# Relativistic symmetry of position-dependent mass particles in a Coulomb field including tensor interaction*

M. Eshghi[a)†], M. Hamzavi[b)], and S. M. Ikhdair[c)d)]

[a)] Physics Department, Imam Hossein Comprehensive University, Tehran, Iran
[b)] Department of Science and Engineering, Abhar Branch, Islamic Azad University, Abhar, Iran
[c)] Department of Electrical and Electronic Engineering, Near East University, 922022, Nicosia, Northern Cyprus, Turkey
[d)] Department of Physics, Faculty of Science, An-Najah National University, Nablus, West Bank, Palestine

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The spatially-dependent mass Dirac equation is solved exactly for attractive scalar and repulsive vector Coulomb potentials, including a tensor interaction under the spin and pseudospin symmetric limits. Closed forms of the energy eigenvalue equation and wave functions are obtained for arbitrary spin–orbit quantum number $\kappa$. Some numerical results are also given, and the effect of tensor interaction on the bound states is presented. It is shown that tensor interaction removes the degeneracy between two states in the spin doublets. We also investigate the effects of the spatially-dependent mass on the bound states under spin symmetric limit conditions in the absence of tensor interaction.

**Keywords:** Dirac equation, spin and pseudospin symmetries, spatially-dependent mass, tensor interaction

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## 1. Introduction

The solutions of the non-relativistic/relativistic wave equations are most frequently used in quantum mechanics,[1–3] and the Dirac equation has become the most appealing relativistic wave equation in the field of mathematical physics for spin-1/2 particles. However, solving such a wave equation is still very challenging, even though it was derived more than 80 years ago and has been utilized profusely. It is useful for investigating relativistic effects.[4–7] For example, in the relativistic treatment of nuclear phenomena, the Dirac equation is used to describe the behavior of the nuclei in the nucleus. It has also been used extensively in the studies of relativistic heavy ion collisions, heavy ion spectroscopy, and more recently laser–matter interaction (for a review, see Ref. [8] and references therein), free electron lasers,[9] and condensed matter physics.[10]

On the other hand, systems with position-dependent mass (PDM) have been found to be very useful in studying the physical properties of various microstructures.[11–18] Recently, there has been increasing interest in searching for analytical solutions of the Dirac equation with PDM or constant mass under spin or pseudo-spin symmetries.[19–36]

In this paper, we consider both PDM and tensor interaction for attractive scalar and repulsive vector Coulomb potentials under the spin and pseudospin symmetry limits. We give some numerical results too.

## 2. Review of the Dirac equation including tensor coupling

The Dirac equation, which describes a nucleon in repulsive vector potential $V(r)$, attractive scalar potential $S(r)$, and tensor potential $U(r)$, is written as

$$\left[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta\left(M(r)c^2+S(r)\right)-\mathrm{i}\beta\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}}U(r)\right]\psi(\boldsymbol{r}) = \left[E-V(r)\right]\psi(\boldsymbol{r}), \tag{1}$$

where $M(r)$ is the effective mass of the fermionic particle, $E$ is the relativistic energy of the system, and $\boldsymbol{p}=-\mathrm{i}\hbar\boldsymbol{\nabla}$ is the three-dimensional momentum operator. $\boldsymbol{\alpha}$ and $\beta$ are the $4\times4$ usual Dirac matrices give by

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}\boldsymbol{I} & 0\\ 0 & -\boldsymbol{I}\end{pmatrix}, \tag{2}$$

where $\boldsymbol{I}$ is the $2\times2$ unitary matrix, and $\boldsymbol{\sigma}$ are three-vector spin matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{3}$$

The total angular momentum operator $\boldsymbol{J}$ and spin–orbit $K=(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, where $\boldsymbol{L}$ is the orbital angular momentum, of the spherical nucleons commute with the Dirac Hamiltonian. The eigenvalues of the spin–orbit coupling operator are $\kappa=(j+1/2)>0$ and $K=-(j+1/2)<0$ for unaligned spin $j=l-1/2$ and aligned spin $j=l+1/2$, respectively. The $(H^2,\ K,\ J^2,\ J_z)$ can be taken as the complete set of the conservative quantities. Thus, the spinor wave functions can be

*Project supported by the Scientific and Technical Research Council of Turkey.
†Corresponding author. E-mail: eshgi54@gmail.com
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classified according to their angular momentum $j$, spin–orbit quantum number $\kappa$, and radial quantum number $n$, and can be written as

$$\psi_{n\kappa}(\boldsymbol{r}) = \begin{pmatrix} f_{n\kappa}(\boldsymbol{r}) \\ g_{n\kappa}(\boldsymbol{r}) \end{pmatrix} = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r)\,\mathrm{Y}^{l}_{jm}(\theta,\varphi) \\ \mathrm{i}G_{n\kappa}(r)\,\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi) \end{pmatrix}, \quad (4)$$

where $f_{n\kappa}(\boldsymbol{r})$ is the upper (large) component and $g_{n\kappa}(\boldsymbol{r})$ is the lower (small) component of the Dirac spinors, $\mathrm{Y}^{l}_{jm}(\theta,\varphi)$ and $\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi)$ are spin and pseudo-spin spherical harmonics, respectively, and $m$ is the projection of the angular momentum on the $z$ axis. Substituting Eq. (4) into Eq. (1) and using the following relations:[37]

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}), \quad (5a)$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{P}) = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{P} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\right), \quad (5b)$$

with the properties

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{L}} \begin{Bmatrix} \mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) \\ \mathrm{Y}^{l}_{jm}(\theta,\phi) \end{Bmatrix} = \begin{cases} (\kappa-1)\mathrm{Y}^{l}_{jm}(\theta,\phi), \\ -(\kappa-1)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi), \end{cases} \quad (6a)$$

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} \begin{Bmatrix} \mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) \\ \mathrm{Y}^{l}_{jm}(\theta,\phi) \end{Bmatrix} = \begin{cases} -\mathrm{Y}^{l}_{jm}(\theta,\phi), \\ -\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi), \end{cases} \quad (6b)$$

we obtain two coupled differential equations for upper and lower radial wave functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right) F_{n\kappa}(r) = \left(M(r)c^2 + E_{n\kappa} - \Delta(r)\right) G_{n\kappa}(r), \quad (7a)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right) G_{n\kappa}(r) = \left(M(r)c^2 - E_{n\kappa} + \Sigma(r)\right) F_{n\kappa}(r), \quad (7b)$$

where

$$\Delta(r) = V(r) - S(r), \quad (8a)$$

$$\Sigma(r) = V(r) + S(r). \quad (8b)$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from Eq. (7), we finally obtain the following two Schrödinger-like differential equations for the upper and the lower radial spinor components:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{\mathrm{d}U(r)}{\mathrm{d}r} - U^2(r)\right] F_{n\kappa}(r) - \frac{\dfrac{\mathrm{d}M(r)}{\mathrm{d}r} - \dfrac{\mathrm{d}\Delta(r)}{\mathrm{d}r}}{M(r) + E_{n\kappa} - \Delta(r)} \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right) F_{n\kappa}(r) = \frac{1}{(\hbar c)^2}\left[\left(M(r)c^2 + E_{n\kappa} - \Delta(r)\right) \times \left(M(r)c^2 - E_{n\kappa} + \Sigma(r)\right)\right] F_{n\kappa}(r), \quad (9a)$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{\mathrm{d}U(r)}{\mathrm{d}r} - U^2(r)\right] G_{n\kappa}(r) - \frac{\dfrac{\mathrm{d}M(r)}{\mathrm{d}r} + \dfrac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}}{M(r) - E_{n\kappa} + \Sigma(r)} \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right) G_{n\kappa}(r) = \frac{1}{(\hbar c)^2}\left[\left(M(r)c^2 + E_{n\kappa} - \Delta(r)\right) \times \left(M(r)c^2 - E_{n\kappa} + \Sigma(r)\right)\right] G_{n\kappa}(r), \quad (9b)$$

where $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1) = l(l+1)$. These radial wave functions are required to satisfy the necessary boundary conditions. The spin–orbit quantum number $\kappa$ is related to the quantum numbers of spin symmetry $l$ and pseudo-spin symmetry $\tilde{l}$ by

$$\kappa = \begin{cases} -(l+1) = -\left(j+\dfrac{1}{2}\right)\ (\mathrm{s}_{1/2}, \mathrm{p}_{3/2}, \ldots), \\ \quad j = l + \dfrac{1}{2},\ \text{aligned spin}, & (\kappa < 0), \\ +l = +\left(j+\dfrac{1}{2}\right)\ (\mathrm{p}_{1/2}, \mathrm{d}_{3/2}, \ldots), \\ \quad j = l - \dfrac{1}{2},\ \text{unaligned spin}, & (\kappa > 0), \end{cases} \quad (10)$$

and the quasi-degenerate doublet structure can be expressed in terms of pseudo-spin angular momentum $\tilde{s} = 1/2$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as

$$\kappa = \begin{cases} -\tilde{l} = -\left(j+\dfrac{1}{2}\right)\ (\mathrm{s}_{1/2}, \mathrm{p}_{3/2}, \ldots), \\ \quad j = \tilde{l} - \dfrac{1}{2},\ \text{aligned pseudospin}, & (\kappa < 0), \\ +(\tilde{l}+1) = +\left(j+\dfrac{1}{2}\right)\ (\mathrm{d}_{3/2}, \mathrm{f}_{5/2}, \ldots), \\ \quad j = \tilde{l} + \dfrac{1}{2},\ \text{unaligned pseudospin}, & (\kappa > 0), \end{cases} \quad (11)$$

where $\kappa = \pm1, \pm2, \ldots$. For example, $\left(1\mathrm{s}_{1/2}, 0\mathrm{d}_{3/2}\right)$ and $\left(1\mathrm{p}_{3/2}, 0\mathrm{f}_{5/2}\right)$ can be considered as pseudo-spin doublets. In the following, we will consider the spin and the pseudospin symmetry cases.

## 3. Relativistic bound state solutions

### 3.1. Spin symmetry case

Equation (9a) cannot be solved analytically because of the last term

$$\frac{\mathrm{d}M(r)/\mathrm{d}r - \mathrm{d}\Delta(r)/\mathrm{d}r}{M(r) + E_{n\kappa} - \Delta(r)} \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right).$$

It is convenient to solve the mathematical relation $\mathrm{d}M(r)c^2/\mathrm{d}r = \mathrm{d}\Delta(r)/\mathrm{d}r = \mathrm{d}V(r)/\mathrm{d}r$.[38,39] By using the above relation, we can exactly solve Eq. (9a). In this stage, we take the vector potential in the form of an attractive Coulomb-like field[22]

$$\Sigma(r) = V(r) = -\frac{\hbar c q_{\mathrm{v}}}{r}, \quad q_{\mathrm{v}} = q, \quad r \neq 0, \quad (12)$$

where $q_{\rm v}$ is a vector dimensionless real coupling constant, and $\hbar c$ is a constant in units of J·fm. The mass function should be taken as[22]

$$M(r)=m_0+\frac{m_1}{r},\quad m_1=m_0\lambda_0 b,\quad \lambda_0=\frac{\hbar}{m_0c}, \tag{13}$$

where $m_0$ and $m_1$ stand for the rest mass of the fermionic particle and the perturbed mass, respectively, $b$ is the dimensionless real constant to be set to zero for the constant mass case, and $\lambda_0$ is the Compton-like wavelength in units of fm. The tensor interaction takes the form

$$U(r)=-\frac{T}{r},\quad T=\frac{Z_{\rm a}Z_{\rm b}e^2}{4\pi\varepsilon_0},\quad r\geq R_{\rm c}, \tag{14}$$

where $R_{\rm c}$ is the Coulomb radius, and $Z_{\rm a}$ and $Z_{\rm b}$ stand for the charges of the projectile particle a and the target nucleus b, respectively.

Substituting Eqs. (12)–(14) into Eq. (9a), considering the spin symmetry, $\Delta(r)=C_{\rm s}=$ const., i.e., $({\rm d}\Delta(r)/{\rm d}r=0)$.[40,41] Thus, the equation obtained for the upper component of the Dirac spinor $F_{nk}(r)$ becomes

$$\begin{aligned}&\left\{\frac{{\rm d}^2}{{\rm d}r^2}-\frac{(\kappa+T)(\kappa+T+1)+b(b-q)}{r^2}\right.\\&+\frac{1}{\hbar c}\frac{\left[q\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right)+b\left(C_{\rm s}-2m_0c^2\right)\right]}{r}\\&\left.-\frac{1}{(\hbar c)^2}\left(m_0c^2-E_{n\kappa}\right)\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right)\right\}F_{n\kappa}(r)=0.\end{aligned} \tag{15}$$

Further, defining new parameters

$$\begin{aligned}\lambda^2&=(\kappa+T)(\kappa+T+1)+b(b-q),\\\delta^2&=\frac{1}{\hbar c}\left[q\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right)+b\left(C_{\rm s}-2m_0c^2\right)\right],\\\varepsilon^2&=\frac{1}{(\hbar c)^2}\left(m_0c^2-E_{n\kappa}\right)\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right),\end{aligned} \tag{16}$$

and introducing $F_{n\kappa}(r)=r^{1/2}\phi(r)$, we turn Eq. (15) into

$$\left[\frac{{\rm d}^2}{{\rm d}r^2}+\frac{1}{r}\frac{\rm d}{{\rm d}r}+\frac{1}{r^2}\left(-\varepsilon^2r^2+\delta^2r-\nu^2\right)\right]\phi(r)=0, \tag{17a}$$

$$\nu^2=\left(\kappa+T+\frac{1}{2}\right)^2+b(b-q). \tag{17b}$$

Following Ref. [42], we can easily find that the energy eigenvalue solution of Eq. (17a) is

$$\varepsilon=\frac{\delta^2}{(1+2n+2\nu)}. \tag{18}$$

Using Eq. (16), we obtain the explicit form of the energy equation

$$\begin{aligned}&\left(m_0c^2-E_{n\kappa}\right)\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right)\\&=\frac{1}{4}\left[\frac{q\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right)+b\left(C_{\rm s}-2m_0c^2\right)}{n+\sqrt{(\kappa+T+1/2)^2+b(b-q)}+1/2}\right]^2,\\&n=0,1,2,\ldots\end{aligned} \tag{19}$$

In the absence of the tensor interaction, i.e. $T=0$, equation (19) reduces to Eq. (34) of Ref. [22] when $q\to 2q$. For the constant mass case, i.e., $b=0$, and no tensor interaction, we have

$$\begin{aligned}&\left(m_0c^2-E_{n\kappa}\right)\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right)\\&=\frac{1}{4}\left[\frac{q\left(m_0c^2+E_{n\kappa}-C_{\rm s}\right)}{n+\kappa+1}\right]^2,\quad n=0,1,2,\ldots\end{aligned} \tag{20}$$

When $C_{\rm s}=0$, $\kappa=l$, $\Sigma(r)=V(r)$, $m_0c^2-E_{n\kappa}\to -E_{nl}$, and $m_0c^2+E_{n\kappa}\to m_0$,, the above equation reduces to the non-relativistic energy limit (in units of $\hbar=1$)

$$E_{nl}=-\frac{2m_0q^2}{(n+l+1)^2}. \tag{21}$$

Further, using Eq. (38) of Ref. [42], we obtain the radial wave function

$$\phi(r)=r^{\nu}\,{\rm e}^{-\varepsilon r}{\rm L}_n^{2\nu}(2\varepsilon r), \tag{22}$$

and hence from $F_{n\kappa}(r)=r^{1/2}\phi(r)$, we find the upper-spinor component of the wave function as

$$\begin{aligned}&F_{n\kappa}(r)\\&=N\,{\rm e}^{-\varepsilon r}r^{\nu+1/2}{\rm L}_n^{2\nu}(2\varepsilon r)\\&=N\frac{\Gamma(n+2\nu+1)}{n!\Gamma(2\nu+1)}\,{\rm e}^{-\varepsilon r}r^{\nu+1/2}{}_1{\rm F}_1\left(-n,2\nu+1,2\varepsilon r\right),\end{aligned} \tag{23}$$

where $N$ is the normalization constant. We use the relation between the Laguerre polynomials and the confluent hypergeometric functions ${\rm L}_n^p(x)=\frac{\Gamma(n+p+1)}{n!\Gamma(p+1)}{}_1{\rm F}_1(-n,p+1,x)$.[43] The normalization constant in Eq. (23) can be found by using the relations[39]

$$\int_0^{\infty}\psi_{n\kappa}^*(r)\psi_{n\kappa}(r)\,{\rm d}r=1, \tag{24}$$

$$\begin{aligned}&\int_0^{\infty}x^{q+1}\,{\rm e}^{-x}{\rm L}_n^q(x){\rm L}_{n'}^q(x)\,{\rm d}x\\&=(2n+q+1)\frac{\Gamma(n+q+1)}{n!}\delta_{nn'},\end{aligned} \tag{25}$$

as shown in Appendix A. The wave function (23) satisfies the boundary conditions, i.e., $F_{n\kappa}(r=0)=0$ and $F_{n\kappa}(r\to\infty)\to 0$. The lower spinor wave function can be obtained via

$$\begin{aligned}G_{n\kappa}(r)&=\frac{1}{\left(m_0c^2+\dfrac{\hbar cb}{r}+E_{n\kappa}-C_{\rm s}\right)}\\&\times\left[\left(-\varepsilon+\frac{\nu+1/2}{r}+\frac{\kappa+T}{r}\right)F_{n\kappa}(r)\right.\\&\left.-2N\varepsilon\,{\rm e}^{-\varepsilon r}r^{\nu+1/2}{\rm L}_{n-1}^{2\nu+1}(2\varepsilon r)\right],\end{aligned} \tag{26}$$

where we have used

$$\frac{\rm d}{{\rm d}x}{\rm L}_n^{\alpha}(x)=-{\rm L}_{n-1}^{(\alpha+1)}(x)=\frac{1}{x}\left[n{\rm L}_n^{\alpha}(x)-(n+\alpha){\rm L}_{n-1}^{\alpha}(x)\right].$$

In Tables 1–3 with $\hbar=c=1$, we give some numerical results for the energy eigenvalues from energy formula (19).

**Table 1.** The bound state energy eigenvalues of the Coulomb potential under the spin symmetry limit for several values of $n$ and $\kappa$ with $m_0 = 5.0\ \mathrm{fm}^{-1}$, $\hbar = c = 1$,, $q = 1$, $T = (0, 1, 5, 10)$, and $C_s = 1$.

| $l$ | $n, \kappa < 0$ | $(l, j = l + 1/2)$ | $m_1$ | $E_{n,\kappa<0}$ | | | |
|---|---|---|---|---|---|---|---|
| | | | | $T = 0$ | $T = 1$ | $T = 5$ | $T = 10$ |
| 1 | 0, –2 | $0p_{3/2}$ | 0.0 | 4.47059 | 3.20000 | 4.86154 | 4.97231 |
| | | | 0.2 | 4.64226 | 3.35948 | 4.91032 | 4.98224 |
| | | | 0.4 | 4.79160 | 3.55969 | 4.94917 | 4.98999 |
| | | | 0.5 | 4.85392 | 3.68198 | 4.96463 | 4.99305 |
| 2 | 0, –3 | $0d_{5/2}$ | 0.0 | 4.75676 | 3.20000 | 4.86154 | 4.96498 |
| | | | 0.2 | 4.84084 | 4.64226 | 4.84084 | 4.97752 |
| | | | 0.4 | 4.90920 | 4.79160 | 4.90920 | 4.98733 |
| | | | 0.5 | 4.93670 | 4.85392 | 4.93670 | 4.99120 |
| 3 | 0, –4 | $0f_{7/2}$ | 0.0 | 4.86154 | 4.75676 | 4.47059 | 4.95431 |
| | | | 0.2 | 4.91032 | 4.84084 | 4.64226 | 4.97065 |
| | | | 0.4 | 4.94917 | 4.9092 | 4.79160 | 4.98345 |
| | | | 0.5 | 4.96463 | 4.9367 | 4.85392 | 4.98850 |
| 4 | 0, –5 | $0g_{9/2}$ | 0.0 | 4.91089 | 4.86154 | 3.20000 | 4.93793 |
| | | | 0.2 | 4.94254 | 4.91032 | 3.35948 | 4.96007 |
| | | | 0.4 | 4.96753 | 4.94917 | 3.55969 | 4.97747 |
| | | | 0.5 | 4.97742 | 4.96463 | 3.68198 | 4.98434 |
| 1 | 1, –2 | $1p_{3/2}$ | 0.0 | 4.75676 | 4.47059 | 4.91089 | 4.97756 |
| | | | 0.2 | 4.83851 | 4.58615 | 4.94243 | 4.98561 |
| | | | 0.4 | 4.90714 | 4.70744 | 4.96743 | 4.99189 |
| | | | 0.5 | 4.93519 | 4.76907 | 4.97735 | 4.99437 |
| 2 | 1, –3 | $1d_{5/2}$ | 0.0 | 4.86154 | 4.75676 | 4.86154 | 4.97231 |
| | | | 0.2 | 4.90991 | 4.83851 | 4.90991 | 4.98223 |
| | | | 0.4 | 4.94881 | 4.90714 | 4.94881 | 4.98999 |
| | | | 0.5 | 4.96437 | 4.93519 | 4.96437 | 4.99305 |
| 3 | 1, –4 | $1f_{7/2}$ | 0.0 | 4.91089 | 4.86154 | 4.75676 | 4.96498 |
| | | | 0.2 | 4.94243 | 4.90991 | 4.83851 | 4.97752 |
| | | | 0.4 | 4.96743 | 4.94881 | 4.90714 | 4.98733 |
| | | | 0.5 | 4.97735 | 4.96437 | 4.93519 | 4.99119 |
| 4 | 1, –5 | $1g_{9/2}$ | 0.0 | 4.93793 | 4.91089 | 4.47059 | 4.95431 |
| | | | 0.2 | 4.96003 | 4.94243 | 4.58615 | 4.97063 |
| | | | 0.4 | 4.97743 | 4.96743 | 4.70744 | 4.98344 |
| | | | 0.5 | 4.98431 | 4.97735 | 4.76907 | 4.98849 |

**Table 2.** The bound state energy eigenvalues of the Coulomb potential under the spin symmetry limit for several values of $n$ and $\kappa$ with $m_0 = 5.0\ \mathrm{fm}^{-1}$, $q = 1$, $\hbar = c = 1$, $T = (0, 1, 5, 10)$, and $C_s = 1$.

| $l$ | $n, \kappa > 0$ | $(l, j = l - 1/2)$ | $m_1$ | $E_{n,\kappa>0}$ | | | |
|---|---|---|---|---|---|---|---|
| | | | | $T = 0$ | $T = 1$ | $T = 5$ | $T = 10$ |
| 2 | 0, 2 | $0d_{3/2}$ | 0.0 | 4.75676 | 4.86154 | 4.96498 | 4.98671 |
| | | | 0.2 | 4.84084 | 4.91032 | 4.97752 | 4.99148 |
| | | | 0.4 | 4.90920 | 4.94917 | 4.98733 | 4.99521 |
| | | | 0.5 | 4.93670 | 4.96463 | 4.99120 | 4.99667 |
| 3 | 0, 3 | $0f_{5/2}$ | 0.0 | 4.86154 | 4.91089 | 4.97231 | 4.98854 |
| | | | 0.2 | 4.91032 | 4.94254 | 4.98224 | 4.99266 |
| | | | 0.4 | 4.94917 | 4.96753 | 4.98999 | 4.99587 |
| | | | 0.5 | 4.96463 | 4.97742 | 4.99305 | 4.99713 |
| 4 | 0, 4 | $0g_{7/2}$ | 0.0 | 4.91089 | 4.93793 | 4.97756 | 4.99001 |
| | | | 0.2 | 4.94254 | 4.96007 | 4.98561 | 4.99360 |
| | | | 0.4 | 4.96753 | 4.97747 | 4.99190 | 4.99640 |
| | | | 0.5 | 4.97742 | 4.98434 | 4.99437 | 4.99750 |
| 5 | 0, 5 | $0h_{9/2}$ | 0.0 | 4.93793 | 4.95431 | 4.98144 | 4.99122 |
| | | | 0.2 | 4.96007 | 4.97065 | 4.98811 | 4.99438 |
| | | | 0.4 | 4.97747 | 4.98345 | 4.99330 | 4.99684 |
| | | | 0.5 | 4.98434 | 4.98850 | 4.99535 | 4.99780 |

**Table 2.** (*Continued*)

| $l$ | $n,\kappa>0$ | $(l, j=l-1/2)$ | $m_1$ | $E_{n,\kappa>0}$ | | | |
|---|---|---|---|---|---|---|---|
| | | | | $T=0$ | $T=1$ | $T=5$ | $T=10$ |
| 2 | 1, 2 | $1d_{3/2}$ | 0.0 | 4.86154 | 4.91089 | 4.97231 | 4.98854 |
| | | | 0.2 | 4.90991 | 4.94243 | 4.98223 | 4.99266 |
| | | | 0.4 | 4.94881 | 4.96743 | 4.98999 | 4.99587 |
| | | | 0.5 | 4.96437 | 4.97735 | 4.99305 | 4.99713 |
| 3 | 1, 3 | $1f_{5/2}$ | 0.0 | 4.91089 | 4.93793 | 4.97756 | 4.99001 |
| | | | 0.2 | 4.94243 | 4.96003 | 4.98561 | 4.99360 |
| | | | 0.4 | 4.96743 | 4.97743 | 4.99189 | 4.99640 |
| | | | 0.5 | 4.97735 | 4.98431 | 4.99437 | 4.99750 |
| 4 | 1, 4 | $1g_{7/2}$ | 0.0 | 4.93793 | 4.95431 | 4.98144 | 4.99122 |
| | | | 0.2 | 4.96003 | 4.97063 | 4.98810 | 4.99438 |
| | | | 0.4 | 4.97743 | 4.98344 | 4.99330 | 4.99684 |
| | | | 0.5 | 4.98431 | 4.98849 | 4.99535 | 4.99780 |
| 5 | 1, 5 | $1h_{9/2}$ | 0.0 | 4.95431 | 4.96498 | 4.98440 | 4.99222 |
| | | | 0.2 | 4.97063 | 4.97752 | 4.99000 | 4.99502 |
| | | | 0.4 | 4.98344 | 4.98733 | 4.99437 | 4.99720 |
| | | | 0.5 | 4.98849 | 4.99119 | 4.99609 | 4.99805 |

**Table 3.** The bound state energy eigenvalues of the Coulomb potential for several states under the spin symmetry limit with $m_0 = 5.0\ \mathrm{fm}^{-1}$, $q=1$, $\hbar=c=1$, $m_1=(0,0.5)$, and $T=(0,10)$.

| $l$ | $n,\kappa<0$ | $(l, j=l+1/2)$ | $C_s$ | $E_{n,\kappa<0}$ | | | |
|---|---|---|---|---|---|---|---|
| | | | | $T=0$ | $T=0$ | $T=10$ | $T=10$ |
| | | | | $m_1=0$ | $m_1=0.5$ | $m_1=0$ | $m_1=0.5$ |
| 1 | 0, –2 | $0p_{3/2}$ | 1.00 | 4.47059 | 4.85392 | 4.97231 | 4.99305 |
| | | | 1.25 | 4.48529 | 4.85798 | 4.97308 | 4.99324 |
| | | | 1.50 | 4.50000 | 4.86204 | 4.97385 | 4.99344 |
| | | | 1.75 | 4.51471 | 4.86610 | 4.97462 | 4.99363 |
| | | | 2.00 | 4.52941 | 4.87015 | 4.97538 | 4.99382 |
| | | | 2.25 | 4.54412 | 4.87421 | 4.97615 | 4.99401 |
| | | | 2.50 | 4.55882 | 4.87827 | 4.97692 | 4.99421 |
| | | | 2.75 | 4.57353 | 4.88233 | 4.97769 | 4.99440 |
| 3 | 0, –4 | $0f_{7/2}$ | 1.00 | 4.86154 | 4.96463 | 4.95431 | 4.98850 |
| | | | 1.25 | 4.86538 | 4.96561 | 4.95558 | 4.98882 |
| | | | 1.50 | 4.86923 | 4.96659 | 4.95685 | 4.98914 |
| | | | 1.75 | 4.87308 | 4.96757 | 4.95812 | 4.98946 |
| | | | 2.00 | 4.87692 | 4.96856 | 4.95939 | 4.98978 |
| | | | 2.25 | 4.88077 | 4.96954 | 4.96066 | 4.99010 |
| | | | 2.50 | 4.88462 | 4.97052 | 4.96193 | 4.99042 |
| | | | 2.75 | 4.88846 | 4.97150 | 4.96320 | 4.99074 |
| 1 | 1, –2 | $1p_{3/2}$ | 1.00 | 4.75676 | 4.93519 | 4.97756 | 4.99437 |
| | | | 1.25 | 4.76351 | 4.93699 | 4.97818 | 4.99453 |
| | | | 1.50 | 4.77027 | 4.93879 | 4.97880 | 4.99468 |
| | | | 1.75 | 4.77703 | 4.94059 | 4.97943 | 4.99484 |
| | | | 2.00 | 4.78378 | 4.94239 | 4.98005 | 4.99499 |
| | | | 2.25 | 4.79054 | 4.94419 | 4.98067 | 4.99515 |
| | | | 2.50 | 4.79730 | 4.94599 | 4.98130 | 4.99531 |
| | | | 2.75 | 4.80405 | 4.94779 | 4.98192 | 4.99546 |
| 3 | 1, –4 | $1f_{7/2}$ | 1.00 | 4.91089 | 4.97735 | 4.96498 | 4.99119 |
| | | | 1.25 | 4.91337 | 4.97797 | 4.96595 | 4.99144 |
| | | | 1.50 | 4.91584 | 4.97860 | 4.96693 | 4.99168 |
| | | | 1.75 | 4.91832 | 4.97923 | 4.96790 | 4.99193 |
| | | | 2.00 | 4.92079 | 4.97986 | 4.96887 | 4.99217 |
| | | | 2.25 | 4.92327 | 4.98049 | 4.96984 | 4.99242 |
| | | | 2.50 | 4.92574 | 4.98112 | 4.97082 | 4.99266 |
| | | | 2.75 | 4.92822 | 4.98175 | 4.97179 | 4.99291 |

### 3.2. Pseudo-spin symmetry case

To avoid the repetition in solving Eq. (9b), we can obtain the negative energy solution for pseudospin symmetry directly from the above positive energy solution for spin symmetry using the parameter mapping[22]

$$F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r); \quad V(r) \to -V(r)(q \to -q);$$
$$\kappa + 1 \to \kappa; \quad E_{n\kappa} \to -E_{n\kappa}; \quad C_{\rm s} \leftrightarrow -C_{\rm ps}. \tag{27}$$

Following the previous results with the above transformations, we finally arrive at the energy equation

$$\left(m_0c^2 + E_{n\kappa}\right)\left(m_0c^2 - E_{n\kappa} + C_{\rm ps}\right) = \frac{1}{4}\left[\frac{q\left(m_0c^2 - E_{n\kappa} + C_{\rm ps}\right) + b\left(C_{\rm ps} + 2m_0c^2\right)}{n + \sqrt{(\kappa + T - 1/2)^2 + b(b+q)} + 1/2}\right]^2, \quad n = 0, 1, 2, \ldots \tag{28}$$

In the absence of the tensor interaction, i.e. $T = 0$, equation (28) reduces to Eq. (52) of Ref. [22] when $q \to 2q$. For the constant mass case, i.e., $b = 0$, and no tensor interaction, we have

$$\left(m_0c^2 + E_{n\kappa}\right)\left(m_0c^2 - E_{n\kappa} + C_{\rm ps}\right) = \frac{1}{4}\left[\frac{q\left(m_0c^2 - E_{n\kappa} + C_{\rm ps}\right)}{n + \kappa}\right]^2, \quad n = 0, 1, 2, \ldots \tag{29}$$

The lower-spinor component of the wave function is

$$G_{n\kappa}(r) = N\mathrm{e}^{-\tilde{\varepsilon}r}r^{\tilde{\nu}+1/2}\mathrm{L}_n^{2\tilde{\nu}}(2\tilde{\varepsilon}r) = N\frac{\Gamma(n+2\tilde{\nu}+1)}{n!\Gamma(2\tilde{\nu}+1)}\mathrm{e}^{-\tilde{\varepsilon}r}r^{\tilde{\nu}+1/2}{}_1\mathrm{F}_1\left(-n, 2\tilde{\nu}+1, 2\tilde{\varepsilon}r\right), \tag{30}$$

where

$$\tilde{\varepsilon}^2 = \frac{1}{(\hbar c)^2}\left(m_0c^2 + E_{n\kappa}\right)\left(m_0c^2 - E_{n\kappa} + C_{\rm ps}\right),$$
$$\tilde{\nu}^2 = \left(\kappa + T - \frac{1}{2}\right)^2 + b(b+q),$$

and $N$ is the normalization constant calculated in Appendix A. The upper-spinor wave function can be obtained via

$$F_{n\kappa}(r) = \frac{1}{\left(m_0c^2 + \dfrac{\hbar cb}{r} - E_{n\kappa} + C_{\rm ps}\right)} \times \left[\left(-\tilde{\varepsilon} + \frac{\tilde{\nu}+1/2}{r} - \frac{\kappa+T}{r}\right)G_{n\kappa}(r) - 2\tilde{N}\tilde{\varepsilon}\,\mathrm{e}^{-\tilde{\varepsilon}r}r^{\tilde{\nu}+1/2}\mathrm{L}_{n-1}^{2\tilde{\nu}+1}(2\tilde{\varepsilon}r)\right]. \tag{31}$$

## 4. Numerical results

In Tables 1–3, we can see that bound states such as $(np_{1/2}, np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{9/2}), \ldots$ (where each pair is considered as a spin doublet) in the absence of the tensor potential are degenerate, but in the presence of the tensor potential the degeneracies are removed. We also investigate the effects of parameters $m_1$ and $C_{\rm s}$ on the bound states under the conditions of the spin symmetry limits for $T = 0$. The results are given in Tables 1–3. It is readily seen that if parameters $m_1$ and $C_{\rm s}$ increase, the bound state energy eigenvalues for several states of this potential increase, and it is shown that the energy eigenvalues decrease with decreasing $m_1$ and increase with increasing tensor strength $T$ for $\kappa > 0$ and $\kappa < 0$. The increase in the energies is slight when strength $T$ is large, and small when strength $T$ is small. In Table 3, for constant values of $C_{\rm s}$ and $T$, the energy increases as $m_1$ increases. The decrease in the energy values is large without the tensor interaction, while small in the presence of the tensor interaction. Furthermore, when $C_{\rm s}$ increases, the energy increases.

## 5. Conclusion

In this paper, the relativistic equation for particles with spin 1/2 was solved exactly with both spatially-dependent mass and tensor interaction for attractive scalar and repulsive vector Coulomb potentials under the spin symmetry limit. Some numerical results were also given. The effects of tensor interaction on the bound states were presented (the tensor interaction removes degeneracy between two states in spin doublets). We also investigated the effects of the spatially-dependent mass on the bound states under the conditions of spin symmetry limits for $T = 0$.

## Appendix A: The normalization constant of the radial wave function

Unlike the non-relativistic case, the normalization condition for the Dirac spinor combines the two individual normalization constants $N_{n\kappa}$ in one single integral.[44] The radial wave functions are normalized according to formula (24), which explicitly implies for the two-spinor components $f_{n\kappa}(r)$ and $g_{n\kappa}(r)$ in Eq. (4) that[44]

$$\int_0^\infty \left(f_{n\kappa}^2(r) + g_{n\kappa}^2(r)\right)r^2\,\mathrm{d}r = \int_0^\infty \left(F_{n\kappa}^2(r) + G_{n\kappa}^2(r)\right)\mathrm{d}r = 1. \tag{A1}$$

The upper full radial wave function is

$$F_{n\kappa}(r) = Nr^{\nu+1/2}\,\mathrm{e}^{-\varepsilon r}{}_1\mathrm{F}_1\left(-n; 2\nu+1; 2\varepsilon r\right), \tag{A2}$$

with $\varepsilon$ and $\nu$ defined in Eqs. (16) and (17b), respectively. Further, the lower radial wave function is

$$G_{n\kappa}(r) = Nr^{\tilde{\nu}+1/2}\,\mathrm{e}^{-\tilde{\varepsilon}r}{}_1\mathrm{F}_1\left(-n; 2\tilde{\nu}+1; 2\tilde{\varepsilon}r\right), \tag{A3}$$

with $\tilde{\varepsilon}$ and $\tilde{\nu}$ following Eq. (30). Therefore, inserting relations (A2) and (A3) into Eq. (A1) gives

$$N^2\left[\left(\frac{n!\Gamma(2\nu+1)}{\Gamma(n+2\nu+1)}\right)^2\int_0^\infty r^{2\nu+1}\mathrm{e}^{-2\varepsilon r}\left[\mathrm{L}_n^{2\nu}(2\varepsilon r)\right]^2\mathrm{d}r + \left(\frac{n!\Gamma(2\tilde{\nu}+1)}{\Gamma(n+2\tilde{\nu}+1)}\right)^2\int_0^\infty r^{2\tilde{\nu}+1}\mathrm{e}^{-2\tilde{\varepsilon}r}\left[\mathrm{L}_n^{2\tilde{\nu}}(2\tilde{\varepsilon}r)\right]^2\mathrm{d}r\right]=1, \tag{A4}$$

and we finally obtain

$$N=\left[\left(\frac{\Gamma(2\nu+1)}{(2\varepsilon)^{\nu+1}}\right)^2\frac{n!(2n+2\nu+1)}{\Gamma(n+2\nu+1)} + \left(\frac{\Gamma(2\tilde{\nu}+1)}{(2\tilde{\varepsilon})^{\tilde{\nu}+1}}\right)^2\frac{n!(2n+2\tilde{\nu}+1)}{\Gamma(n+2\tilde{\nu}+1)}\right]^{-1/2}. \tag{A5}$$

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