The squeezing entangled state of two particles with unequal mass*

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For two unequal-mass particles, we construct the entangled state representation and then derive the corresponding squeezing operator. This squeezing operator has a natural realization in the entangled state representation, which exhibits the intrinsic relation between squeezing and quantum entanglement. This squeezing operator involves both two-mode squeezing and the direct product of two single-mode squeezings. The maximum squeezing occurs when the two particles possess equal mass. When the two particles' mass difference becomes large, the component of the two single-mode squeezings becomes dominant.

Keywords: squeezing operator, integration within an ordered product (IWOP) technique, entangled state representation

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1. Introduction

Quantum entangled states are being paid increasing attention by physicists due to their potential uses in quantum communication.[1–4] The concept of entanglement also plays a key role in understanding some of the fundamental problems in quantum mechanics and quantum optics.[5–8] In an entangled state, a measurement performed on one part of the system provides information about the remaining part, and this has now been known as a basic feature of quantum mechanics, despite the fact that it seems strange. The entanglement was first revealed in the pioneer paper of Einstein, Podolsky, and Rosen (EPR)[9] by explicitly writing the wave function of two particles with their relative position $x_0 = X_1 - X_2$ and total momentum $p_0 = P_1 + P_2 = 0$, i.e., $\psi(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{\mathrm{i}p(x_1 - x_2 + x_0)}$. In Refs. [10] and [11], the simultaneous eigenstate $|\eta\rangle$ of commutative operators $X_1 - X_2$ and $P_1 + P_2$ in the two-mode Fock space is found

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle_{12}, \quad (1)$$

where $\eta = (\eta_1 + \mathrm{i}\eta_2)/\sqrt{2}$ is a complex number, $|00\rangle$ is the two-mode vacuum state; a_i and $a_i^\dagger$ $(i = 1, 2)$ are two-mode Bose annihilation and creation operators in the Fock space, $[a_i, a_j^\dagger] = \delta_{i,j}$, they are related to (X_i, P_i) by

$$X_i = (a_i + a_i^\dagger)/\sqrt{2}, \quad P_i = (a_i - a_i^\dagger)/(\sqrt{2}\mathrm{i}). \quad (2)$$

The $|\eta\rangle$ state obeys the eigenvector equations

$$\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta\,|\eta\rangle, \quad \left(a_2 - a_1^\dagger\right)|\eta\rangle = -\eta^*\,|\eta\rangle. \quad (3)$$

It then follows

$$(X_1 - X_2)\,|\eta\rangle = \eta_1\,|\eta\rangle, \quad (P_1 + P_2)\,|\eta\rangle = \eta_2\,|\eta\rangle. \quad (4)$$

On the other hand, the conjugate state to $|\eta\rangle$, which is the common eigenvector of $X_1 + X_2$ and $P_1 - P_2$, is

$$|\xi\rangle = \exp\left[-\frac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_2^\dagger a_1^\dagger\right]|00\rangle, \quad (5)$$
$$(\xi = \xi_1 + \mathrm{i}\xi_2),$$

which spans a complete and orthonormal set

$$\int \frac{\mathrm{d}^2\eta}{\pi}\,|\xi\rangle\langle\xi| = 1,$$
$$\langle\xi'|\xi\rangle = \pi\delta(\xi_1' - \xi_1)(\xi_2' - \xi_2). \quad (6)$$

It is now known that EPR light fields with bipartite entanglement can be built from a two-single-mode squeezed vacuum state combined at a 50:50 beamsplitter (BS), i.e., two light fields maximally squeezed in X_i and P_i (in opposite quadratures), respectively, entering the two input ports of a 50:50 BS produce a pair of entangled light beams at the output of the BS. In this work, we pay attention to the common eigenstate $|\xi\rangle_{\mu_1}$ of the two commuting operators $X_{\mathrm{c.m}} = \mu_1 X_1 + \mu_2 X_2$ (the center-of-mass coordinate when two particles have different masses m_1 and m_2, $\mu_1 = m_1/(m_1 + m_2)$, $\mu_2 = m_2/(m_1 + m_2)$) and $P_{\mathrm{r}} = \mu_2 P_1 - \mu_1 P_2$ (the mass dependent relative momentum), since $[X_{\mathrm{c.m}}, P_{\mathrm{r}}] = 0$. We will discuss the squeezing operator for $|\xi\rangle_{\mu_1} \to |\xi/\nu\rangle_{\mu_1}$ and its characteristic. By doing so, we hope to better understand the intrinsic relation between squeezing and quantum entanglement.

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2. Obtaining $|\xi\rangle_{\mu_1}$

By introducing a rotation operator[12]

$$R = \exp\left[\left(a_1^\dagger a_2^\dagger\right)\ln\begin{pmatrix}\frac{1}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}}(\mu_1-\mu_2)\\ \frac{1}{\sqrt{\lambda}}(\mu_2-\mu_1) & \frac{1}{\sqrt{\lambda}}\end{pmatrix}\times\begin{pmatrix}a_1\\ a_2\end{pmatrix}\right], \tag{7}$$
$$\lambda\equiv 2\left(\mu_1^2+\mu_2^2\right),\quad \mu_1+\mu_2=1,$$

which plays the role of transforming

$$a_1^\dagger\to Ra_1^\dagger R^{-1}=\frac{1}{\sqrt{\lambda}}a_1^\dagger+\frac{(\mu_2-\mu_1)}{\sqrt{\lambda}}a_2^\dagger\equiv b_1^\dagger,$$
$$a_2^\dagger\to Ra_2^\dagger R^{-1}=\frac{(\mu_1-\mu_2)}{\sqrt{\lambda}}a_1^\dagger+\frac{1}{\sqrt{\lambda}}a_2^\dagger\equiv b_2^\dagger, \tag{8}$$

with

$$a_1\to Ra_1R^{-1}=\frac{1}{\sqrt{\lambda}}a_1+\frac{(\mu_2-\mu_1)}{\sqrt{\lambda}}a_2\equiv b_1,$$
$$a_2\to Ra_2R^{-1}=\frac{(\mu_1-\mu_2)}{\sqrt{\lambda}}a_1+\frac{1}{\sqrt{\lambda}}a_2\equiv b_2, \tag{9}$$

so that $[b_1,b_1^\dagger]=[b_2,b_2^\dagger]=1$, $[b_1,b_2^\dagger]=[b_2,b_1^\dagger]=0$, we perform the transformation for $|\xi\rangle$ in Eq. (5) and obtain

$$\begin{aligned}R|\xi\rangle &= \exp\left\{-\frac{1}{2}|\xi|^2+\frac{1}{\sqrt{\lambda}}\left[\xi+(\mu_1-\mu_2)\xi^*\right]a_1^\dagger\right.\\ &\quad+\frac{1}{\sqrt{\lambda}}\left[\xi^*-(\mu_1-\mu_2)\xi\right]a_2^\dagger\\ &\quad\left.+\frac{1}{\lambda}\left[(\mu_2-\mu_1)\left(a_1^{\dagger2}-a_2^{\dagger2}\right)-4\mu_1\mu_2a_2^\dagger a_1^\dagger\right]\right\}|00\rangle\\ &\equiv|\xi\rangle_{\mu_1},\end{aligned} \tag{10}$$

Operating a_i on $|\xi\rangle_{\mu_1}$, we have

$$\begin{aligned}a_1|\xi\rangle_{\mu_1} &= \left\{\frac{1}{\sqrt{\lambda}}\left[\xi+(\mu_1-\mu_2)\xi^*\right]\right.\\ &\quad\left.+\frac{2}{\lambda}\left[(\mu_2-\mu_1)a_1^\dagger-4\mu_1\mu_2a_2^\dagger\right]\right\}|\xi\rangle_{\mu_1},\end{aligned} \tag{11}$$

$$\begin{aligned}a_2|\xi\rangle_{\mu_1} &= \left\{\frac{1}{\sqrt{\lambda}}\left[\xi^*-(\mu_1-\mu_2)\xi\right]\right.\\ &\quad\left.-\frac{2}{\lambda}\left[(\mu_2-\mu_1)a_2^\dagger-4\mu_1\mu_2a_1^\dagger\right]\right\}|\xi\rangle_{\mu_1},\end{aligned} \tag{12}$$

which means

$$\begin{aligned}&(\mu_1a_1+\mu_2a_2)|\xi\rangle_{\mu_1}\\ &=\left[\sqrt{\lambda}\xi_1-\left(\mu_1a_1^\dagger+\mu_2a_2^\dagger\right)\right]|\xi\rangle_{\mu_1},\end{aligned} \tag{13}$$

$$\begin{aligned}&(\mu_1a_2-\mu_2a_1)|\xi\rangle_{\mu_1}\\ &=\left[-\mathrm{i}\sqrt{\lambda}\xi_1-\left(\mu_2a_1^\dagger-\mu_1a_2^\dagger\right)\right]|\xi\rangle_{\mu_1}.\end{aligned} \tag{14}$$

Therefore, $|\xi\rangle_{\mu_1}$ is really the common eigenstate of $X_{\mathrm{c.m}}$ and P_{r}

$$\begin{aligned}X_{\mathrm{c.m}}|\xi\rangle_{\mu_1} &= \frac{1}{\sqrt{2}}\left[\mu_1\left(a_1+a_1^\dagger\right)+\mu_2\left(a_2+a_2^\dagger\right)\right]|\xi\rangle_{\mu_1}\\ &=\sqrt{\frac{\lambda}{2}}\xi_1|\xi\rangle_{\mu_1},\end{aligned} \tag{15}$$

$$\begin{aligned}P_{\mathrm{r}}|\xi\rangle_{\mu_1} &= \frac{\mathrm{i}}{\sqrt{2}}\left[\mu_1\left(a_2-a_2^\dagger\right)-\mu_2\left(a_1-a_1^\dagger\right)\right]|\xi\rangle_{\mu_1}\\ &=\sqrt{\frac{\lambda}{2}}\xi_2|\xi\rangle_{\mu_1}.\end{aligned} \tag{16}$$

Using the technique of the integration within an ordered product (IWOP)[13–15] of operators, we can show

$$\begin{aligned}&\int\frac{\mathrm{d}^2\eta}{\pi}|\xi\rangle_{\mu_1\,\mu_1}\langle\xi|=1,\\ &{}_{\mu_1}\langle\xi'|\xi\rangle_{\mu_1}=\pi\delta\left(\xi_1'-\xi_1\right)\left(\xi_2'-\xi_2\right).\end{aligned} \tag{17}$$

3. The generalized squeezing operator for $|\xi\rangle_{\mu_1}\to|\xi/\nu\rangle_{\mu_1}$

Now we construct the following ket–bra operator:

$$S_{\mu_1}\equiv\int\frac{\mathrm{d}^2\xi}{\pi\nu}|\xi/\nu\rangle_{\mu_1\,\mu_1}\langle\xi|. \tag{18}$$

Using the normal product form of $|00\rangle\langle00|$

$$|00\rangle\langle00|=:\exp\left[-a_1^\dagger a_1-a_2^\dagger a_2\right]: \tag{19}$$

and the IWOP technique, we have

$$\begin{aligned}S_{\mu_1} &= \int\frac{\mathrm{d}^2\xi}{\pi\nu}:\exp\left\{-\frac{(1+\nu^2)}{2\nu^2}|\xi|^2+\xi\frac{1}{\sqrt{\lambda}}\left[\left(\frac{1}{\nu}a_1^\dagger+a_2\right)\right.\right.\\ &\quad\left.+(\mu_1-\mu_2)\left(-\frac{1}{\nu}a_2^\dagger+a_1\right)\right]\\ &\quad+\xi^*\frac{1}{\sqrt{\lambda}}\left[\left(\frac{1}{\nu}a_2^\dagger+a_1\right)+(\mu_1-\mu_2)\left(\frac{1}{\nu}a_1^\dagger-a_2\right)\right]\\ &\quad+\frac{1}{\lambda}(\mu_2-\mu_1)\left(a_1^{\dagger2}-a_2^{\dagger2}+a_1^2-a_2^2\right)\\ &\quad\left.-\frac{4\mu_1\mu_2}{\lambda}(a_1^\dagger a_2^\dagger+a_1a_2)-a_1^\dagger a_1-a_2^\dagger a_2\right\}:\\ &=\exp\left\{\left[\frac{\mu_1-\mu_2}{\lambda}\left(a_2^{\dagger2}-a_1^{\dagger2}\right)-\frac{4\mu_1\mu_2}{\lambda}a_1^\dagger a_2^\dagger\right]\tanh\gamma\right\}\\ &\quad\times\exp\left[\left(a_1^\dagger a_1+a_2^\dagger a_2+1\right)\ln\mathrm{sech}\,\gamma\right]\\ &\quad\times\exp\left\{\left[\frac{\mu_1-\mu_2}{\lambda}\left(a_1^2-a_2^2\right)\right.\right.\\ &\quad\left.\left.+\frac{4\mu_1\mu_2}{\lambda}a_1a_2\right]\tanh\gamma\right\},\end{aligned} \tag{20}$$

where

$$\nu=\mathrm{e}^\gamma,\quad\mathrm{sech}\,\gamma=\frac{2\nu}{1+\nu^2},\quad\tanh\gamma=\frac{\nu^2-1}{\nu^2+1}, \tag{21}$$

and in the last step we have used the identity

$$\mathrm{e}^{fa_1^\dagger a_1}=:\exp\left[\left(\mathrm{e}^f-1\right)a_1^\dagger a_1\right]:. \tag{22}$$

Since in Eq. (20)

$$\frac{4\mu_1\mu_2}{\lambda}=\frac{2\mu_1\mu_2}{\mu_1^2+\mu_2^2}\le1, \tag{23}$$

thus the maximum squeezing occurs when $\mu_1 = \mu_2$. In this case, S_{μ_1} reduces to the usual two-mode squeezing operator

$$\begin{aligned} S_{\mu_1=0} &= \exp\left(-a_1^\dagger a_2^\dagger \tanh\gamma\right) \\ &\quad \times \exp[(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)\ln\operatorname{sech}\gamma] \\ &\quad \times \exp\left(a_1 a_2 \tanh\gamma\right). \end{aligned} \tag{24}$$

On the other hand, when $m_2 \to 0$, $\mu_1 \to 1$,

$$\begin{aligned} S_{\mu_1\to 1} &\to \exp\left[\frac{1}{2}\left(a_2^{\dagger 2} - a_1^{\dagger 2}\right)\tanh\gamma\right] \\ &\quad \times \exp\left[\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)\ln\operatorname{sech}\gamma\right] \\ &\quad \times \exp\left[\frac{1}{2}\left(a_1^2 - a_2^2\right)\tanh\gamma\right], \end{aligned} \tag{25}$$

which is the direct product of two single-mode squeezing operators. Thus we can see that squeezing operator S_{μ_1} involves both two-mode squeezing and the direct product of two single-mode squeezings. When the two particles' mass difference becomes large, the component of the two single-mode squeezings becomes dominant.

4. The transformation property of S_{μ_1}

Using the Baker–Hausdroff formula

$$\begin{aligned} \mathrm{e}^A B \mathrm{e}^{-A} &= B + [A,B] + \frac{1}{2!}[A,[A,B]] \\ &\quad + \frac{1}{3!}[A,[A,[A,B]]] + \cdots, \end{aligned} \tag{26}$$

we can see that

$$\begin{aligned} &S_{\mu_1} a_1 S_{\mu_1}^{-1} \\ &= a_1 \cosh\gamma + \left[\frac{4\mu_1\mu_2}{\lambda} a_2^\dagger + 2a_1^\dagger \frac{(\mu_1-\mu_2)}{\lambda}\right]\sinh\gamma, \end{aligned} \tag{27}$$

$$\begin{aligned} &S_{\mu_1} a_2 S_{\mu_1}^{-1} \\ &= a_2 \cosh\gamma + \left[\frac{4\mu_1\mu_2}{\lambda} a_1^\dagger - 2a_2^\dagger \frac{(\mu_1-\mu_2)}{\lambda}\right]\sinh\gamma. \end{aligned} \tag{28}$$

It then follows

$$\begin{aligned} &S_{\mu_1} X_1 S_{\mu_1}^{-1} \\ &= X_1 \cosh\gamma + \left[\frac{4\mu_1\mu_2}{\lambda} X_2 + 2\frac{(\mu_1-\mu_2)}{\lambda} X_1\right]\sinh\gamma, \end{aligned} \tag{29}$$

$$\begin{aligned} &S_{\mu_1} X_2 S_{\mu_1}^{-1} \\ &= X_2 \cosh\gamma + \left[\frac{4\mu_1\mu_2}{\lambda} X_1 - 2\frac{(\mu_1-\mu_2)}{\lambda} X_2\right]\sinh\gamma. \end{aligned} \tag{30}$$

So under the S_{μ_1} transformation, the center-of-mass coordinate and the mass dependent relative momentum, respectively, become

$$S_{\mu_1} X_{\mathrm{c.m}} S_{\mu_1}^{-1} = X_{\mathrm{c.m}}\, \mathrm{e}^{\gamma}, \tag{31}$$

$$S_{\mu_1} P_{\mathrm{r}} S_{\mu_1}^{-1} = P_{\mathrm{r}}\, \mathrm{e}^{\gamma}, \tag{32}$$

this coincides with

$$S_{\mu_1} |\xi\rangle_{\mu_1} = \frac{1}{\nu}\left|\frac{\xi}{\nu}\right\rangle_{\mu_1}, \quad \nu = \mathrm{e}^{\gamma}. \tag{33}$$

5. The compact form of the squeezing operator

In Eq. (20), let

$$A^\dagger = \frac{(\mu_1-\mu_2)}{\lambda}\left(a_2^{\dagger 2} - a_1^{\dagger 2}\right) - \frac{4\mu_1\mu_2}{\lambda} a_1^\dagger a_2^\dagger, \tag{34}$$

$$B^\dagger = a_1^\dagger a_1 + a_2^\dagger a_2 + 1, \tag{35}$$

$$A = \frac{(\mu_1-\mu_2)}{\lambda}\left(a_2^2 - a_1^2\right) - \frac{4\mu_1\mu_2}{\lambda} a_1 a_2, \tag{36}$$

then

$$\begin{aligned} S_{\mu_1} &= \int \frac{\mathrm{d}^2\xi}{\pi\nu} |\xi\rangle_{\mu_1\,\mu_1}\langle\xi| \\ &= \mathrm{e}^{A^\dagger \tanh\gamma}\, \mathrm{e}^{B\ln\operatorname{sech}\gamma}\, \mathrm{e}^{-A\tanh\gamma}. \end{aligned} \tag{37}$$

Using

$$[A^\dagger, B] = -2A^\dagger, \quad [A,B] = 2A, \quad [A^\dagger, A] = -B, \tag{38}$$

we have

$$\begin{aligned} \frac{\partial S_{\mu_1}}{\partial\gamma} &= A^\dagger S_{\mu_1} \operatorname{sech}^2\gamma - \exp\left(A^\dagger\tanh\gamma\right) \\ &\quad \times \tanh\gamma \exp\left(B\ln\operatorname{sech}\gamma\right) B \exp\left(-A\tanh\gamma\right) \\ &\quad - S_{\mu_1}\operatorname{sech}^2\gamma A \\ &= S_{\mu_1}\left(A^\dagger - A\right), \end{aligned} \tag{39}$$

so

$$S_{\mu_1} = \mathrm{e}^{\gamma\left(A^\dagger - A\right)}. \tag{40}$$

This is the compact form of S_{μ_1}.

6. Conclusion

In summary, for two unequal-mass particles, we have constructed the entangled state $|\xi\rangle_{\mu_1}$ and derived the corresponding squeezing operator. This squeezing operator has a natural realization in the $|\xi\rangle_{\mu_1}$ representation, which exhibits the intrinsic relation between squeezing and quantum entanglement. This squeezing operator involves both two-mode squeezing and the direct product of two single-mode squeezings. It is also shown that the maximum squeezing occurs when the two particles possess equal mass. When the two particles' mass difference becomes large, the component of the two single-mode squeezings becomes dominant.

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