Stability analysis and control synthesis of uncertain Roesser-type discrete-time two-dimensional systems*

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We study the stability analysis and control synthesis of uncertain discrete-time two-dimensional (2D) systems. The mathematical model of the discrete-time 2D system is established upon the well-known Roesser model, and the uncertainty phenomenon, which appears typically in practical environments, is modeled by a convex bounded (polytope type) uncertain domain. The stability analysis and control synthesis of uncertain discrete-time 2D systems are then developed by applying the Lyapunov stability theory. In the processes of stability analysis and control synthesis, the obtained stability/stabilzaition conditions become less conservative by applying some novel relaxed techniques. Moreover, the obtained results are formulated in the form of linear matrix inequalities, which can be easily solved via standard numerical software. Finally, numerical examples are given to demonstrate the effectiveness of the obtained results.

Keywords: uncertain two-dimensional systems, Roesser model, Lyapunov stability theory, control synthesis

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1. Introduction

A two-dimensional (2D) model system could represent a wide range of practical systems, such as those in water stream heating, thermal processes, biomedical imaging, gas absorption, river pollution modeling, the process of gas filtration, grid based wireless sensor networks, data processing and transmission, and many others.[1,2] Therefore, a considerable interest in stability analysis and control synthesis of 2D systems has emerged.[3–5] Moreover, the 2D system theory is also frequently developed as an analysis tool for new problems, for example, the problem of H_∞ control for 2D discrete state delay systems described by the second Fornasini–Marchesini (FM) state-space model has been investigated in Ref. [6], and the H_∞ filtering design for 2D Markovian jump systems has also been investigated in Ref. [7]. The stability analysis of 2D discrete systems described by the FM second model with state saturation was studied in Ref. [8]. However, all the above results are only for linear certain 2D systems. As is well known, most of the practical 2D systems in the real world belong to uncertain systems, and the above results fail to work when uncertain parameters affect the systems.

On the other hand, the Lyapunov stability theory has become an efficient tool for dealing with the problem of stability analysis and control synthesis of usual 1D uncertain systems.[9–21] Indeed, the earlier results on stability analysis and control synthesis of uncertain 1D systems are based on the so-called common quadratic Lyapunov function (CQLF).[22] However, this uses a single Lyapunov matrix for all uncertain sub-models and hence the underlying results are rather conservative. With the purpose of further reducing the conservatism, the so-called affine parameter-dependent Lyapunov function (APDLF) was proposed in Ref. [23], and its extension versions have been developed in Refs. [24]–[29]. It is worth noting that the information in the Roesser-type discrete-time 2D system propagates along two independent directions and this fact makes the problem of stability analysis and control synthesis more complicated than those in the usual 1D uncertain systems. Due to the complexity of mathematical analysis, there has been little work on the stability analysis and control synthesis of uncertain Roesser-type discrete-time two-dimensional systems in the existing literature so far. Therefore, this problem needs to be further investigated.

Motivated by the above discussion, we investigate the stability analysis and control synthesis of uncertain Roesser-type discrete-time two-dimensional systems applying the Lyapunov stability theory in this paper. The mathematical model of the discrete-time 2D system is established upon the well-known Roesser model, and the uncertainty phenomenon, which appears typically in practical environments, is modeled by a convex bounded (polytope type) uncertain domain. Firstly, linear matrix inequality (LMI) based stability conditions are proposed by applying a common quadratic Lyapunov function for the uncertain Roesser-type discrete-time two-dimensional systems. With the purpose of further reducing the conservatism, an affine parameter-dependent Lyapunov function which suits the Roesser-type discrete-time two-dimensional systems is proposed, and the relaxed LMI-based stability conditions are obtained. Secondly, the control synthesis of the uncertain Roesser-type discrete-time two-dimensional systems is investigated using

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the affine parameter-dependent Lyapunov function. Finally, numerical examples are given to demonstrate the effectiveness of the proposed results.

For simplicity, the notations used are fairly standard. For example, $\boldsymbol{X} > 0$ ($\boldsymbol{X} \geq 0$) means matrix $\boldsymbol{X}$ is symmetric and positive definite (symmetric and positive semidefinite), and $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of $\boldsymbol{X}$. Symbol $\boldsymbol{I}$ represents an identity matrix with appropriate dimensions. For a matrix $\boldsymbol{X}$, $\min(\boldsymbol{X})$ ($\max(\boldsymbol{X})$) means the smallest (largest) eigenvalue of $\boldsymbol{X}$. A star $*$ in a symmetric matrix denotes the transposed element in the symmetric position.

2. System description and preliminaries

Consider a class of uncertain Roesser-type discrete-time 2D systems described by

$$\boldsymbol{x}^{+}(k,l) = \boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{x}(k,l) + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{u}(k,l), \tag{1}$$

with

$$\boldsymbol{x}(k,l) = \begin{bmatrix} \boldsymbol{x}^{\mathrm{h}}(k,l) \\ \boldsymbol{x}^{\mathrm{v}}(k,l) \end{bmatrix}, \quad \boldsymbol{x}^{+}(k,l) = \begin{bmatrix} \boldsymbol{x}^{\mathrm{h}}(k+1,l) \\ \boldsymbol{x}^{\mathrm{v}}(k,l+1) \end{bmatrix},$$

where $\boldsymbol{x}^{\mathrm{h}}$ is the horizonal state in $\mathbf{R}^{n_1}$, $\boldsymbol{x}^{\mathrm{v}}$ is the vertical state in $\mathbf{R}^{n_2}$, n_1 and n_2 are the dimensions of the horizonal and the vertical state vectors, respectively, $\boldsymbol{u}$ is the control input in $\mathbf{R}^{n_3}$, and k, l are two integers in $\mathbb{Z}^{+}$. The coefficient matrices $\boldsymbol{A}(\boldsymbol{\alpha})$ and $\boldsymbol{B}(\boldsymbol{\alpha})$ are not precisely known, but have the following structures:

$$\boldsymbol{A}(\boldsymbol{\alpha}) = \begin{bmatrix} \boldsymbol{A}^{11}(\boldsymbol{\alpha}) & \boldsymbol{A}^{12}(\boldsymbol{\alpha}) \\ \boldsymbol{A}^{21}(\boldsymbol{\alpha}) & \boldsymbol{A}^{22}(\boldsymbol{\alpha}) \end{bmatrix}, \quad \boldsymbol{B}(\boldsymbol{\alpha}) = \begin{bmatrix} \boldsymbol{B}^{1}(\boldsymbol{\alpha}) \\ \boldsymbol{B}^{2}(\boldsymbol{\alpha}) \end{bmatrix},$$

with $\boldsymbol{A}^{11}(\boldsymbol{\alpha}) \in \mathbf{R}^{n_1\times n_1}$, $\boldsymbol{A}^{12}(\boldsymbol{\alpha}) \in \mathbf{R}^{n_1\times n_2}$, $\boldsymbol{A}^{21}(\boldsymbol{\alpha}) \in \mathbf{R}^{n_2\times n_1}$, $\boldsymbol{A}^{22}(\boldsymbol{\alpha}) \in \mathbf{R}^{n_2\times n_2}$, $\boldsymbol{B}^{1}(\boldsymbol{\alpha}) \in \mathbf{R}^{n_1\times n_3}$, $\boldsymbol{B}^{2}(\boldsymbol{\alpha}) \in \mathbf{R}^{n_2\times n_3}$. Furthermore, matrices $\boldsymbol{A}^{11}(\boldsymbol{\alpha})$, $\boldsymbol{A}^{12}(\boldsymbol{\alpha})$, $\boldsymbol{A}^{21}(\boldsymbol{\alpha})$, $\boldsymbol{A}^{22}(\boldsymbol{\alpha})$, $\boldsymbol{B}^{1}(\boldsymbol{\alpha})$, and $\boldsymbol{B}^{2}(\boldsymbol{\alpha})$ belong to a convex bounded (polytope type) uncertain domain $\mathscr{P}$ given by

$$\begin{aligned} \mathscr{P} :=\ & \big\{(\boldsymbol{A}^{11}, \boldsymbol{A}^{12}, \boldsymbol{A}^{21}, \boldsymbol{A}^{22}, \boldsymbol{B}^{1}, \boldsymbol{B}^{2})(\boldsymbol{\alpha}) : \\ & (\boldsymbol{A}^{11}, \boldsymbol{A}^{12}, \boldsymbol{A}^{21}, \boldsymbol{A}^{22}, \boldsymbol{B}^{1}, \boldsymbol{B}^{2})(\boldsymbol{\alpha}) \\ =\ & \sum_{i=1}^{r} \alpha_i (\boldsymbol{A}_i^{11}, \boldsymbol{A}_i^{12}, \boldsymbol{A}_i^{21}, \boldsymbol{A}_i^{22}, \boldsymbol{B}_i^{1}, \boldsymbol{B}_i^{2});\ \boldsymbol{\alpha} \in \Delta_r \big\}, \end{aligned} \tag{2}$$

where Δ_r is the unit simplex given by

$$\Delta_r = \Big\{ \boldsymbol{\alpha} \in \mathbf{R}^{r} : \sum_{i=1}^{r} \alpha_i = 1,\ \alpha_i \geq 0;\ i = 1, \ldots, r \Big\}. \tag{3}$$

Moreover, the boundary conditions along two independent directions are defined as $\boldsymbol{x}^{\mathrm{h}}(0,l) = \boldsymbol{f}(l)$ and $\boldsymbol{x}^{\mathrm{v}}(k,0) = \boldsymbol{g}(k)$, where $\boldsymbol{f}(l)$ and $\boldsymbol{g}(k)$ are the corresponding boundary conditions along the two independent directions.

Denote $X_N = \sup\{\|\boldsymbol{x}(k,l)\| : N = k + l\}$, then we give the definition of robust asymptotical stability for an uncertain Roesser-type discrete-time 2D system (1).

Definition 1[30] The uncertain Roesser-type discrete-time 2D system (1) is robust and asymptotically stable if $\lim\limits_{k\to\infty,\ l\to\infty} X_N = 0$ with the initial conditions and the boundary conditions $\boldsymbol{x}^{\mathrm{h}}(0,l) = \boldsymbol{f}(l)$ and $\boldsymbol{x}^{\mathrm{v}}(k,0) = \boldsymbol{g}(k)$.

We end this section with a useful lemma.

Lemma 1[31] For two symmetric matrices $\boldsymbol{P} > 0$, $\boldsymbol{P}_{+} > 0$, the inequality $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{+}\boldsymbol{A} - \boldsymbol{P} < 0$ holds if there exists a matrix $\boldsymbol{G}$ such that $\begin{bmatrix} \boldsymbol{P} & (*) \\ \boldsymbol{G}\boldsymbol{A} & \boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}} - \boldsymbol{P}_{+} \end{bmatrix} > 0$.

3. Main results

3.1. Stability analysis of uncertain Roesser-type discrete-time 2D systems

In this subsection, LMI-based robust stability conditions for an uncertain Roesser-type discrete-time 2D system (1) with $\boldsymbol{u}(k,l) = 0$ are developed by means of the Lyapunov stability theory. Firstly, LMI-based stability conditions are given by applying a common quadratic Lyapunov function for the uncertain Roesser-type discrete-time two-dimensional system.

Theorem 1 The uncertain Roesser-type discrete-time 2D system (1) with $\boldsymbol{u}(k,l) = 0$ is robust and asymptotically stable if an appropriate-dimensional matrix $\boldsymbol{P} > 0$ exists with

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}^{1} & * \\ \boldsymbol{P}^{3} & \boldsymbol{P}^{2} \end{bmatrix},\ \boldsymbol{P}^{1} \in \mathbf{R}^{n_1\times n_1},\ \boldsymbol{P}^{2} \in \mathbf{R}^{n_2\times n_2},\ \boldsymbol{P}^{3} \in \mathbf{R}^{n_2\times n_1}$$

such that the following LMIs hold:

$$\begin{bmatrix} \boldsymbol{A}_i^{11} & \boldsymbol{A}_i^{12} \\ \boldsymbol{A}_i^{21} & \boldsymbol{A}_i^{22} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P}^{1} & * \\ \boldsymbol{P}^{3} & \boldsymbol{P}^{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_i^{11} & \boldsymbol{A}_i^{12} \\ \boldsymbol{A}_i^{21} & \boldsymbol{A}_i^{22} \end{bmatrix} - \begin{bmatrix} \boldsymbol{P}^{1} & * \\ \boldsymbol{P}^{3} & \boldsymbol{P}^{2} \end{bmatrix} < 0,\ i = 1, 2, \ldots, r; \tag{4}$$

$$\mathrm{He}\left(\begin{bmatrix} \boldsymbol{A}_i^{11} & \boldsymbol{A}_i^{12} \\ \boldsymbol{A}_i^{21} & \boldsymbol{A}_i^{22} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P}^{1} & * \\ \boldsymbol{P}^{3} & \boldsymbol{P}^{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_j^{11} & \boldsymbol{A}_j^{12} \\ \boldsymbol{A}_j^{21} & \boldsymbol{A}_j^{22} \end{bmatrix} \right) - 2 \begin{bmatrix} \boldsymbol{P}^{1} & \boldsymbol{P}^{2} \\ \boldsymbol{P}^{3} & \boldsymbol{P}^{2} \end{bmatrix} < 0,\ 1 \leq i < j \leq r. \tag{5}$$

Proof Consider the following Lyapunov function for the uncertain Roesser-type discrete-time 2D system (1):

$$V(\boldsymbol{x}(k,l)) = \boldsymbol{x}^{\mathrm{T}}(k,l)\boldsymbol{P}\boldsymbol{x}(k,l), \tag{6}$$

where the Lyapunov matrix $\boldsymbol{P}$ has the form

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}^{1} & * \\ \boldsymbol{P}^{3} & \boldsymbol{P}^{2} \end{bmatrix}.$$

Then, the variation of $V(\boldsymbol{x}(k,l))$ can be written as

$$\Delta V(\boldsymbol{x}(k,l)) = \boldsymbol{x}^{\mathrm{T}}(k,l)(\boldsymbol{A}(\boldsymbol{\alpha})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}(\boldsymbol{\alpha}) - \boldsymbol{P})\boldsymbol{x}(k,l). \tag{7}$$

From Eq. (7), it can be concluded that $\Delta V(\boldsymbol{x}(k,l)) < 0$ holds if

$$\boldsymbol{A}(\boldsymbol{\alpha})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}(\boldsymbol{\alpha}) - \boldsymbol{P} < 0. \tag{8}$$

On the other hand, we have

$$\boldsymbol{A}(\boldsymbol{\alpha})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}(\boldsymbol{\alpha})-\boldsymbol{P}=\sum_{i=1}^{r}\alpha_i^2\left\{\begin{bmatrix}\boldsymbol{A}_i^{11} & \boldsymbol{A}_i^{12}\\ \boldsymbol{A}_i^{21} & \boldsymbol{A}_i^{22}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{P}^1 & *\\ \boldsymbol{P}^3 & \boldsymbol{P}^2\end{bmatrix}\begin{bmatrix}\boldsymbol{A}_i^{11} & \boldsymbol{A}_i^{12}\\ \boldsymbol{A}_i^{21} & \boldsymbol{A}_i^{22}\end{bmatrix}-\begin{bmatrix}\boldsymbol{P}^1 & *\\ \boldsymbol{P}^3 & \boldsymbol{P}^2\end{bmatrix}\right\}$$
$$+\sum_{i=1}^{r-1}\sum_{j>i}\alpha_i\alpha_j\left\{\mathrm{He}\left(\begin{bmatrix}\boldsymbol{A}_i^{11} & \boldsymbol{A}_i^{12}\\ \boldsymbol{A}_i^{21} & \boldsymbol{A}_i^{22}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{P}^1 & *\\ \boldsymbol{P}^3 & \boldsymbol{P}^2\end{bmatrix}\begin{bmatrix}\boldsymbol{A}_j^{11} & \boldsymbol{A}_j^{12}\\ \boldsymbol{A}_j^{21} & \boldsymbol{A}_j^{22}\end{bmatrix}\right)-2\begin{bmatrix}\boldsymbol{P}^1 & \boldsymbol{P}^2\\ \boldsymbol{P}^3 & \boldsymbol{P}^2\end{bmatrix}\right\}. \tag{9}$$

Thus, if the LMI-based stability conditions (4) and (5) hold, inequality (8) obviously holds, which guarantees the robust asymptotical stability of the uncertain Roesser-type discrete-time 2D system (1). This completes the proof.

Remark 1 In Theorem 1, LMI-based stability conditions for the uncertain Roesser-type discrete-time 2D system (1) are developed by means of a common quadratic Lyapunov function. However, it uses a single Lyapunov matrix $\boldsymbol{P}$ for all the r sub-models, hence the obtained result is rather conservative. In the following, a new parameter-dependent Lyapunov function for the uncertain Roesser-type discrete-time 2D system (1) will be proposed with the purpose of further reducing the conservatism.

Theorem 2 The uncertain Roesser-type discrete-time 2D system (1) with $\boldsymbol{u}(k,l)=0$ is robust and asymptotically stable if appropriate-dimensional matrices exist $\boldsymbol{P}_i>0$, $\boldsymbol{G}_i$ ($i\in\{1,2,\ldots,r\}$) with

$$\boldsymbol{P}_i=\begin{bmatrix}\boldsymbol{P}_i^1 & *\\ \boldsymbol{P}_i^3 & \boldsymbol{P}_i^2\end{bmatrix},\ \boldsymbol{P}_i^1\in\mathbf{R}^{n_1\times n_1},\ \boldsymbol{P}_i^2\in\mathbf{R}^{n_2\times n_2},\ \boldsymbol{P}_i^3\in\mathbf{R}^{n_2\times n_1},$$
$$\boldsymbol{G}_i=\begin{bmatrix}\boldsymbol{G}_i^1 & \boldsymbol{G}_i^2\\ \boldsymbol{G}_i^3 & \boldsymbol{G}_i^4\end{bmatrix},\ \boldsymbol{G}_i^1\in\mathbf{R}^{n_1\times n_1},\ \boldsymbol{G}_i^2\in\mathbf{R}^{n_1\times n_2},\ \boldsymbol{G}_i^3\in\mathbf{R}^{n_2\times n_1},$$
$$\boldsymbol{G}_i^4\in\mathbf{R}^{n_2\times n_2}$$

such that the following LMIs hold:

$$\begin{bmatrix}\boldsymbol{P}_i & *\\ \boldsymbol{G}_i\boldsymbol{A}_i & \boldsymbol{G}_i+\boldsymbol{G}_i^{\mathrm{T}}-\boldsymbol{P}_i\end{bmatrix}>0,\ i=1,2,\ldots,r; \tag{10}$$
$$\begin{bmatrix}\boldsymbol{P}_i+\boldsymbol{P}_j & *\\ \boldsymbol{G}_j\boldsymbol{A}_i+\boldsymbol{G}_i\boldsymbol{A}_j & \boldsymbol{G}_i+\boldsymbol{G}_j+\boldsymbol{G}_i^{\mathrm{T}}+\boldsymbol{G}_j^{\mathrm{T}}-\boldsymbol{P}_i-\boldsymbol{P}_j\end{bmatrix}>0,$$
$$1\leqslant i<j\leqslant r. \tag{11}$$

Proof Consider a new Lyapunov function for the uncertain Roesser-type discrete-time 2D system (1)

$$V(\boldsymbol{x}(k,l))=\boldsymbol{x}^{\mathrm{T}}(k,l)\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{x}(k,l), \tag{12}$$

where the applied Lyapunov matrix $\boldsymbol{P}(\boldsymbol{\alpha})$ has the form

$$\boldsymbol{P}(\boldsymbol{\alpha})=\sum_{i=1}^{r}\alpha_i\boldsymbol{P}_i=\sum_{i=1}^{r}\alpha_i\begin{bmatrix}\boldsymbol{P}_i^1 & *\\ \boldsymbol{P}_i^3 & \boldsymbol{P}_i^2\end{bmatrix}.$$

Firstly, let us check the validity of the affine parameter-dependent Lyapunov function (12). Notice that if the LMI-based stability conditions of Theorem 2 hold, it follows from inequality (10) that $\boldsymbol{P}(\boldsymbol{\alpha})>0$ always holds, which ensures the fact that $\min(\boldsymbol{P}_g(h))>0$ exists. Therefore, we can write

$$\underline{\lambda}\boldsymbol{x}^{\mathrm{T}}(k,l)\boldsymbol{x}(k,l)\leqslant V(\boldsymbol{x}(k,l))\leqslant\overline{\lambda}\boldsymbol{x}^{\mathrm{T}}(k,l)\boldsymbol{x}(k,l), \tag{13}$$

where $\underline{\lambda}=\min(\boldsymbol{P}(\boldsymbol{\alpha}))$ and $\overline{\lambda}=\max(\boldsymbol{P}(\boldsymbol{\alpha}))$. Thus, inequality (10) ensures $V(\boldsymbol{x}(k,l))$ is a candidate Lyapunov function.

Then, the variation of $V(\boldsymbol{x}(k,l))$ can be written as

$$\Delta V(\boldsymbol{x}(k,l))=\boldsymbol{x}^{\mathrm{T}}(k,l)(\boldsymbol{A}(\boldsymbol{\alpha})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{A}(\boldsymbol{\alpha})-\boldsymbol{P}(\boldsymbol{\alpha}))\times\boldsymbol{x}(k,l). \tag{14}$$

From equality (14), it can be concluded that $\Delta V(\boldsymbol{x}(k,l))<0$ holds if

$$\boldsymbol{A}(\boldsymbol{\alpha})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{A}(\boldsymbol{\alpha})-\boldsymbol{P}(\boldsymbol{\alpha})<0. \tag{15}$$

Using Lemma 1 with $\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{\alpha})$ and $\boldsymbol{G}=\boldsymbol{G}(\boldsymbol{\alpha})$, inequality (15) is ensured by the following inequality:

$$\begin{bmatrix}\boldsymbol{P}(\boldsymbol{\alpha}) & *\\ \boldsymbol{G}(\boldsymbol{\alpha})\boldsymbol{A}(\boldsymbol{\alpha}) & \boldsymbol{G}(\boldsymbol{\alpha})+\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{\alpha})-\boldsymbol{P}(\boldsymbol{\alpha})\end{bmatrix}>0. \tag{16}$$

On the other hand, we have

$$\begin{bmatrix}\boldsymbol{P}(\boldsymbol{\alpha}) & *\\ \boldsymbol{G}(\boldsymbol{\alpha})\boldsymbol{A}(\boldsymbol{\alpha}) & \boldsymbol{G}(\boldsymbol{\alpha})+\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{\alpha})-\boldsymbol{P}(\boldsymbol{\alpha})\end{bmatrix}$$
$$=\sum_{i=1}^{r}\alpha_i^2\begin{bmatrix}\boldsymbol{P}_i & *\\ \boldsymbol{G}_i\boldsymbol{A}_i & \boldsymbol{G}_i+\boldsymbol{G}_i^{\mathrm{T}}-\boldsymbol{P}_i\end{bmatrix}+\sum_{i=1}^{r-1}\sum_{j>i}\alpha_i\alpha_j\begin{bmatrix}\boldsymbol{P}_i+\boldsymbol{P}_j & *\\ \boldsymbol{G}_j\boldsymbol{A}_i+\boldsymbol{G}_i\boldsymbol{A}_j & \boldsymbol{G}_i+\boldsymbol{G}_j+\boldsymbol{G}_i^{\mathrm{T}}+\boldsymbol{G}_j^{\mathrm{T}}-\boldsymbol{P}_i-\boldsymbol{P}_j\end{bmatrix}. \tag{17}$$

Thus, if the LMI-based stability conditions (10) and (11) hold, inequality (15) obviously holds, which guarantees the robust asymptotical stability of the uncertain Roesser-type discrete-time 2D system (1). This completes the proof.

3.2. Control synthesis of uncertain Roesser-type discrete-time 2D systems

A new state-feedback control law for the uncertain Roesser-type discrete-time 2D system (1) is proposed as follows:

$$\boldsymbol{u}(k,l)=\boldsymbol{F}\boldsymbol{G}^{-1}\boldsymbol{x}(k,l), \tag{18}$$

where $\boldsymbol{F}=\begin{bmatrix}\boldsymbol{F}_1 & \boldsymbol{F}_2\end{bmatrix}$ with $\boldsymbol{F}_1\in\mathbf{R}^{n_3\times n_1}$ and $\boldsymbol{F}_2\in\mathbf{R}^{n_3\times n_2}$, $\boldsymbol{G}=\begin{bmatrix}\boldsymbol{G}^1 & \boldsymbol{G}^2\\ \boldsymbol{G}^3 & \boldsymbol{G}^2\end{bmatrix}$ with $\boldsymbol{G}^1\in\mathbf{R}^{n_1\times n_1}$, $\boldsymbol{G}^2\in\mathbf{R}^{n_1\times n_2}$, $\boldsymbol{G}^3\in\mathbf{R}^{n_2\times n_1}$, $\boldsymbol{G}^4\in\mathbf{R}^{n_2\times n_2}$. Thus, the uncertain Roesser-type discrete-time 2D system (1) has another closed-loop form

$$\boldsymbol{x}^{+}(k,l)=(\boldsymbol{A}(\boldsymbol{\alpha})+\boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F}\boldsymbol{G}^{-1})\boldsymbol{x}(k,l). \tag{19}$$

Theorem 3 Under the control of Eq. (18), the uncertain Roesser-type discrete-time 2D system (1) is robust and asymptotically stable if there exists any given number $d \in \mathbb{Z}^+$ appropriate-dimensional matrices $\boldsymbol{P}_i > 0$ $(i = 1, \ldots, r)$, $\boldsymbol{F}$ and $\boldsymbol{G}$ with

$$\boldsymbol{P}_i = \begin{bmatrix} \boldsymbol{P}_i^1 & * \\ \boldsymbol{P}_i^3 & \boldsymbol{P}_i^2 \end{bmatrix},\ \boldsymbol{P}_i^1 \in \mathbf{R}^{n_1 \times n_1},\ \boldsymbol{P}_i^2 \in \mathbf{R}^{n_2 \times n_2},\ \boldsymbol{P}_i^3 \in \mathbf{R}^{n_2 \times n_1},$$
$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 & \boldsymbol{F}_2 \end{bmatrix},\ \boldsymbol{F}_1 \in \mathbf{R}^{n_3 \times n_1},\ \boldsymbol{F}_2 \in \mathbf{R}^{n_3 \times n_2},$$
$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}^1 & \boldsymbol{G}^2 \\ \boldsymbol{G}^3 & \boldsymbol{G}^2 \end{bmatrix},\ \boldsymbol{G}^1 \in \mathbf{R}^{n_1 \times n_1},\ \boldsymbol{G}^2 \in \mathbf{R}^{n_1 \times n_2},\ \boldsymbol{G}^3 \in \mathbf{R}^{n_2 \times n_1},$$
$$\boldsymbol{G}^4 \in \mathbf{R}^{n_2 \times n_2}$$

such that the following LMIs hold:

$$\begin{bmatrix} \boldsymbol{P}_i & * \\ \boldsymbol{A}_i\boldsymbol{G} + \boldsymbol{B}_i\boldsymbol{F} & \boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}} - \boldsymbol{P}_i \end{bmatrix} > 0,\ i = 1, 2, \ldots, r. \tag{20}$$

Proof Consider a new Lyapunov function for a closed-loop system (19)

$$V(\boldsymbol{x}(k,l)) = \boldsymbol{x}^{\mathrm{T}}(k,l)\boldsymbol{G}^{-T}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{G}^{-1}\boldsymbol{x}(k,l), \tag{21}$$

where the Lyapunov matrix $\boldsymbol{P}$ is a positive definite matrix with the structure

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}^1 & * \\ \boldsymbol{P}^3 & \boldsymbol{P}^2 \end{bmatrix},\ \boldsymbol{P}^1 \in \mathbf{R}^{n_1 \times n_1},\ \boldsymbol{P}^2 \in \mathbf{R}^{n_2 \times n_2},\ \boldsymbol{P}^3 \in \mathbf{R}^{n_2 \times n_1}.$$

Then, the variation of $V(\boldsymbol{x}(k,l))$ could be written as

$$\begin{aligned} \Delta V(\boldsymbol{x}(k,l)) = \boldsymbol{x}^{\mathrm{T}}(k,l)\big(&\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{G}^{-T}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{G}^{-1}\boldsymbol{\Sigma} \\ &-\boldsymbol{G}^{-\mathrm{T}}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{G}^{-1}\big)\boldsymbol{x}(k,l), \end{aligned} \tag{22}$$

where $\boldsymbol{\Sigma} = \boldsymbol{A}(\boldsymbol{\alpha}) + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F}\boldsymbol{G}^{-1}$.

Now, it is easy to see that the closed-loop system (19) is asymptotically stable if the following inequality holds:

$$\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{G}^{-T}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{G}^{-1}\boldsymbol{\Sigma} - \boldsymbol{G}^{-\mathrm{T}}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{G}^{-1} < 0. \tag{23}$$

By multiplying left by $\boldsymbol{G}^{\mathrm{T}}$ and right by $\boldsymbol{G}$ to inequality (23), it is easy to verify that the closed-loop system (19) is asymptotically stable if the following inequality holds:

$$\begin{aligned} &(\boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{G} + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F})^{\mathrm{T}}\boldsymbol{G}^{-T}\boldsymbol{P}(\boldsymbol{\alpha})\boldsymbol{G}^{-1}(\boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{G} + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F}) \\ &-\boldsymbol{P}(\boldsymbol{\alpha}) < 0. \end{aligned} \tag{24}$$

Using Lemma 1 with $\boldsymbol{A} = \boldsymbol{G}^{-1}(\boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{G} + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F})$, a sufficient condition ensuring inequality (24) is

$$\begin{bmatrix} \boldsymbol{P}(\boldsymbol{\alpha}) & * \\ \boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{G} + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F} & \boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}} - \boldsymbol{P}(\boldsymbol{\alpha}) \end{bmatrix} > 0. \tag{25}$$

On the other hand, reordering the expression of

$$\begin{bmatrix} \boldsymbol{P}(\boldsymbol{\alpha}) & * \\ \boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{G} + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F} & \boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}} - \boldsymbol{P}(\boldsymbol{\alpha}) \end{bmatrix},$$

we get

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{P}(\boldsymbol{\alpha}) & * \\ \boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{G} + \boldsymbol{B}(\boldsymbol{\alpha})\boldsymbol{F} & \boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}} - \boldsymbol{P}(\boldsymbol{\alpha}) \end{bmatrix} \\ &= \sum_{i=1}^{r} \alpha_i \left(\begin{bmatrix} \boldsymbol{P}_i & * \\ \boldsymbol{A}_i\boldsymbol{G} + \boldsymbol{B}_i\boldsymbol{F} & \boldsymbol{G} + \boldsymbol{G}^{\mathrm{T}} - \boldsymbol{P}_i \end{bmatrix} \right). \end{aligned} \tag{26}$$

From expressions (23)–(26), if the LMI-based stabilization condition (20) holds, inequality (23) obviously holds, which guarantees the robust asymptotical stability of the uncertain Roesser-type discrete-time 2D system (1). This completes the proof.

4. Numerical example

Consider the following nonlinear differential equation:[32]

$$\begin{aligned} \frac{\partial^2 q(x,t)}{\partial x \partial t} = &\ a_1 \frac{\partial q(x,t)}{\partial t} + a_2 \frac{\partial q(x,t)}{\partial x} \\ &+ a_0 \sin^2(\beta) q(x,t) + b u_{\mathrm{c}}(x,t), \end{aligned}$$

where the initial and the boundary conditions $q(x,0) = q_1(x)$ and $q(0,t) = q_2(t)$, $q(x,t)$ are the variable function, a_0, a_1, a_2, b are real coefficients, $u_{\mathrm{c}}(x,t)$ is the input function, and β is an uncertain value.

First, we establish the state space model for the above nonlinear differential equation. Let us define

$$\begin{aligned} x_{\mathrm{c}}^{\mathrm{h}}(x,t) &= \frac{\partial q(x,t)}{\partial t} - a_2 q(x,t), \\ x_{\mathrm{c}}^{\mathrm{v}}(x,t) &= q(x,t), \end{aligned}$$

then the following 2D state space model can be easily obtained:

$$\begin{aligned} \begin{bmatrix} \partial x_{\mathrm{c}}^{\mathrm{h}}(x,t)/\partial x \\ \partial x_{\mathrm{c}}^{\mathrm{v}}(x,t)/\partial t \end{bmatrix} &= \begin{bmatrix} a_1 & a_1 a_2 + a_0 \sin^2(\beta) \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{h}}(x,t) \\ x_{\mathrm{c}}^{\mathrm{v}}(x,t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u_{\mathrm{c}}(x,t) \\ &= \alpha_1 \begin{bmatrix} a_1 & a_1 a_2 + a_0 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{h}}(x,t) \\ x_{\mathrm{c}}^{\mathrm{v}}(x,t) \end{bmatrix} + \alpha_1 \begin{bmatrix} b \\ 0 \end{bmatrix} u_{\mathrm{c}}(x,t) + \alpha_2 \begin{bmatrix} a_1 & a_1 a_2 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{h}}(x,t) \\ x_{\mathrm{c}}^{\mathrm{v}}(x,t) \end{bmatrix} + \alpha_2 \begin{bmatrix} b \\ 0 \end{bmatrix} u_{\mathrm{c}}(x,t), \end{aligned}$$

where $\alpha_1 = \sin^2(\beta)$ and $\alpha_2 = 1 - \sin^2(\beta)$.

The above uncertain Roesser-type 2D system can be discretized with sampling times T_1 and T_2 corresponding to variables x and t, respectively. The obtained discrete-time uncertain Roesser-type 2D system is given by

$$\begin{bmatrix} x^{\mathrm{h}}(k+1,l) \\ x^{\mathrm{v}}(k,l+1) \end{bmatrix} = \sum_{i=1}^{2} \alpha_i \left(\boldsymbol{A}_i \begin{bmatrix} x^{\mathrm{h}}(k,l) \\ x^{\mathrm{v}}(k,l) \end{bmatrix} + \boldsymbol{B}_i u_{\mathrm{c}}(k,l) \right),$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 + a_1 T_1 & (a_1 a_2 + a_0) T_1 \\ T_2 & 1 + a_2 T_2 \end{bmatrix},$$
$$\boldsymbol{A}_2 = \begin{bmatrix} 1 + a_1 T_1 & a_1 a_2 T_1 \\ T_2 & 1 + a_2 T_2 \end{bmatrix},\quad \boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} bT_1 \\ 0 \end{bmatrix}.$$

Consider the following parameters: $a_0 = -2$, $a_1 = -3$, $b = -1$, $T_1 = 0.3$, $T_2 = 0.5$, then we can calculate the feasible parameter intervals by evaluating the feasibility of the associ-

ated LMI problems with Theorems 1–4 for varying values of a_2 . Moreover, the initial conditions and the boundary conditions in this numerical example are set as $x^{\mathrm{h}}(0,l) = \cos(l)$ for $l < 50$ and $x^{\mathrm{v}}(k,0) = \sin(k)$ for $k < 50$; $x^{\mathrm{h}}(0,l) = -0.15$ for $l \geq 50$ and $x^{\mathrm{v}}(k,0) = 0.1$ for $k \geq 50$.

4.1. Stability analysis

With $\boldsymbol{u}(s,l) = 0$, Table 1 shows the parameter feasible intervals of a_2 in which the uncertain Roesser-type discrete-time 2D system's asymptotically stability can be ensured by applying the approaches provided by Theorems 1 and 2 respectively. From Table 1, it can be seen that Theorem 2 provides a less conservative result than Theorem 1. Figures 1 and 2 show the state trajectories of the system state variables $x^{\mathrm{h}}(k,l)$ and $x^{\mathrm{v}}(k,l)$ with $a_2 = -2.179$, respectively. From Table 1, we can find that $a_2 = -2.179$ is out of the calculated stable region of Theorem 1 but within that of Theorem 2. The state trajectories of the system state variables are asymptotically stable, as shown in Figs. 1 and 2. Therefore, the effectiveness of the proposed approach given in Theorem 2 is illustrated in the numerical example.

Table 1. Feasible stability parameter intervals of a_2 via methods proposed in this paper.

Method	Theorem 1	Theorem 2
Interval	$[-2.172,\ -0.423]$	$[-2.179,\ -0.415]$

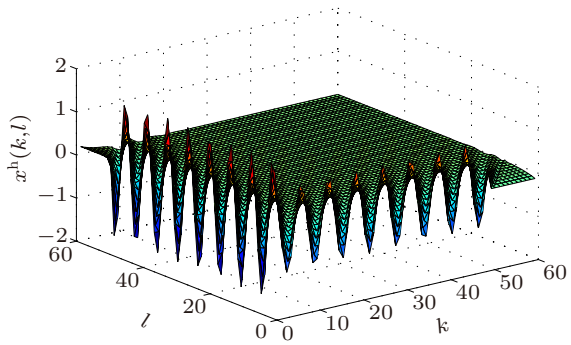


Fig. 1. (color online) State trajectory of system state variable $x^{\mathrm{h}}(k,l)$ with $a_2 = -2.179$.

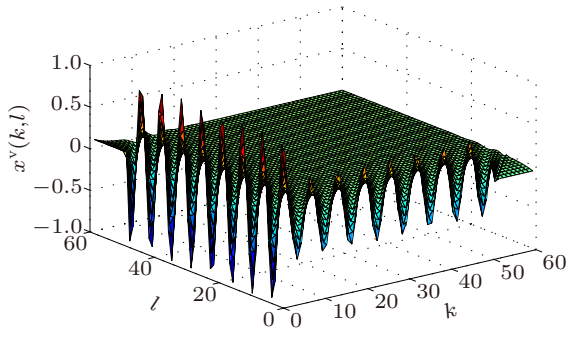


Fig. 2. (color online) State trajectory of the system state variable $x^{\mathrm{v}}(k,l)$ with $a_2 = -2.179$.

4.2. Control synthesis

From Table 1, we can find that $a_2 = -2.60$ is out of the calculated stable regions of both theorems. Indeed, the state trajectories of the system state variables are not asymptotically stable, as shown in Figs. 3 and 4.

Let $a_2 = -2.60$, the following control matrices can be obtained by solving the LMI-based stabilization condition (20):

$$\boldsymbol{F} = \begin{bmatrix} 38.8010 & 234.7276 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} 32.3952 & 4.1180 \\ 4.1180 & 34.3169 \end{bmatrix}.$$

Under the control of Eq. (19) with the above matrix parameters, the state trajectories of the system state variables $x^{\mathrm{h}}(k,l)$ and $x^{\mathrm{v}}(k,l)$ with $a_2 = -2.60$ are shown in Figs. 5 and 6, respectively. The state trajectories of the system state variables are asymptotically stable, as shown in Figs. 5 and 6. Therefore, the effectiveness of the proposed approach given in Theorem 3 is illustrated in the numerical example.

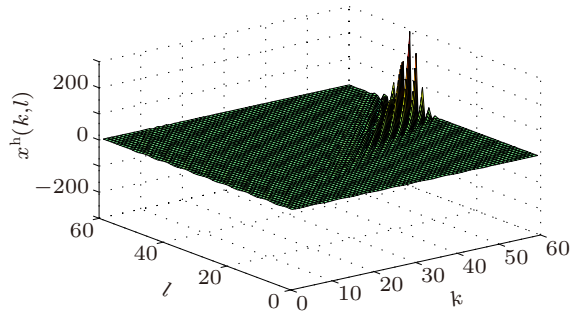


Fig. 3. (color online) State trajectory of system state variable $x^{\mathrm{h}}(k,l)$ with $a_2 = -2.60$.

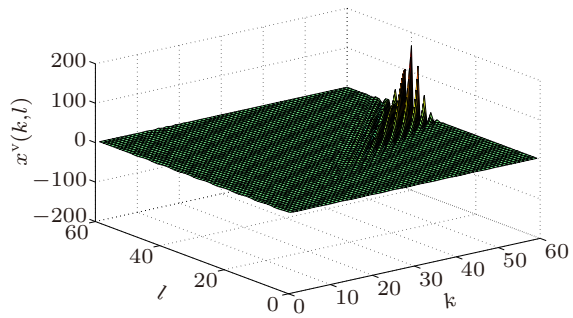


Fig. 4. (color online) State trajectory of system state variable $x^{\mathrm{v}}(k,l)$ with $a_2 = -2.60$.

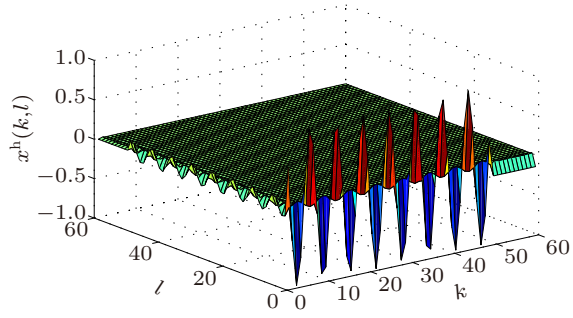


Fig. 5. (color online) State trajectory of system state variable $x^{\mathrm{h}}(k,l)$ with $a_2 = -2.60$.

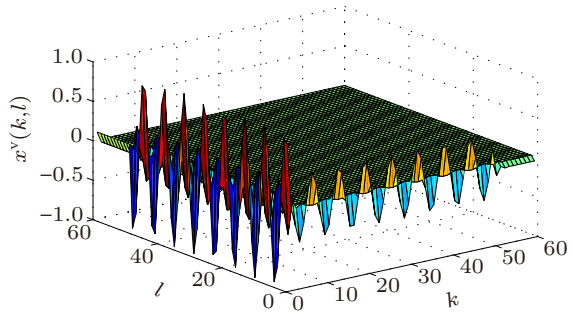


Fig. 6. (color online) State trajectory of system state variable $x^v(k,l)$ with $a_2 = -2.60$.

5. Conclusion

The problem of stability analysis and control synthesis of uncertain Roesser-type 2D systems has been studied by using the Lyapunov stability theory. The mathematical model of the discrete-time 2D system is established upon the well-known Roesser model, and the uncertainty phenomenon, which appears typically in practical environments, is modeled by a convex bounded (polytope type) uncertain domain. By applying the Lyapunov stability theory, LMI-based stability and stabilization conditions of uncertain discrete-time 2D systems are developed, which can be easily solved via the standard numerical software. In the processes of stability analysis and control synthesis, the obtained stability/stabilzaition conditions become less conservative by applying some novel relaxed techniques. Finally, numerical examples are given to demonstrate the effectiveness of the proposed results.

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