\langle\tilde{n}|I\rangle = \langle m|a^j\rho_0 a^{\dagger j}|n\rangle, \tag{18}$$

so equation (15) reduces to

$$\begin{aligned} |\rho(t)\rangle &= \sum_{i,j,m,n=0}^{\infty}\sqrt{\frac{(m+i)!(n+i)!}{m!n!}\Gamma_z^{m+n+1}(m,n)} \\ &\quad \times \frac{\Gamma_+^i(m,n)\Gamma_-^j(m,n)}{i!j!}\exp[\mathrm{i}g(m-n)t + k(2\bar{n}+1)t] \\ &\quad \times |m+i,\tilde{n}+i\rangle\langle m|a^j\rho_0 a^{\dagger j}|n\rangle. \end{aligned} \tag{19}$$

Using Eq. (17) again, we can rewrite Eq. (19) as

$$\begin{aligned} |\rho(t)\rangle &= \sum_{i,j,m,n=0}^{\infty}\sqrt{\frac{(m+i)!(n+i)!}{m!n!}}\frac{\Gamma_+^i(m,n)\Gamma_-^j(m,n)}{i!j!} \\ &\quad \times \sqrt{\Gamma_z^{m+n+1}(m,n)}\exp[\mathrm{i}g(m-n)t + k(2\bar{n}+1)t] \\ &\quad \times \langle m|a^j\rho_0 a^{\dagger j}|n\rangle|m+i\rangle\langle\tilde{n}+i|I\rangle. \end{aligned} \tag{20}$$

After depriving $|I\rangle$ from both sides of Eq. (20), the solution of master equation (1) appears as

$$\begin{aligned} \rho(t) &= \sum_{i,j,m,n=0}^{\infty}\frac{\Gamma_+^i(m,n)\Gamma_-^j(m,n)}{i!j!}\sqrt{\Gamma_z^{m+n+1}(m,n)} \\ &\quad \times \exp[\mathrm{i}gt(m-n) + k(2\bar{n}+1)t] \\ &\quad \times a^{\dagger i}|m\rangle\langle m|a^j\rho_0 a^{\dagger j}|n\rangle\langle n|a^i, \end{aligned} \tag{21}$$

which is just the infinitive operator-sum representation of density operator $\rho(t)$ in terms of the super-operator theory describing the evolution of the density operator from an initial $\rho_0$ into a final state $\rho(t)$.[20] Therefore, once the initial state $\rho_0$ of the diffusive AHO is given, the evolution density operator $\rho(t)$ is easily calculated from Eq. (21). Further, we can formally express Eq. (21) as

$$\rho(t) = \sum_{i,j,m,n=0}^{\infty} M_{m,n,i,j}\rho_0\mathfrak{M}_{m,n,i,j}^\dagger, \tag{22}$$

where the two operators $M_{m,n,i,j}$ and $\mathfrak{M}_{m,n,i,j}^\dagger$ are defined as

$$\begin{aligned} M_{m,n,i,j} &= \sqrt{\frac{\Gamma_+^i(m,n)\Gamma_-^j(m,n)\Gamma_z^{1/2}(m,n)}{i!j!}}\exp[k(\bar{n}+1/2)t] \\ &\quad \times \exp[m(\mathrm{i}gt + \ln\sqrt{\Gamma_z(m,n)})]a^{\dagger i}|m\rangle\langle m|a^j, \end{aligned} \tag{23}$$

$$\begin{aligned} \mathfrak{M}_{m,n,i,j}^\dagger &= \left\{\sqrt{\frac{\Gamma_+^i(m,n)\Gamma_-^j(m,n)\Gamma_z^{1/2}(m,n)}{i!j!}}\exp[k(\bar{n}+1/2)t]\right. \\ &\quad \left.\times \exp[n(\mathrm{i}gt + \ln\sqrt{\Gamma_z(m,n)})]a^{\dagger i}|n\rangle\langle n|a^j\right\}^\dagger. \end{aligned} \tag{24}$$

The operators $M_{m,n,i,j}$ and $\mathfrak{M}_{m,n,i,j}^\dagger$ are not Hermite conjugate to each other. However, they still hold the normalization condition, i.e.,

$$\sum_{i,j,m,n=0}^{\infty} M_{m,n,i,j}\mathfrak{M}_{m,n,i,j}^\dagger = 1, \tag{25}$$

which means that they are trace-preserving in a general sense, so $M_{m,n,i,j}$ and $\mathfrak{M}^{\dagger}_{m,n,i,j}$ can be called the generalized Kraus operators. In particular, on the condition that $k \to 0$, $\bar{n} \to \infty$, and the product $\kappa = k\bar{n}$ is finite, let

$$\Lambda = \mathrm{i}g(m-n) + 2\kappa, \quad \Delta = [\Lambda^2 - 4\kappa^2]^{1/2}, \tag{26}$$

$$\Gamma(t) = \frac{2\kappa \sinh \Delta t}{\Delta \cosh \Delta t + \Lambda \sinh \Delta t},$$

$$\Gamma_0(t) = \frac{\Delta}{\Delta \cosh \Delta t + \Lambda \sinh \Delta t}, \tag{27}$$

we have

$$\begin{aligned}\rho(t) = & \sum_{i,j,m,n=0}^{\infty} \frac{\sqrt{(m+i)!(n+i)!(m+j)!(n+j)!}}{m!n!i!j!} \\ & \times \Gamma^{i+j}(t)\Gamma_0^{m+n+1}(t)\exp[\mathrm{i}g(m-n)t + 2\kappa t] \\ & \times \langle m+j|\rho_0|n+j\rangle |m+i\rangle\langle n+i|, \end{aligned} \tag{28}$$

which is just in agreement with the exact time evolution of the density operator of the diffusive AHO in the condition of the diffusive limit.[12] In addition, when $\bar{n} = 0$ and $k$ is kept finite, equation (21) reduces to the density operator's evolution governed by master equation (3) for a dissipative cavity with a Kerr medium, i.e.,

$$\begin{aligned}\rho(t) = & \sum_{j,m,n=0}^{\infty} \frac{\Pi^j(t)}{j!}\exp[-\mathrm{i}g(m^2-n^2)t \\ & -(m+n)kt]|m\rangle\langle m|a^j\rho_0 a^{\dagger j}|n\rangle\langle n|, \end{aligned} \tag{29}$$

where $\Pi(t) = k\{1 - \mathrm{e}^{-2[\mathrm{i}g(m-n)+k]t}\}/[\mathrm{i}g(m-n)+k]$.

## 4. The evolution of WF for diffusive AHO

In the phase space, the WF description of quantum states of light is a very powerful tool to study nonclassical effects, such as quantum squeezing, interference, and entanglement. In this section we shall use two TESRs to obtain the time evolution of the WF of the diffusive AHO. In order to achieve this goal, we first derive a new expression of WF in the TESR. The WF of density operator $\rho$ is given as

$$W(\alpha,\alpha^*) = \mathrm{Tr}[\Delta(\alpha,\alpha^*)\rho], \tag{30}$$

where $\Delta(\alpha,\alpha^*)$ is the single-mode Wigner operator in the coherent state representation, whose normal ordering form is defined as[21]

$$\begin{aligned}\Delta(\alpha,\alpha^*) &= \frac{1}{\pi}\colon \exp[-2(\alpha-a)(\alpha^*-a^{\dagger})]\colon \\ &= \frac{1}{\pi}D(2\alpha)(-1)^{a^{\dagger}a}. \end{aligned} \tag{31}$$

From Eqs. (8), (30), and (31) we derive the WF in the TESR

$$\begin{aligned}W(\alpha,\alpha^*) &= \sum_{m,n=0}^{\infty}\langle n,\tilde{n}|\Delta(\alpha,\alpha^*)\rho|m,\tilde{m}\rangle \\ &= \frac{1}{\pi}\langle \eta = 0|D(2\alpha)(-1)^{a^{\dagger}a}\rho|\eta = 0\rangle \\ &= \frac{1}{\pi}\langle \eta = -2\alpha|(-1)^{a^{\dagger}a}|\rho\rangle \\ &= \frac{1}{\pi}\langle \xi = 2\alpha|\rho\rangle. \end{aligned} \tag{32}$$

Equation (32) is the WF in the context of thermal dynamics, in which the WF for the density operator is simplified as an overlap between two pure states in an enlarged Fock space, rather than using the ensemble average in the system-mode space. This seems to be a new approach and is very convenient for calculating the time evolution of WFs. Inserting the overcomplete relation (9) into the inner product between $\langle \xi = 2\alpha|$ and $|\rho(t)\rangle$, we derive the time evolution of the WF of the diffusive AHO

$$W(\alpha,\alpha^*,t) = 4\int \frac{\mathrm{d}^2\beta}{\pi}K(\alpha,\beta^*,t)W(\beta,\beta^*,0), \tag{33}$$

where $W(\beta,\beta^*,0)$ is the WF at the initial time

$$W(\beta,\beta^*,0) = \frac{1}{\pi}\langle \xi = 2\beta|\rho_0\rangle, \tag{34}$$

and

$$\begin{aligned}& K(\alpha,\beta^*,t) \\ &= \langle \xi = 2\alpha|\exp[\mathrm{i}gtK_0 + k(2\bar{n}+1)t]\exp(\Gamma_+K_+) \\ &\quad \times \exp[(2\ln\sqrt{\Gamma_z})K_z]\exp(\Gamma_-K_-)|\xi' = 2\beta\rangle. \end{aligned} \tag{35}$$

From Eq. (15) we can rewrite $K(\alpha,\beta^*,t)$ as

$$\begin{aligned}& K(\alpha,\beta^*,t) \\ &= \sum_{m,n=0}^{\infty}\sqrt{\Gamma_z^{m+n+1}(m,n)}\exp[\mathrm{i}gt(m-n) + k(2\bar{n}+1)t] \\ &\quad \times \langle \xi = 2\alpha|\exp[\Gamma_+(m,n)a^{\dagger}\tilde{a}^{\dagger}]|m,\tilde{n}\rangle \\ &\quad \times \langle m,\tilde{n}|\exp[\Gamma_-(m,n)a\tilde{a}]|\xi' = 2\beta\rangle. \end{aligned} \tag{36}$$

To obtain the matrix element in Eq. (36), for convenience we use the generating function formula of the two-variable Hermite polynomials[16,17] $\mathrm{H}_{m,n}(\eta,\eta^*)$ to expand the entangled state $|\xi\rangle$ as

$$|\xi\rangle = \mathrm{e}^{-|\xi|^2/2}\sum_{m,n=0}^{\infty}\frac{1}{\sqrt{m!n!}}\mathrm{H}_{m,n}(\xi,\xi^*)|m,\tilde{n}\rangle, \tag{37}$$

where $|m,\tilde{n}\rangle = \dfrac{a^{\dagger m}\tilde{a}^{\dagger n}}{\sqrt{m!n!}}|0,\tilde{0}\rangle$ is the two-mode Fock state. Thus, from Eq. (37) we derive the overlap

$$\langle \xi|m,\tilde{n}\rangle = \frac{1}{\sqrt{m!n!}}\mathrm{H}_{m,n}(\xi^*,\xi)\ \mathrm{e}^{-|\xi|^2/2}. \tag{38}$$

Then, using Eqs. (36) and (38), we can arrive at

$$\begin{aligned}& K(\alpha,\beta^*,t) \\ &= \sum_{m,n=0}^{\infty}\sqrt{\Gamma_z^{m+n+1}(m,n)}\exp[\mathrm{i}gt(m-n) \\ &\quad + k(2\bar{n}+1)t]\sum_{k,l=0}^{\infty}\frac{\Gamma_+^k(m,n)\Gamma_-^l(m,n)}{k!l!} \end{aligned}$$

$$\times\langle\xi=2\alpha|a^{\dagger k}\tilde{a}^{\dagger k}|m,\tilde{n}\rangle\langle m,\tilde{n}|a^{l}\tilde{a}^{l}|\xi'=2\beta\rangle$$
$$=\sum_{m,n=0}^{\infty}\sqrt{\Gamma_z^{m+n+1}(m,n)}\exp[\mathrm{i}gt(m-n)+k(2\bar{n}+1)t]$$
$$\times\sum_{k,l=0}^{\infty}\frac{\Gamma_+^k(m,n)\Gamma_-^l(m,n)}{k!l!m!n!}\,\mathrm{e}^{-2|\alpha|^2-2|\beta|^2}$$
$$\times\mathrm{H}_{m+k,n+k}(2\alpha^*,2\alpha)\mathrm{H}_{m+l,n+l}(2\beta,2\beta^*). \quad (39)$$

Further, using a new sum formula

$$\sum_{l=0}^{\infty}\frac{x^l}{l!}\mathrm{H}_{m+l,n+l}(y,z)$$
$$=\frac{\mathrm{e}^{xyz/(x+1)}}{(x+1)^{(m+n+2)/2}}\mathrm{H}_{m,n}\left(\frac{y}{\sqrt{x+1}},\frac{z}{\sqrt{x+1}}\right), \quad (40)$$

we can reduce Eq. (39) to

$$K(\alpha,\beta^*,t)$$
$$=\sum_{m,n=0}^{\infty}\frac{\sqrt{\Gamma_z^{m+n+1}(m,n)}}{m!n!\sqrt{\{[\Gamma_+(m,n)+1][\Gamma_-(m,n)+1]\}^{(m+n+2)}}}$$
$$\times\exp[\mathrm{i}gt(m-n)+k(2\bar{n}+1)t]$$
$$\times\exp\left\{2\frac{\Gamma_+(m,n)-1}{\Gamma_+(m,n)+1}|\alpha|^2+2\frac{\Gamma_-(m,n)-1}{\Gamma_-(m,n)+1}|\beta|^2\right\}$$
$$\times\mathrm{H}_{m,n}\left[\frac{2\alpha^*}{\sqrt{\Gamma_+(m,n)+1}},\frac{2\alpha}{\sqrt{\Gamma_+(m,n)+1}}\right]$$
$$\times\mathrm{H}_{m,n}\left[\frac{2\beta}{\sqrt{\Gamma_-(m,n)+1}},\frac{2\beta^*}{\sqrt{\Gamma_-(m,n)+1}}\right]. \quad (41)$$

Finally, substituting Eqs. (33) and (41) leads to the exact evolution formula of $W(\alpha,\alpha^*,t)$ for the diffusive AHO. It is obvious that when $k\to 0$, $\bar{n}\to\infty$, the $\kappa=k\bar{n}$ product is kept finite, and equation (33) becomes the time evolution of the WF of the diffusive AHO in the condition of a diffusive limit as expected in Ref. [12], i.e.,

$$W(\alpha,\alpha^*,t)$$
$$=\sum_{m,n=0}^{\infty}\frac{4\Gamma_0^{m+n+1}(t)}{m!n![\Gamma(t)+1]^{(m+n+2)}}$$
$$\times\exp\left[\mathrm{i}gt(m-n)+2\kappa t+2\frac{\Gamma(t)-1}{\Gamma(t)+1}|\alpha|^2\right]$$
$$\times\mathrm{H}_{m,n}\left[\frac{2\alpha^*}{\sqrt{\Gamma(t)+1}},\frac{2\alpha}{\sqrt{\Gamma(t)+1}}\right]$$
$$\times\int\frac{\mathrm{d}^2\beta}{\pi}\exp\left\{2\frac{\Gamma(t)-1}{\Gamma(t)+1}|\beta|^2\right\}$$
$$\times\mathrm{H}_{m,n}\left[\frac{2\beta}{\sqrt{\Gamma(t)+1}},\frac{2\beta^*}{\sqrt{\Gamma(t)+1}}\right]W(\beta,\beta^*,0). \quad (42)$$

However, when $\bar{n}\to 0$ and $k$ is kept finite, equation (43) reduces to

$$W(\alpha,\alpha^*,t)$$
$$=\sum_{m,n=0}^{\infty}\frac{4\mathrm{H}_{m,n}(2\alpha^*,2\alpha)}{m!n![\Pi(t)+1]^{(m+n+2)/2}}\exp[-\mathrm{i}gt(m^2-n^2)$$
$$-k(m+n)t-2|\alpha|^2]\int\frac{\mathrm{d}^2\beta}{\pi}\exp\left(2\frac{\Pi(t)-1}{\Pi(t)+1}|\beta|^2\right)$$
$$\times\mathrm{H}_{m,n}\left[\frac{2\beta}{\sqrt{\Pi(t)+1}},\frac{2\beta^*}{\sqrt{\Pi(t)+1}}\right]W(\beta,\beta^*,0), \quad (43)$$

which is just the evolution of the WF in a dissipative channel with a Kerr medium.

Now we want to discuss the applications of the WF evolution formula (33). We take the coherent state $|z\rangle$ as the initial state, whose WF reads $W(\beta,\beta^*)=\frac{1}{\pi}\exp[-2|\beta-z|^2]$. Substituting $W(\beta,\beta^*)$ into Eq. (33) and considering Eq. (41), we immediately arrive at the evolution of the WF for an initial coherent state

$$W(\alpha,\alpha^*,t)$$
$$=\sum_{m,n=0}^{\infty}\frac{\sqrt{\Gamma_z^{m+n+1}(m,n)}}{\pi m!n!\sqrt{[\Gamma_+(m,n)+1]^{m+n+2}}}$$
$$\times\exp[\mathrm{i}gt(m-n)+k(2\bar{n}+1)t]\exp\left[2\frac{\Gamma_+(m,n)-1}{\Gamma_+(m,n)+1}|\alpha|^2\right]$$
$$\times\mathrm{H}_{m,n}\left[\frac{2\alpha^*}{\sqrt{\Gamma_+(m,n)+1}},\frac{2\alpha}{\sqrt{\Gamma_+(m,n)+1}}\right]$$
$$\times z^m z^{*n}\exp\{[\Gamma_-(m,n)-1]|z|^2\}, \quad (44)$$

which indicates that it is very convenient to obtain the evolution of the WF for the diffusive AHO by using evolution formula (33) if the WF for the initial quantum state is given. In particular, when $t=0$, $\Gamma_z(m,n)=1$, and $\Gamma_+(m,n)=\Gamma_-(m,n)=0$, equation (44) becomes

$$W(\alpha,\alpha^*,0)=\frac{\mathrm{e}^{-2|\alpha|^2-|z|^2}}{\pi}\sum_{m,n=0}^{\infty}\frac{z^m z^{*n}}{m!n!}\mathrm{H}_{m,n}(2\alpha^*,2\alpha). \quad (45)$$

Using the generating function formula of the two-variable Hermite polynomials, we find that equation (45) just returns to the WF for the initial coherent state. On the other hand, using the WF's evolution formula, we derive the time evolution of the PND of the diffusive AHO. In terms of the thermo-field dynamics, the PND $p(s,t)=\mathrm{Tr}[\rho(t)|s\rangle\langle s|]$ can be converted into the following matrix element:

$$p(s,t)=\sum_{m=0}^{\infty}\langle s,\tilde{s}|\rho(t)|m,\tilde{m}\rangle=\langle s,\tilde{s}|\rho(t)\rangle. \quad (46)$$

Inserting the overcomplete relation of $|\xi\rangle$ in Eq. (9) into Eq. (46) leads to

$$p(s,t)=\int\frac{\mathrm{d}^2\xi}{\pi}\langle s,\tilde{s}|\xi\rangle\langle\xi|\rho(t)\rangle. \quad (47)$$

From Eq. (32), we can see that $\langle\xi|\rho(t)\rangle/\pi$ is the WF of density operator $\rho(t)$, and $\langle s,\tilde{s}|\xi\rangle/\pi$ is the WF of the number state,[22] i.e, $W_{|s\rangle\langle s|}(\alpha,\alpha^*)=(-1)^s\mathrm{e}^{-2|\alpha|^2}L_s(4|\alpha|^2)/\pi$. So the relation between the PND and the WF of the density operator $\rho(t)$ is

$$p(s,t)=4\int\mathrm{d}^2\alpha(-1)^s\mathrm{e}^{-2|\alpha|^2}L_s(4|\alpha|^2)W(\alpha,\alpha^*,t). \quad (48)$$

Substituting Eq. (33) into Eq. (48) and performing the integration, we arrive at the exact time evolution of the PND of $\rho(t)$ for the diffusive AHO

$$
\begin{aligned}
& p(s,t) \\
= & \sum_{m,n=0}^{\infty} \frac{4(-1)^{(3m+n)/2} s! \sqrt{\Gamma_z^{m+n+1}(m,n)\Gamma_+^{(2s-n-m)}(m,n)}}{m!n!(s-m)!\sqrt{[\Gamma_-(m,n)+1]^{(m+n+2)}}} \\
& \times \exp[\mathrm{i}gt(m-n)+k(2\bar{n}+1)t] \\
& \times \int \mathrm{d}^2\beta \exp\left(2\frac{\Gamma_-(m,n)-1}{\Gamma_-(m,n)+1}|\beta|^2\right) \\
& \times \mathrm{H}_{m,n}\left[\frac{2\beta}{\sqrt{\Gamma_-(m,n)+1}}, \frac{2\beta^*}{\sqrt{\Gamma_-(m,n)+1}}\right] W(\beta,\beta^*,0), \quad (49)
\end{aligned}
$$

which shows that the evolution of the PND can be directly obtained from Eq. (49) if the initial WF is known.

## 5. Conclusion

In summary, by virtue of two newly introduced TESRs, we presented a new approach to solve the quantum master equation of density operator $\rho(t)$ of the diffusive AHO, and obtained the exact time evolution of density operator $\rho(t)$ with its infinitive operator-sum representation. In our approach, the WF for any density operator is simplified as an overlap between two pure states in a two-mode enlarged Fock space, rather than using the ensemble average in the system-mode space. In this aspect, our approach seems to be self-consistent and concise. Using this new approach to derive the WF, we have obtained a new formula for calculating the time evolution of the WF for the diffusive AHO, which brings much convenience to the investigation of the WF's evolution for any given initial state. In addition, as the applications of this formula, we discussed the analytical expression of WF for the initial coherent state and the evolution of the PND for any initial state of the diffusive AHO. In terms of the above discussion, we believe that the TESR is useful in treating other master equations in quantum optics theory.

## References


[1] Landau L D and Lifschitz E M 1976 *Mechanics* (Oxford: Pergamon Press)
[2] Wang L M, Cai J and Qiao H X 2011 *Chin. Phys. B* **20** 073202
[3] He L, Zhang F M, Yang Y F and Li C F 2010 *Chin. Phys. B* **19** 040306
[4] Tanaś R, Miranowicz A and Kielich S 1991 *Phys. Rev. A* **43** 4014
[5] Chung N N and Chew L Y 2007 *Phys. Rev. A* **76** 032113
[6] Jentschura U D, Surzhykov A and Zinn-Justin J 2009 *Phys. Rev. Lett.* **102** 011601
[7] Ho K C, Liu Y T, Lo C F, Liu K L, Kwok W M and Shiu M L 1996 *Phys. Rev. A* **53** 1280
[8] Luo H G and Wang S J 2000 *Phys. Rev. B* **62** 5341
[9] Abraham K J and Vary J P 2003 *Phys. Rev.* C **67** 054313
[10] Núñez M A 2003 *Phys. Rev. E* **68** 016703
[11] Claudon J, Zazunov A, Hekking F W J and Buisson O 2008 *Phys. Rev. B* **78** 184503
[12] Peixoto de Faria J G 2007 *Eur. Phys. J. D* **42** 153
[13] Daniel D J and Milburn G J 1989 *Phys. Rev. A* **39** 4628
[14] Oliveira A C, Peixoto de Faria J G and Nemes M C 2006 *Phys. Rev. E* **73** 046207
[15] Chaturvedi S and Srinivasan V 1991 *Phys. Rev. A* **43** 4054
[16] Jiang N Q, Fan H Y and Hu L Y 2011 *J. Phys A: Math. Theor.* **44** 195302
[17] Meng X G, Wang J S and Liang B L 2011 *Chin. Phys. B* **20** 014204
[18] Hu L Y and Fan H Y 2009 *Chin. Phys. B* **18** 1674
[19] Wodkiewicz K and Eberly J H 1985 *J. Opt. Soc. Am. B* **2** 458
[20] Moya-Cessa H 2006 *Phys. Rep.* **432** 1
[21] Hu L Y and Fan H Y 2009 *Opt. Commun.* **282** 4379
[22] Takahashi Y and Umezawa H 1975 *Collect. Phenom.* **2** 55