Modified Burgers' equation by the local discontinuous Galerkin method*

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In this paper, we present the local discontinuous Galerkin method for solving Burgers' equation and the modified Burgers' equation. We describe the algorithm formulation and practical implementation of the local discontinuous Galerkin method in detail. The method is applied to the solution of the one-dimensional viscous Burgers' equation and two forms of the modified Burgers' equation. The numerical results indicate that the method is very accurate and efficient.

Keywords: local discontinuous Galerkin method, modified Burgers' equation

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1. Introduction

In this work, we apply the local discontinuous Galerkin (LDG) methods to copy the nonlinear convection-diffusion equation of the form[1]

$$u_t + f(u)_x - \nu u_{xx} = 0, \quad (L_0 \le x \le L_1, t > t_0), \tag{1}$$

with Dirichlet boundary conditions. The equations under consideration are the Burgers' equations[2] of the form

$$u_t + uu_x - \nu u_{xx} = 0.$$

The two forms of the modified Burgers' equation (denoted by I[3] and II,[4] respectively) are given by

$$u_t + u^\delta u_x - \nu u_{xx} = 0,$$
$$u_t + (v+u)u_x - \nu u_{xx} = 0,$$

where v and ν are constants, and δ is a positive integer with $\delta \ge 2$. Burgers' equation was initially given by Bateman in 1915.[5] This equation was introduced in particular by Burgers[6] as the simplest differential model for a fluid flow, and is therefore often called the viscous Burgers' equation. The equation is applicable for various fields such as gas dynamics, heat conduction, and elasticity. When ν approaches zero, it becomes the inviscid Burgers' equation, which is a prototype of equations that develop shock waves and has important applications in physics.[7] The modified Burgers' equation I has strong nonlinear aspects and has been used in many practical transport problems, such as nonlinear waves in a medium with low-frequency pumping or absorption, ion reflection at quasi-perpendicular shocks, turbulence transport, the transport and dispersion of pollutants in rivers, and sediment transport.[3] The modified Burgers' equation II was proposed in Ref. [4] to judge whether the numerical method is able to resolve the large gradient regions.

In recent years, a class of powerful mathematical methods has been presented to solve the Burgers-type equation.[8,9] Many researchers have also constructed various numerical schemes for MBE to obtain approximate solutions. Recently, Bratsos proposed a fourth-order numerical scheme for MBE in Ref. [3]. Ramadan and El-Danaf[10] studied a numerical solution of MBE by the collocation method with quintic splines. Burgers' equation and MBE were solved with the use of cubic B-spline Galerkin finite-element methods,[11] and a quintic B-spline collocation method.[12] Duan *et al.* introduced a lattice Boltzmann model to solve the MBE.[13] In this paper, we develop a local discontinuous Galerkin method for solving the MBE. The LDG method was introduced by Cockburn and Shu[1] in 1998 based on the method of Bassi and Rebay.[14] This method has several advantages such as flexibility for arbitrarily unstructured meshes, program accessibility, high parallelizability, and the ability to easily accommodate h–p adaptivity. Hence, in recent years, LDG has been widely used in a broader range of applications, including viscous fluid flows, advection–diffusion problems, viscous magnetohydrodynamics, and semi-conductor modeling.[15] The idea of the LDG method is to rewrite Eq. (1) with second-order derivatives into systems of first-order equations. The systems are then discretized using the DG space discretization technique. Finally, an explicit and total variation diminishing (TVD) Runge–Kutta method is applied to solve the ordinary differential equa-

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tion systems obtained from the LDG spatial discretization.

The rest of the paper is organized as follows. Section 2 is devoted to a concise description of the local discontinuous Galerkin (LDG) spatial discretization and TVD time discretization. In Section 3 we discuss the test problem formulations, and then the numerical results obtained by LDG are compared with the exact solution. Finally, we provide conclusions in Section 4.

2. The LDG method

In this section, we illustrate the essential ideas of the LDG method in detail. To this purpose, we introduce an auxiliary variable, $q=\sqrt{\nu}u_x$, and rewrite Eq. (1) as follows:

$$u_t+f(u)_x-(\sqrt{\nu}q)_x=0,\quad (L_0\le x\le L_1, t>t_0), \tag{2}$$

$$q-(\sqrt{\nu}u)_x=0,\quad (L_0\le x\le L_1, t>t_0). \tag{3}$$

The computational domain $[L_0,L_1]$ is divided into N elements $L_0=x_{\frac{1}{2}}<x_{\frac{3}{2}}<\cdots<x_{N+\frac{1}{2}}=L_1$. Let us denote the cell by $I_j=[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]$ with the center of the cell $x_j=(1/2)(x_{j-\frac{1}{2}}+x_{j+\frac{1}{2}})$ and the size of each cell $\Delta x_j=x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}$. The solution and test function space are defined by a local orthogonal basis over I_j, $V_h^k=v_l^j(x)$ $(l=0,1,\ldots,k)$, and the basis functions are defined here by scaled Legendre polynomials:

$$v_0^j(x)=1,\quad v_1^j(x)=\frac{x-x_j}{\Delta x_j/2},$$
$$v_2^j(x)=\frac{3}{2}\left(\frac{x-x_j}{\Delta x_j/2}\right)^2-\frac{1}{2},\ldots$$

Then the numerical solution in the space V_h^k can be written as follows:

$$u_{\rm h}(x,t)=\sum_{l=0}^{k}u_j^l(t)v_1^j(x)=\boldsymbol{V}^{\rm T}(x)\boldsymbol{U}_{\rm h}(t), \tag{4}$$

for $x\in I_j$. The $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ are values of u at $x_{j+\frac{1}{2}}$ from the right cell, I_{j+1}, and the left cell, I_j, respectively.

The semi-discrete LDG method is applied to the system expressed by Eqs. (2) and (3). Since $u_{\rm h},q_{\rm h}\in V_{\rm h}^k$, for all test functions $v,w\in V_{\rm h}^k$, we have

$$\int_{I_j}(u_{\rm h})_t v\,{\rm d}x-\int_{I_j}(f(u_{\rm h})-\sqrt{\nu}q_{\rm h})v_x\,{\rm d}x$$
$$+(\hat{f}-\sqrt{\nu}\hat{q})_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}}-(\hat{f}-\sqrt{\nu}\hat{q})_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}}=0, \tag{5}$$
$$\int_{I_j}q_{\rm h}w\,{\rm d}x+\int_{I_j}\sqrt{\nu}u_{\rm h}w_x\,{\rm d}x$$
$$-\sqrt{\nu}\hat{u}_{j+\frac{1}{2}}w^-_{j+\frac{1}{2}}+\sqrt{\nu}\hat{u}_{j-\frac{1}{2}}w^+_{j-\frac{1}{2}}=0, \tag{6}$$

for all cells $1\le j\le N$. The "hat" terms in Eqs. (5) and (6) are numerical fluxes, namely single-valued functions defined at the cell interfaces, which typically depend on the discontinuous numerical solutions from both sides of the interface. There are two types of numerical fluxes to be defined. One is the diffusion numerical fluxes, $\hat{u}$, and $\hat{q}$, and the other the convection numerical flux, $\hat{f}$, which was discussed in detail in Ref. [16]. The numerical fluxes associated with diffusion are given as follows:[17]

$$\hat{q}=0.5(q^++q^-)-C_{11}(u^--u^+)-C_{12}(q^--q^+), \tag{7}$$
$$\hat{u}=0.5(u^++u^-)+C_{12}(u^--u^+). \tag{8}$$

It should be noticed that the numerical flux on boundary nodes $j=1$ and $j=N+1$ must take the boundary condition into consideration, and the flux takes the following form:

$$\hat{u}_{\frac{1}{2}}=u(L_0,t),\quad \hat{u}_{N+\frac{1}{2}}=u(L_1,t),$$
$$\hat{q}_{\frac{1}{2}}=\hat{q}^+_{\frac{1}{2}}-C_{11}(u^-_{\frac{1}{2}}-u^+_{\frac{1}{2}}),$$
$$\hat{q}_{N+\frac{1}{2}}=\hat{q}^-_{N+\frac{1}{2}}-C_{11}(u^-_{N+\frac{1}{2}}-u^+_{N+\frac{1}{2}}).$$

When $C_{11}=0$ and $C_{12}=0.5$ or -0.5, the fluxes are called alternating fluxes,[1] while the fluxes are called central fluxes when $C_{11}=C_{12}=0$. The convective numerical flux, $\hat{f}=\hat{f}(u^-,u^+)$, should be chosen to satisfy: (i) locally Lipschitz continuous and consistent $(\hat{f}(u,u)=f(u))$, (ii) non-decreasing in the first argument and non-increasing in the second argument, and (iii) allowance for a local resolution $q_{\rm h}$ in terms of $u_{\rm h}$. Here, we adopt the simple local Lax–Friedrichs flux

$$\hat{f}(u^-,u^+)=\frac{1}{2}(f(u^-)+f(u^+)-\alpha(u^+-u^-)),$$
$$\alpha=\max_{\min(u^-,u^+)\le u\le\max(u^-,u^+)}|f'(u)|. \tag{9}$$

The essence of the local discontinuous Galerkin method is that we can solve $q_{\rm h}$ explicitly and locally (in cell I_j) in terms of $u_{\rm h}$ by inverting the element mass matrix inside cell I_j. Thus, we eliminate the equation for $q_{\rm h}$, and obtain the combined ordinary differential equation system for freedoms $\boldsymbol{U}_{\rm h}$ as follows:

$$\frac{{\rm d}}{{\rm d}t}\boldsymbol{U}_{\rm h}=L(\boldsymbol{U}_{\rm h},t). \tag{10}$$

This semi-discrete scheme is discretized in time by the third-order TVD Runge–Kutta method as follows:

$$\boldsymbol{U}_{\rm h}^{(1)}=\boldsymbol{U}_{\rm h}^n+\Delta tL(\boldsymbol{U}_{\rm h}^n,t^n), \tag{11}$$
$$\boldsymbol{U}_{\rm h}^{(2)}=\frac{3}{4}\boldsymbol{U}_{\rm h}^n+\frac{1}{4}\boldsymbol{U}_{\rm h}^{(1)}+\frac{1}{4}\Delta tL(\boldsymbol{U}_{\rm h}^{(1)},t^n+\Delta t), \tag{12}$$
$$\boldsymbol{U}_{\rm h}^{(3)}=\frac{1}{3}\boldsymbol{U}_{\rm h}^n+\frac{2}{3}\boldsymbol{U}_{\rm h}^{(2)}+\frac{2}{3}\Delta tL\left(\boldsymbol{U}_{\rm h}^{(2)},t^n+\frac{1}{2}\Delta t\right). \tag{13}$$

3. Results and discussion

In order to test the ability of the LDG method to solve the nonlinear convection–diffusion equation, we consider three

different problems. The first problem is the viscous Burgers' equation, the second the modified Burgers' equation I with δ=2 and 3, and the third is the modified Burgers' equation II. All three problems have exact solutions, which can be used to check our method rigorously. In the first subsection, we discuss the problem formulations, followed by the numerical results.

3.1. Problem formulations

Problem 1 The test problem of the one-dimensional viscous Burgers' equation is given by

$$u_t + uu_x - \nu u_{xx} = 0, \quad (0 < x < 1, t > 0), \tag{14}$$

with initial and boundary conditions

$$\begin{aligned} &u(x,0) = sin(\pi x), \quad (0 \le x \le 1), \\ &u(0,t) = u(1,t) = 0, \quad (t > 0). \end{aligned} \tag{15}$$

The exact solution of this equation is in the form of the infinite series[18]

$$u(x,t) = 4\pi\nu \frac{C}{D+E}, \tag{16}$$

where

$$\begin{aligned} C &= \sum_{n=1}^{\infty} nI_n\left(\frac{1}{2\pi\nu}\right)\sin(n\pi x)\exp(-n^2\pi^2\nu t), \\ D &= 2\sum_{n=1}^{\infty} I_n\left(\frac{1}{2\pi\nu}\right)\cos(n\pi x)\exp(-n^2\pi^2\nu t), \\ E &= I_0\left(\frac{1}{2\pi\nu}\right), \end{aligned}$$

and $I_n(x)$ is the first type of the n-th modified Bessel function. When $n = 50$, it is used as an approximation to the infinite sum (16).

Problem 2 The modified Burgers' equation I with $\delta = 2$ and $\delta = 3$ will be denoted as MBE2 and MBE3, respectively.

The MBE2 equation

$$u_t + u^2 u_x - \nu u_{xx} = 0, \quad (0 < x < 1, t \ge 1) \tag{17}$$

has the analytical solution

$$u(x,t) = \frac{x}{t}\left[1 + \frac{t}{c_0}\exp\left(\frac{x^2}{4\nu t}\right)\right]^{-1}, \quad (0 \le x \le 1, t \ge 1), \tag{18}$$

where $c_0 \in (0,1)$ is constant. In our numerical solution, the case of $c_0 = 0.5$ is considered, and the initial condition $u(x,1)$ is given based on the exact solution (18).

The MBE3 equation

$$u_t + u^3 u_x - \nu u_{xx} = 0, \quad (0 < x < \pi, t \ge 0) \tag{19}$$

has an asymptotic solution with the following initial condition and boundary condition:

$$u(x,0) = \sin(x), \quad u(0,t) = u(\pi,t) = 0.$$

For a detailed description of the asymptotic solution, we refer the readers to Ref. [9].

Problem 3 For the third test problem, we consider the modified Burgers' equation II

$$u_t + (v+u)u_x - \nu u_{xx} = 0, \quad (-\infty < x < \infty, t > 0). \tag{20}$$

The exact solution of this problem is a shock wave

$$u(x,t) = -\tanh\left(\frac{x - x_0 - vt}{2\nu}\right). \tag{21}$$

The initial and boundary conditions are determined by Eq. (19). For numerical purposes, we can consider a finite domain due to the exponential decay of the solution. Consequently, for $\nu = 0.01$, $x_0 = -0.25$, $v = 1$, and $0 \le t \le 0.5$, we can choose the computational domain as $x \in [-1,1]$ with Dirichlet boundary conditions.

3.2. Numerical results

We now obtain the numerical solutions of the modified Burgers' equation by the LDG method. So far, all the problems are solved by the third-order scheme (piecewise quadratic cases), namely $k = 2$ in Eq. (4). The coefficient $C_{11} = 0$ and $C_{12} = 0$ can give the optimal convergence result from numerous numerical tests. For the time step, we emphasize that the size of the time step must be strictly restricted. To illustrate the performance of this method, we define L_∞ and L_2 norms as

$$\begin{aligned} L_\infty &= \max_{1 \le j \le N}\{|u(x_j,t^n) - u_{\rm h}(x_j,t^n)\}, \\ L_2 &= \sqrt{h\sum_{j=1}^{N}(u(x_j,t^n) - u_{\rm h}(x_j,t^n))^2}, \end{aligned}$$

where

$$u_{\rm h}(x_j,t^n) = \sum_{l=0}^{k} u_j^l(t)v_1^j(x_j) = u_j^0(t^n) - \frac{1}{2}u_j^2(t^n),$$

due to $v_0^j(x_j) = 1$, $v_1^j(x_j) = 0$, and $v_2^j(x_j) = -1/2$.

Problem 1 We show in Table 1 the L_∞ and L_2 errors and orders of accuracy for problem 1 at t=1.0. From the table we can see that a higher than $(k+1)$-th order of accuracy can be achieved with spatial mesh refining. We plot the numerical and exact solutions with $\nu = 0.01$ at $t = 0.1$ and $t = 1.0$, respectively, in Fig. 1. Figure 2 shows the profiles for $\nu = 0.01$ and $\nu = 0.001$ at $t = 0.1$, 0.5, 1.0, 2.0, and 5.0.

Table 1. L_2 and L_∞ errors and orders of accuracy for Problem 1 with $\nu = 0.01$ at $t = 1.0$.

h	L_2 error	order	L_∞ error	order
$\frac{1}{20}$	2.87×10^{-3}	–	9.52×10^{-3}	–
$\frac{1}{40}$	6.64×10^{-4}	2.1	3.85×10^{-3}	1.3
$\frac{1}{80}$	4.65×10^{-5}	3.8	3.78×10^{-4}	3.3
$\frac{1}{160}$	5.10×10^{-6}	3.2	2.03×10^{-5}	4.2
$\frac{1}{320}$	3.15×10^{-7}	4.0	1.23×10^{-6}	4.0

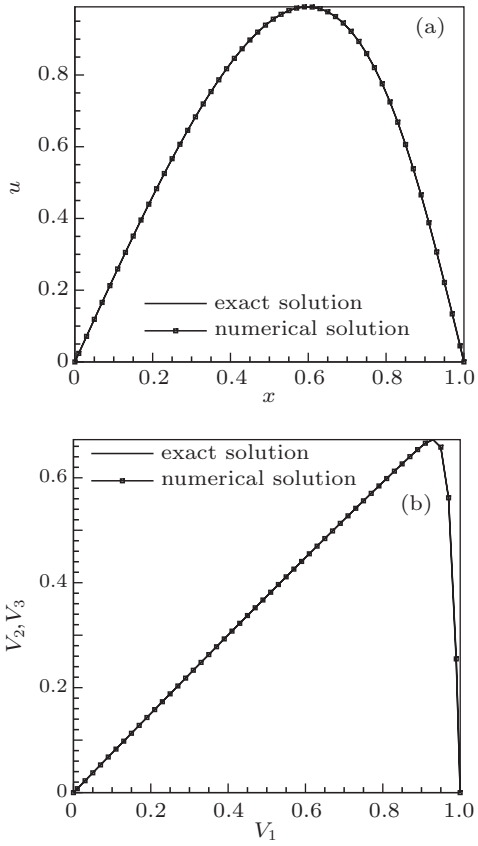


Fig. 1. Comparison between the LDG solution and the exact solution for Problem 1. (a) $t = 0.1$, (b) $t = 1.0$.

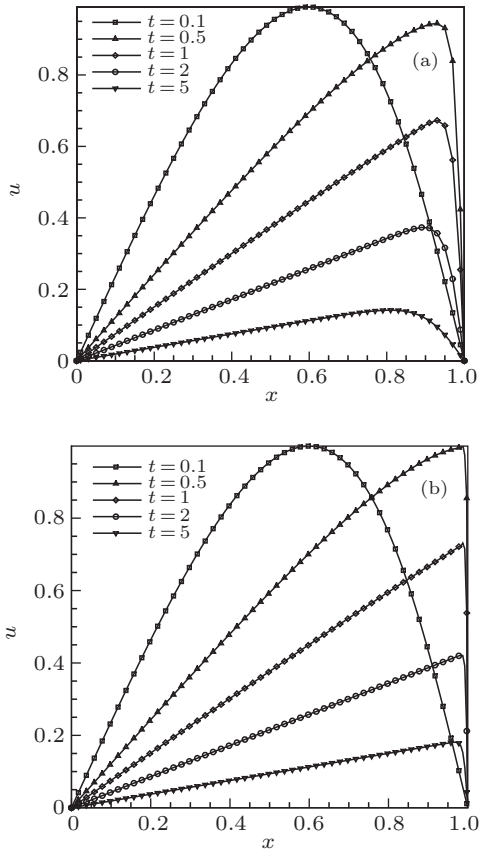


Fig. 2. Numerical solutions of Problem 1 at different time levels. (a) $\nu = 0.01$, (b) $\nu = 0.001$.

Problem 2 The numerical experiments are conducted for the MBE2 equation (17) with 80 cells. In Table 2, we show L_∞ and L_2 errors, accompanying the corresponding results in Refs. [3], [10], and [12] with $\nu = 0.01$ and $\nu = 0.001$. Note that we can obtain the equivalent accuracy to those in Ref. [3] with fewer cells. The curves of the numerical solution and analytical solution are plotted in Fig. 3 for $\nu = 0.01$ and $\nu = 0.005$, respectively, at various time levels.

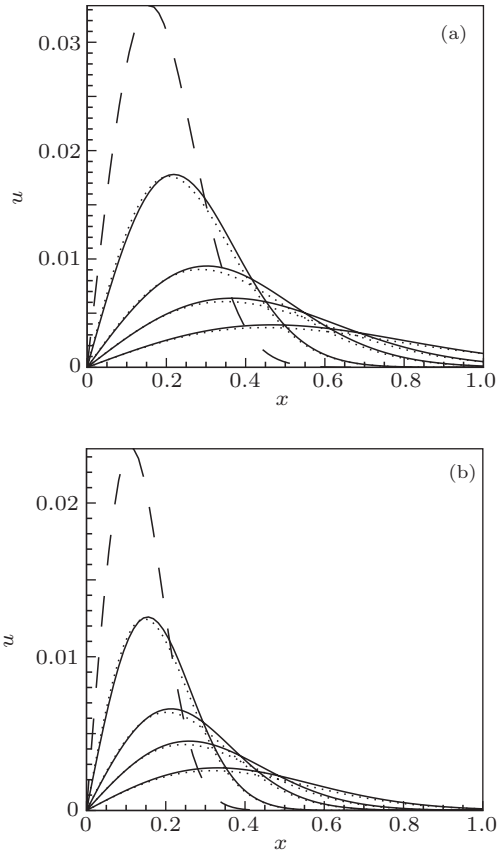


Fig. 3. The exact solution of the MBE2 equation at $t = 1.0$ (dashed line); the numerical solution at $t = 2$, 4, 6, 10 (solid line); and the corresponding exact solution (dotted line). (a) $\nu = 0.01$, (b) $\nu = 0.005$.

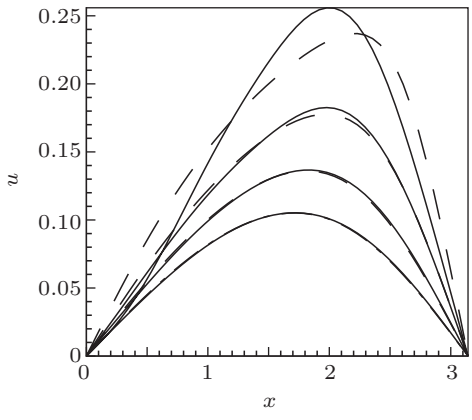


Fig. 4. The theoretical solution of the MBE3 equation (solid line); and the numerical solution at $t = 100$, 150, 200, 250 (dashed line, from top to bottom) with $\nu = 0.005$.

Table 2. Comparison of errors for the MBE2 equation (17) resulting from different methods.

	Error	$t=2$		$t=6$		$t=10$	
		L_∞	L_2	L_∞	L_2	L_∞	L_2
$\nu=0.01$	LDG	3.79×10^{-4}	8.16×10^{-4}	2.73×10^{-4}	4.65×10^{-4}	1.93×10^{-4}	3.02×10^{-4}
	Ref. [3]	3.79×10^{-4}	8.17×10^{-4}	2.73×10^{-4}	4.65×10^{-4}	1.93×10^{-4}	3.01×10^{-4}
	Ref. [10]	5.23×10^{-4}	1.22×10^{-3}	4.90×10^{-4}	7.22×10^{-4}	6.40×10^{-4}	1.28×10^{-3}
	Ref. [12]	3.79×10^{-4}	8.17×10^{-4}	3.26×10^{-4}	5.26×10^{-4}	5.47×10^{-4}	1.28×10^{-3}
$\nu=0.01$	LDG	3.79×10^{-4}	8.16×10^{-4}	2.73×10^{-4}	4.65×10^{-4}	1.93×10^{-4}	3.02×10^{-4}
	Ref. [3]	3.79×10^{-4}	8.17×10^{-4}	2.73×10^{-4}	4.65×10^{-4}	1.93×10^{-4}	3.01×10^{-4}
	Ref. [10]	5.23×10^{-4}	1.22×10^{-3}	4.90×10^{-4}	7.22×10^{-4}	6.40×10^{-4}	1.28×10^{-3}
	Ref. [12]	3.79×10^{-4}	8.17×10^{-4}	3.26×10^{-4}	5.26×10^{-4}	5.47×10^{-4}	1.28×10^{-3}

We obtain the numerical solution of the MBE3 equation (19) with 50 cells. The profiles of the numerical solution and the analytical solution for $\nu=0.01$ at different time levels are shown in Fig. 4. It is seen that there is no obvious difference between the profiles during the time considered.

Problem 3 This problem has large gradients moving with the constant velocity $v=1.0$ and small diffusion constant $\nu=0.01$. The performance of a numerical method can be judged from its ability to resolve the large gradient regions. For the MBE equation (20), we gain the numerical solution with 160 cells at various time levels, and show the results in Fig. 5. The L_∞ and L_2 errors at $t=0.5$ are 1.4031×10^{-4} and 9.0428×10^{-4}.

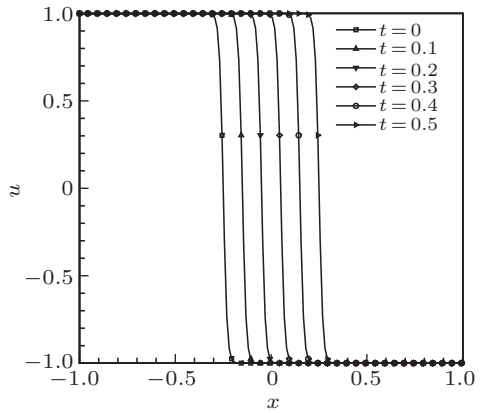


Fig. 5. The numerical solution for Problem 3 with $\nu=0.01$ and $v=1$ at $t=0.1, 0.2, 0.3, 0.4, 0.5$.

4. Conclusion

In this paper, the LDG method was proposed to solve the viscous Burgers' equation and modified Burgers' equation. Compared with the analytical solution and other numerical schemes, the LDG scheme can give more accurate results, especially at long time intervals. Due to the efficiency of this method, we will solve the generalized Burgers' equations, such as the Burgers–Fisher equation and the Burgers–Huxley equation, in our future work.

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