Asymptotic solution of a weak nonlinear model for the mid-latitude stationary wind field of a two-layer barotropic ocean*

Lin Wan-Tao(林万涛)[a)†], Zhang Yu(张 宇)[b)], and Mo Jia-Qi(莫嘉琪)[c)]

[a)] State Key Laboratory of Numerical Modeling for Atmospheric Sciences and Geophysical Fluid Dynamics, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, China

[b)] University of the Chinese Academy of Sciences, Beijing 100049, China

[c)] Department of Mathematics, Anhui Normal University, Wuhu 241003, China

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A weak nonlinear model of a two-layer barotropic ocean with Rayleigh dissipation is built. The analytic asymptotic solution is derived in the mid-latitude stationary wind field, and the physical meaning of the corresponding problem is discussed.

Keywords: two-layer barotropic ocean, ocean model asymptotic solution

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1. Introduction

The variations of the coupled air–sea system have been widely investigated in recent years.[1–7] An unusual atmospheric wind will cause anomalous ocean flow. Many scholars have discussed this phenomenon. Chase *et al.*[8] considered the wind stress-driven sine Sverdrup balance and the first-order baroclinic Rossby wave as well as its impact on the Atlantic Ocean. Qiu and Chen[9] used a wind stress-driven reduced gravity model to discuss the dynamic mechanism of temporal and spatial variations of the sea surface height in the South Pacific. The effect of mid-latitude surface west wind stress on the sea flow is anomalous, which means that the mid-latitudes have the largest wind stress which decays gradually toward the north and the south. Therefore, wind stress in the meridional shows nonuniform distribution. In order to overcome the limitations of a one-dimensional horizontal model, Lu *et al.*[10] and Zhang *et al.*[11,12] built a two-dimensional barotropic quasi-equilibrium ocean linear model under wind forcing, which involves the Rayleigh friction. The analytic solution is given under the influence of a mid-latitude large-scale wind field. The results show that, under the force of the westerly jet stream, in the upper flow field from the east to west coast of the ideal ocean, there is a stronger eastward flow in the westerly forcing, while near the west coast, there are streams with cyclonic curvature and anti-cyclonic curvature on both sides of eastward flow.

Recently, many scholars have studied the nonlinear problems. Many approximate methods have been developed, including the boundary layer method, the methods of matched asymptotic expansion, and the multiple scales.[13–18] Using the perturbative and other methods, the authors[19–24] also considered a series of nonlinear problems. In this paper, we build a two-dimensional barotropic quasi-equilibrium weak nonlinear ocean model under wind force, which takes Rayleigh friction into consideration, and solve its analytic asymptotic solution.

2. Wind field of a two-layer barotropic ocean model

Consider the following weakly nonlinear model in the absence of base flow, mass change between two levels, as well as Rayleigh friction induced by vertical stationary atmospheric wind stress:[10–12]

$$(f_0+\beta y)v_k-\frac{\partial(a_k\Phi_1+\Phi_2)}{\partial x}-\mu\delta_k(u_1-u_2)=\varepsilon F_k(u_k,v_k), \quad (1)$$

$$(f_0+\beta y)u_k+\frac{\partial(a_k\Phi_1+\Phi_2)}{\partial y}+\mu\delta_k(v_1-v_2)=\varepsilon G_k(u_k,v_k), \quad (2)$$

$$\frac{\partial u_k}{\partial x}+\frac{\partial v_k}{\partial y}=0, \quad (3)$$

where t denotes time; x is the zonal variable; y is the meridional variable; $k=1,2$ denotes the up and down level, respectively; u_k is the zonal wind speed; v_k is the meridional wind speed; μ is the Rayleigh friction factor (here, it is a constant); f_0 is the Coriolis parameter at y_0 ($\beta=(\partial f/\partial y)_{y=y_0}$); y_0 is the center position of the β channel; Φ is the geopotential height at which $a_1=1$, $0<a_2=\rho_1/\rho_2<1$; ρ_1 and ρ_2 are the density of the up and down level, respectively; tracer coefficients δ_k are defined as $\delta_1=1$ and $\delta_2=-1$; εF_k and εG_k are nonlinear perturbation wind stress which are sufficiently smooth bounded functions; ε is a small positive parameter.

In this article, we consider an ideal ocean climate in middle-high latitude under a stationary wind force, and βchannel in the x direction is a semi-infinite space. Let the

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†Corresponding author. E-mail: linwt@lasg.iap.ac.cn

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west coast be a rigid wall at ($x=0$), and there exists a rigid wall of the β channel at $y=y_0\pm M$. Thus, the boundary conditions of this model are

$$u_k|_{x=0}=(-1)^k a\cos\frac{\pi}{2M}(y-y_0, v_k|_{x=0}=0, \tag{4}$$

$$\frac{\partial u_k}{\partial y}\Big|_{y=y_0\pm M}=\frac{\partial v_k}{\partial y}\Big|_{y=y_0\pm M}=0, \tag{5}$$

where a and M are constants.

3. Perturbative solutions for a weak nonlinear model

Let

$$u_k=\sum_{i=0}^{\infty}u_{ki}\varepsilon^i,\quad v_k=\sum_{i=0}^{\infty}v_{ki}\varepsilon^i,\quad (k=1,2). \tag{6}$$

Substituting Eq. (6) into Eqs. (1)–(5), expanding nonlinear terms in ε, equating the coefficients of the same powers of ε and let the coefficient of ε^0 equal zero, we obtain

$$(f_0+\beta y)v_{k0}-\frac{\partial(a_k\Phi_1+\Phi_2)}{\partial x}-\mu\delta_k(u_{10}-u_{20})=0, \tag{7}$$

$$(f_0+\beta y)u_{k0}+\frac{\partial(a_k\Phi_1+\Phi_2)}{\partial y}+\mu\delta_k(v_{10}-v_{20})=0, \tag{8}$$

$$\frac{\partial u_{k0}}{\partial x}+\frac{\partial v_{k0}}{\partial y}=0, \tag{9}$$

$$u_{k0}|_{x=0}=(-1)^k a\cos\frac{\pi}{2M}(y-y_0),\quad v_{k0}|_{x=0}=0, \tag{10}$$

$$\frac{\partial u_{k0}}{\partial y}\Big|_{y=y_0\pm M}=\frac{\partial v_{k0}}{\partial y}\Big|_{y=y_0\pm M}=0. \tag{11}$$

Based on Eq. (9), we set

$$u_{k0}=-\frac{\partial W_{k0}}{\partial y},\quad v_{k0}=\frac{\partial W_{k0}}{\partial x},\quad (k=1,2). \tag{12}$$

Substituting Eq. (12) into Eqs. (7) and (8), we have

$$(f_0+\beta y)\frac{\partial W_{k0}}{\partial x}-\frac{\partial(a_k\Phi_1+\Phi_2)}{\partial x}+\mu\delta_k\Big(\frac{\partial W_{10}}{\partial y}-\frac{\partial W_{2o}}{\partial y}\Big)=0, \tag{13}$$

$$-(f_0+\beta y)\frac{\partial W_{k0}}{\partial y}+\frac{\partial(a_k\Phi_1+\Phi_2)}{\partial y}+\mu\delta_k\Big(\frac{\partial W_{10}}{\partial x}-\frac{\partial W_{20}}{\partial x}\Big)=0. \tag{14}$$

Taking the derivation of Eq. (13) with respect to y and Eq. (14) to x, and then add up them for the two sides, it follows

$$\beta\frac{\partial W_{k0}}{\partial x}+\mu\delta_k\Delta(W_{10}-W_{20})=0.$$

Equivalently,

$$\beta\frac{\partial W_{10}}{\partial x}+\mu\Delta(W_{10}-W_{20})=0, \tag{15}$$

$$\beta\frac{\partial W_{20}}{\partial x}-\mu\Delta(W_{10}-W_{20})=0. \tag{16}$$

From Eqs. (15) and (16), we have

$$\beta\frac{\partial(W_{10}+W_{20})}{\partial x}=0, \tag{17}$$

$$\beta\frac{\partial}{\partial x}(W_{10}-W_{20})+2\mu\Delta(W_{10}-W_{20})=0. \tag{18}$$

Let

$$W_{10}-W_{20}=X_0(x)\sin\Big[\frac{\pi}{2M}(y-y_0)\Big]. \tag{19}$$

Substituting Eq. (19) into Eq. (18), we have

$$\frac{\mathrm{d}^2X_0}{\mathrm{d}x^2}+\frac{\beta}{2\mu}\frac{\mathrm{d}X_0}{\mathrm{d}x}-\Big(\frac{\pi}{2M}\Big)^2X_0=0. \tag{20}$$

Obviously, equation (19) has a bounded solution

$$X_0(x)=C_0\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big],$$

where C_0 is an arbitrary constant. Substituting the above equation into Eq. (19), we obtain

$$W_{10}-W_{20}=C_0\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big]\times\sin\frac{\pi}{2M}(y-y_0). \tag{21}$$

Taking the derivation of Eq. (12) with respective to y, considering Eq. (12), and substituting $x=0$ into Eq. (10), then we obtain $C_0=-4aM/\pi$. Substituting it into Eq. (21), we have

$$W_{10}-W_{20}=-\frac{4aM}{\pi}\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big]\times\sin\frac{\pi}{2M}(y-y_0). \tag{22}$$

Taking the derivation of Eq. (22) with respective to y and x, respectively, we have

$$-u_{10}+u_{20}=2a\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big]\times\cos\frac{\pi}{2M}(y-y_0), \tag{23}$$

$$v_{20}=v_{10}-\frac{4aM}{\pi}\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)\times\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big]\times\sin\frac{\pi}{2M}(y-y_0). \tag{24}$$

From Eq. (17), we obtain

$$v_{10}=-v_{20}. \tag{25}$$

Thus, from Eqs. (24) and (25), we have

$$v_{k0}=(-1)^{k+1}\frac{2aM}{\pi}\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{(\frac{\beta}{2\mu})^2+(\frac{\pi}{M})^2}\Big)\times\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big]\times\sin\frac{\pi}{2M}(y-y_0),\quad (k=1,2). \tag{26}$$

From Eq. (17), considering the condition (10), we have

$$u_{10}+u_{20}=0. \tag{27}$$

From Eq. (23), we obtain

$$u_{k0} = (-1)^k a \exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big] \times \cos\frac{\pi}{2M}(y-y_0), \quad (k=1,2). \tag{28}$$

Remark Substituting Eq. (24) into Eqs. (7)–(11), we also derive functions Φ_1 and Φ_2. Nevertheless, their constitutions are omitted.

Substituting Eq. (6) into Eqs. (1)–(5), expanding nonlinear terms in ε, and equating the coefficients of the same powers of ε. Let the coefficients of $\varepsilon^i (i=1,2,\ldots)$ equal zero. We obtain

$$(f_0+\beta y)v_{ki} - \frac{\partial(a_k\Phi_1+\Phi_2)}{\partial x} - \mu\delta_k(u_{1i}-u_{2i}) = \bar{F}_{ki}, \tag{29}$$

$$(f_0+\beta y)u_{ki} + \frac{\partial(a_k\Phi_1+\Phi_2)}{\partial y} + \mu\delta_k(v_{1i}-v_{2i}) = \bar{G}_{ki}, \tag{30}$$

$$\frac{\partial u_{ki}}{\partial x}+\frac{\partial v_{ki}}{\partial y}=0, \tag{31}$$

$$u_{ki}|_{x=0} = v_{ki}|_{x=0} = \frac{\partial u_{ki}}{\partial y}\Big|_{y=y_0\pm M} = \frac{\partial v_{ki}}{\partial y}\Big|_{y=y_0\pm M} = 0, \tag{32}$$

where

$$\bar{F}_{ki} = \frac{1}{(i-1)!}\Big[\frac{\partial^{i-1}}{\partial\varepsilon^{i-1}}F_k\Big(\sum_{j=0}^{\infty}u_k\varepsilon^j,\sum_{j=0}^{\infty}v_k\varepsilon\Big)\Big]_{\varepsilon=0},$$

$$\bar{G}_{ki} = \frac{1}{(i-1)!}\Big[\frac{\partial^{i-1}}{\partial\varepsilon^{i-1}}G_k\Big(\sum_{j=0}^{\infty}u_k\varepsilon^j,\sum_{j=0}^{\infty}v_k\varepsilon\Big)\Big]_{\varepsilon=0},$$

$$(k=1,2, \quad i=1,2,\ldots).$$

These are successively determined functions. From Eq. (31), we set

$$u_{ki} = -\frac{\partial W_{ki}}{\partial y}, \quad v_{k0} = \frac{\partial W_{ki}}{\partial x}, \quad (k=1,2). \tag{33}$$

Substituting Eq. (33) into Eqs. (29) and (30), we have

$$(f_0+\beta y)\frac{\partial W_{ki}}{\partial x} - \frac{\partial(a_k\Phi_1+\Phi_2)}{\partial x} + \mu\delta_k\Big(\frac{\partial W_{1i}}{\partial y}-\frac{\partial W_{2i}}{\partial y}\Big) = \bar{F}_{ki}, \tag{34}$$

$$-(f_0+\beta y)\frac{\partial W_{ki}}{\partial y} + \frac{\partial(a_k\Phi_1+\Phi_2)}{\partial y} + \mu\delta_k\Big(\frac{\partial W_{1i}}{\partial x}-\frac{\partial W_{2i}}{\partial x}\Big) = \bar{G}_{ki}. \tag{35}$$

We take the derivation of Eq. (34) with respective to y, and that of Eq. (35) with respective to x. Then, summing the two resulting equations leads to

$$\beta\frac{\partial W_{ki}}{\partial x} + \mu\delta_k\Delta(W_{1i}-W_{2i}) = \frac{\partial\bar{F}_{ki}}{\partial y}+\frac{\partial\bar{G}_{ki}}{\partial x}.$$

That is,

$$\beta\frac{\partial W_{1i}}{\partial x} + \mu\Delta(W_{1i}-W_{2i}) = \frac{\partial\bar{F}_{1i}}{\partial y}+\frac{\partial\bar{G}_{1i}}{\partial x}, \tag{36}$$

$$\beta\frac{\partial W_{2i}}{\partial x} - \mu\Delta(W_{1i}-W_{2i}) = \frac{\partial\bar{F}_{2i}}{\partial y}+\frac{\partial\bar{G}_{2i}}{\partial x}. \tag{37}$$

From Eqs. (36) and (37), we obtain

$$\beta\frac{\partial(W_{1i}+W_{2i})}{\partial x} = \frac{\partial}{\partial y}(\bar{F}_{1i}+\bar{F}_{2i})+\frac{\partial}{\partial x}(\bar{G}_{1i}+\bar{G}_{2i}), \tag{38}$$

$$\beta\frac{\partial}{\partial x}(W_{1i}-W_{2i}) + 2\mu\Delta(W_{1i}-W_{2i}) = \frac{\partial}{\partial y}(\bar{F}_{1i}-\bar{F}_{2i}) + \frac{\partial}{\partial x}(\bar{G}_{1i}-\bar{G}_{2i}). \tag{39}$$

From Eqs. (33) and (38), we obtain

$$v_{1i}+v_{2i} = \frac{1}{\beta}\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}+\bar{F}_{2i})+\frac{\partial}{\partial x}(\bar{G}_{1i}+\bar{G}_{2i})\Big]. \tag{40}$$

Let

$$W_{1i}-W_{2i} = X_i(x)\sin\frac{\pi}{2M}(y-y_0). \tag{41}$$

Substituting Eq. (41) into Eq. (39), we have

$$\frac{\mathrm{d}^2X_i}{\mathrm{d}x^2}+\frac{\beta}{2\mu}\frac{\mathrm{d}X_i}{\mathrm{d}x}-(\frac{\pi}{2M})^2X_i = \frac{1}{2\mu}\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}-\bar{F}_{2i})+\frac{\partial}{\partial x}(\bar{G}_{1i}-\bar{G}_{2i})\Big]. \tag{42}$$

Obviously, equation (42) has a bounded solution

$$X_i(x) = C_i\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big] + \frac{1}{2\mu}\int_0^x\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}(\xi,y)-\bar{F}_{2i}(\xi,y)) + \frac{\partial}{\partial\xi}(\bar{G}_{1i}(\xi,y)-\bar{G}_{2i}(\xi,y))\Big] \times\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)(x-\xi)\Big]\mathrm{d}\xi,$$

where C_i is the arbitrary constant. Substituting the above expression into Eq. (41), we obtain

$$W_{1i}-W_{2i} = \Big[C_i\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big] + H_i(x,y)\Big]\sin\frac{\pi}{2M}(y-y_0), \tag{43}$$

where

$$H_i(x,y) = \frac{1}{2\mu}\int_0^x\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}(\xi,y)-\bar{F}_{2i}(\xi,y)) + \frac{\partial}{\partial\xi}(\bar{G}_{1i}(\xi,y)-\bar{G}_{2i}(\xi,y))\Big] \times\exp\Big[-(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2})(x-\xi)\Big]\mathrm{d}\xi.$$

Taking the derivation of Eq. (48) with respective to y and considering Eq. (33), we have

$$-u_{1i}+u_{2i} = \frac{\pi}{2M}\Big[C_i\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big] + H_i(x,y)\Big]\cos\frac{\pi}{2M}(y-y_0)+\frac{\partial H_i}{\partial y}\sin\frac{\pi}{2M}(y-y_0).$$

Substituting $x=0$, and considering the condition (32), we have $C_i=0$. Substituting it into the above expression, then we obtain

$$-u_{1i}+u_{2i} = \frac{\pi}{2M}[H_i(x,y)]\cos\frac{\pi}{2M}(y-y_0) + \frac{\partial H_i}{\partial y}\sin\frac{\pi}{2M}(y-y_0). \tag{44}$$

From Eq. (38), we have

$$\beta\Big(\frac{\partial u_{1i}}{\partial x}+\frac{\partial u_{2i}}{\partial x}\Big)=-\frac{\partial}{\partial y}\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}+\bar{F}_{2i})+\frac{\partial}{\partial x}(\bar{G}_{1i}+\bar{G}_{2i})\Big].$$

Similarly, we have

$$\begin{aligned}u_{1i}+u_{2i} &= -\frac{1}{\beta}\frac{\partial}{\partial y}\int_0^x\Big[\frac{\partial}{\partial y}\big(\bar{F}_{1i}(\xi,y)+\bar{F}_{2i}(\xi,y)\big)\\&\quad+\frac{\partial}{\partial x}\big(\bar{G}_{1i}(\xi,y)+\bar{G}_{2i}(\xi,y)\big)\Big]\mathrm{d}\xi. \qquad (45)\end{aligned}$$

Thus, according to Eqs. (44) and (45), we then obtain

$$\begin{aligned}u_{ki}=&-\frac{1}{2\beta}\frac{\partial}{\partial y}\int_0^x\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}(\xi,y)+\bar{F}_{2i}(\xi,y))\\&+\frac{\partial}{\partial x}(\bar{G}_{1i}(\xi,y)+\bar{G}_{2i}(\xi,y))\Big]\mathrm{d}\xi\\&+(-1)^k\frac{\pi}{4M}[H_i(x,y)]\cos\frac{\pi}{2M}(y-y_0)\\&+\frac{(-1)^k}{2}\frac{\partial H_i}{\partial y}\sin\frac{\pi}{2M}(y-y_0),\ (k=1,2,\ i=1,2,\ldots).\qquad (46)\end{aligned}$$

Taking the derivation of Eq. (43) with respective to x and considering Eqs. (32), (33), and (40), and $C_i=0$, we have

$$\begin{aligned}v_{ki} &= \frac{1}{2\beta}\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}+\bar{F}_{2i})+\frac{\partial}{\partial x}(\bar{G}_{1i}+\bar{G}_{2i})\Big]\\&\quad-\frac{(-1)^k}{2}\frac{\partial H_i}{\partial x}\sin\frac{\pi}{2M}(y-y_0),\\&\quad(k=1,2,\ i=1,2,\ldots).\qquad (47)\end{aligned}$$

Based on Eqs. (26), (28), (46), and (47), substituting u_k and v_{ki} into Eq. (6), we obtain perturbative solutions u_{kmper} and v_{kmper} for the original problems (1)–(5) as follows:

$$\begin{aligned}u_{kmper} &= (-1)^k a\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big]\cos\frac{\pi}{2M}(y-y_0)\\&\quad+\sum_{i=1}^{m}\Big[-\frac{1}{2\beta}\frac{\partial}{\partial y}\int_0^x\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}(\xi,y)+\bar{F}_{2i}(\xi,y))+\frac{\partial}{\partial x}(\bar{G}_{1i}(\xi,y)+\bar{G}_{2i}(\xi,y))\Big]\mathrm{d}\xi\\&\quad+(-1)^k\frac{\pi}{4M}[H_i(x,y)]\cos\frac{\pi}{2M}(y-y_0)+\frac{(-1)^k}{2}\frac{\partial H_i}{\partial y}\sin\frac{\pi}{2M}(y-y_0)\Big]\varepsilon^i+O(\varepsilon^{m+1}),\qquad (48)\\v_{kmper} &= (-1)^k\frac{2aM}{\pi}\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{(\frac{\beta}{2\mu})^2+(\frac{\pi}{M})^2}\Big)\exp\Big[-\Big(\frac{\beta}{4\mu}+\frac{1}{2}\sqrt{\Big(\frac{\beta}{2\mu}\Big)^2+\Big(\frac{\pi}{M}\Big)^2}\Big)x\Big]\sin\frac{\pi}{2M}(y-y_0)\\&\quad+\sum_{i=1}^{m}\Big[\frac{1}{2\beta}\Big[\frac{\partial}{\partial y}(\bar{F}_{1i}+\bar{F}_{2i})+\frac{\partial}{\partial x}(\bar{G}_{1i}+\bar{G}_{2i})\Big]-\frac{(-1)^k}{2}\frac{\partial H_i}{\partial x}\sin\frac{\pi}{2M}(y-y_0)\Big]\varepsilon^i+O(\varepsilon^{m+1}).\ (k=1,2,0<\varepsilon\ll 1).\qquad (49)\end{aligned}$$

4. Example

Observations show that there is max surface westerly intensity nearby 40° N over the west Pacific Ocean. Thus, we set the center of β channel as 40° N, half width as $M=10^3$ km, $\mu=3\times10^{-7}$ km$^{-1}\cdot$s^{-1} and $\beta=1.7536\times10^{-8}km^{-1}\cdot$s^{-1}.[12] For convenience, we set the nonlinear perturbative wind stress as $F_k=\varepsilon\beta\exp(-u_k^2)$, $G_k=0$, geopotential height $\Phi_i=0$ ($i=1,2$), and $a=1$. Thus, the weak nonlinear maritime climate model under the atmospheric wind stress (1)–(5) can transform into

$$\begin{aligned}&(f_0+1.7536\times10^{-8}y)v_k-3\times10^{-7}\delta_k(u_1-u_2)\\&=\varepsilon\beta\exp(-u_k^2),\qquad (50)\\&(f_0+1.7536\times10^{-8}y)u_k+3\times10^{-7}\delta_k(v_1-v_2)=0,\qquad (51)\\&\frac{\partial u_k}{\partial x}+\frac{\partial v_k}{\partial y}=0,\qquad (52)\\&u_k|_{x=0}=(-1)^k\cos(y-y_0),\quad v_k|_{x=0}=0,\qquad (53)\\&\frac{\partial u_k}{\partial y}|_{y=y_0\pm10^6}=\frac{\partial v_k}{\partial y}|_{y=y_0\pm10^6}=0,\qquad (54)\end{aligned}$$

where $k=1,2$. Then, let us solve the approximate perturbative solutions for the model (50)–(54). Let

$$u_{k\mathrm{per}}=\sum_{i=0}^{\infty}u_{ki}\varepsilon^i,\quad v_{k\mathrm{per}}=\sum_{i=0}^{\infty}v_{ki}\varepsilon^i,\quad (k=1,2),\qquad (55)$$

Let the coefficient of ε^0 be equal to zero, through substituting Eq. (55) into Eqs. (50)–(54), expanding nonlinear terms in ε, and equating the coefficients of the same powers of ε, we obtain

$$\begin{aligned}&(f_0+1.7536\times10^{-8}y)v_{k0}-3\times10^{-7}\delta_k(u_{10}-u_{20})=0,\qquad (56)\\&(f_0+1.7536\times10^{-8}y)u_{k0}+3\times10^{-7}\delta_k(v_{10}-v_{20})=0,\qquad (57)\\&\frac{\partial u_{k0}}{\partial x}+\frac{\partial v_{k0}}{\partial y}=0,\qquad (58)\\&u_{k0}|_{x=0}=(-1)^k\cos(y-y_0),\quad v_{k0}|_{x=0}=0,\qquad (59)\\&\frac{\partial u_{k0}}{\partial y}|_{y=y_0\pm10^5}=\frac{\partial v_{k0}}{\partial y}|_{y=y_0\pm10^6}=0.\qquad (60)\end{aligned}$$

From Eqs. (26) and (28), the solutions u_{k0} and v_{k0} for Eqs. (56)–(60) are

$$\begin{aligned}u_{k0} &= (-1)^k\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\\&\quad\times\cos\frac{\pi}{2\times10^3}(y-y_0),\qquad (61)\\v_{k0} &= (-1)^k\frac{2\times10^3}{\pi}\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)\\&\quad\times\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\\&\quad\times\sin\frac{\pi}{2\times10^3}(y-y_0).\qquad (62)\end{aligned}$$

Let the coefficient of ε^1 equal to zero. Substituting Eq. (55) into Eqs. (50)–(54), expanding nonlinear terms in ε, and equating the coefficients of the same powers of ε, we obtain

$$(f_0+1.7536\times10^{-8}y)v_{k1}-3\times10^{-7}\delta_k(u_{11}-u_{21}) = \beta\exp(-u_{k0}^2), \tag{63}$$

$$(f_0+1.7536\times10^{-8}y)u_{k1}-3\times10^{-7}\delta_k(v_{11}-v_{21})=0, \tag{64}$$

$$\frac{\partial u_{k1}}{\partial x}+\frac{\partial v_{k1}}{\partial y}=0, \tag{65}$$

$$u_{k1}|_{x=0}=v_{k1}|_{x=0}=\frac{\partial u_{k1}}{\partial y}|_{y=y_0\pm10^6}=\frac{\partial v_{k1}}{\partial y}|_{y=y_0\pm10^6}=0. \tag{66}$$

From Eqs. (46) and (47), we obtain the solutions u_{k1} and v_{k1} for the problem (63)–(66) as

$$\begin{aligned}u_{k1}=&\frac{\pi}{10^3}u_{k0}\exp(-u_{k0}^2)\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\sin\frac{\pi}{2\times10^3}(y-y_0)\\&-\frac{(-1)^k\pi\cos\frac{\pi}{2\times10^3}(y-y_0)}{\frac{1.7536}{60}\Big(\frac{1.7536}{60}+\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)}\Big[1-\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\Big]\\&-(-1)^k\frac{1.7536}{60}\sin\frac{\pi}{2\times10^3}(y-y_0),\end{aligned} \tag{67}$$

$$v_{ki}=\frac{1}{2\beta}\Big[\frac{\partial}{\partial y}(\bar F_{1i}+\bar F_{2i})+\frac{\partial}{\partial x}(\bar G_{1i}+\bar G_{2i})\Big]-\frac{(-1)^k}{2}\frac{\partial H_i}{\partial x}\sin\frac{\pi}{2M}(y-y_0),$$

$$\begin{aligned}v_{k1}=&\frac{\pi}{10^3}u_{k0}[\exp(-u_{k0}^2)]\times\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\sin\frac{\pi}{2\times10^3}(y-y_0)\\&+\frac{(-1)^k1.7536(y-y_0)}{60\Big(\frac{1.7536}{60}+\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)}\exp\Big[-\Big(\frac{1.7536}{60}+\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\times\sin\frac{\pi}{2\times10^3}(y-y_0),\end{aligned} \tag{68}$$

where u_{k0} is determined by Eqs. (61) and (62).

Thus, from Eqs. (55), (61), (62), (67), and (68), we then obtain the first approximate perturbative solutions $u_{k1\mathrm{per}}$ and $v_{k1\mathrm{per}}$ for the weak nonlinear maritime climate model (48)–(52) under stationary atmospheric wind stress as follows:

$$\begin{aligned}u_{k1\mathrm{per}}=&(-1)^k\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\cos\frac{\pi}{2\times10^3}(y-y_0)\\&\times\varepsilon\Big[\frac{\pi}{10^3}\Big[(-1)^k\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\cos\frac{\pi}{2\times10^3}(y-y_0)\Big]\\&\times\exp\Big(-\Big[\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\cos\frac{\pi}{2\times10^3}(y-y_0)\Big]^2\Big)\\&\times\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\sin\frac{\pi}{2\times10^3}(y-y_0)\\&-\frac{(-1)^k\pi\cos\frac{\pi}{2\times10^3}(y-y_0)}{\frac{1.7536}{60}\Big(\frac{1.7536}{60}+\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)}\Big[1-\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\Big]\\&-(-1)^k\frac{1.7536}{60}\sin\frac{\pi}{2M}(y-y_0)\Big]+O(\varepsilon^2),\quad(0<\varepsilon\ll1,k=1,2),\end{aligned} \tag{69}$$

$$\begin{aligned}v_{k1\mathrm{per}}=&(-1)^k\frac{2\times10^3}{\pi}\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)\\&\times\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\sin\frac{\pi}{2\times10^3}(y-y_0)\\&+\varepsilon\Big[\frac{\pi}{10^3}u_{k0}[\exp(-u_{k0}^2)]\times\exp\Big[-\Big(\frac{1.7536}{120}+\frac{1}{2}\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\sin\frac{\pi}{2\times10^3}(y-y_0)\\&+\frac{(-1)^k1.7536(y-y_0)}{60\Big(\frac{1.7536}{60}+\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)}\exp\Big[-\Big(\frac{1.7536}{60}+\sqrt{\Big(\frac{1.7536}{60}\Big)^2+\Big(\frac{\pi}{10^3}\Big)^2}\Big)x\Big]\\&\times\sin\frac{\pi}{2\times10^3}(y-y_0)\Big]+O(\varepsilon^2),\quad(0<\varepsilon\ll1,\ k=1,2).\end{aligned} \tag{70}$$

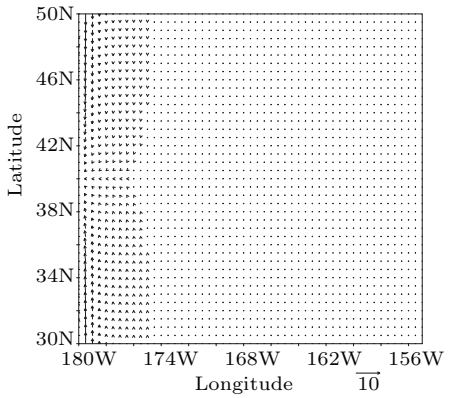


Fig. 1. Numerical solution of $(u–v)_{10}$.

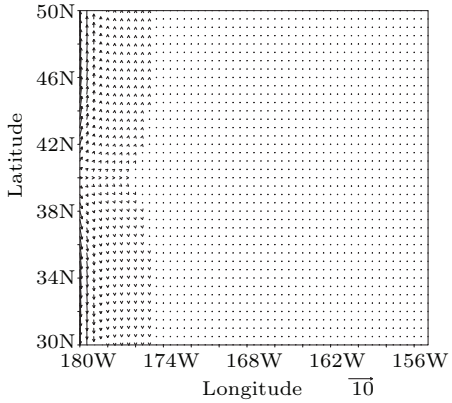


Fig. 2. Numerical solution of $(u–v)_{20}$.

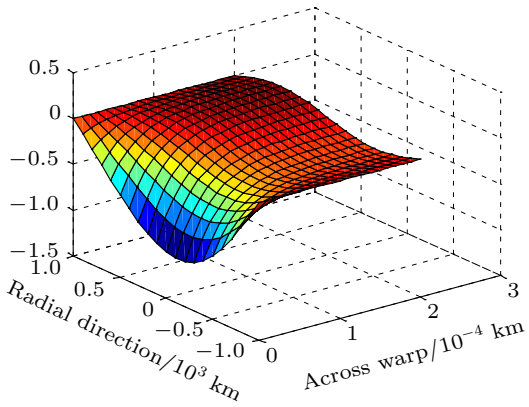


Fig. 3. (color online) Perturbative solution of $u_{11\text{per}}$.

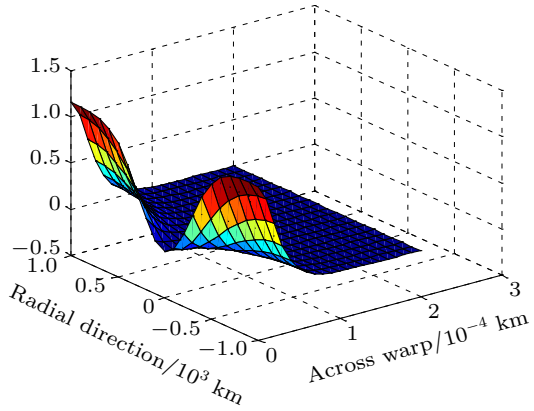


Fig. 4. (color online) Perturbative solution of $v_{11\text{per}}$.

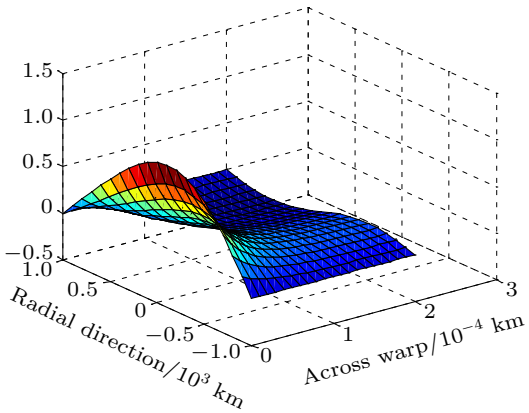


Fig. 5. (color online) Perturbative solution of $u_{21\text{per}}$.

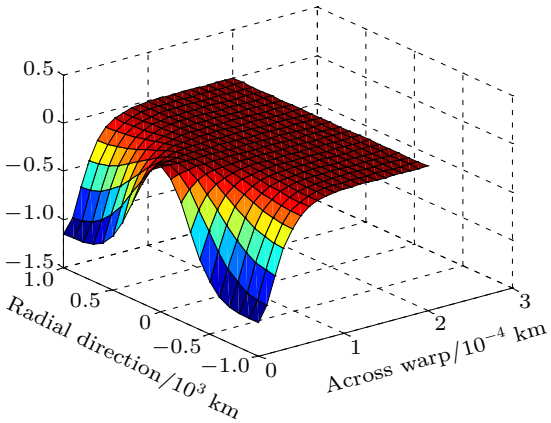


Fig. 6. (color online) Perturbative solution of $v_{21\text{per}}$.

According to Eqs. (69) and (70), the simulated figures of approximate perturbative solutions for the weak nonlinear maritime climate model (50)–(54) under stationary atmospheric wind stress are shown in Figs. 1–6 ($\varepsilon = 0$ and $\varepsilon = 0.5$).

5. Physical significance of the asymptotic solution

The results of the asymptotic solution calculate a strong flow near the west boundary and fit better with the actual flow situation. The flow in the upper layers converges to the center dimensions (40° N) and then moves to the east. The asymptotic solutions of the low-layer flow reverse their direction but have the same size as the upper-layer flow. The asymptotic solution of the flow are qualitatively consistent with the real situation, because the asymptotic solution close to the west boundary is given a strong partial flow. By contrast, the interior gyres appear much weaker.

6. Summary

The anomalous atmospheric mid-latitude western wind field will cause anomalous mid-latitude upper layer ocean flow. This is a complicated natural phenomenon. In order

to study the nonlinear model for the mid-latitude upper layer ocean flow, we sometimes need to solve them using the approximate method. In this paper, the perturbation method is a simple and valid method. This method differs from general numerical methods. The obtained analytic solution from the asymptotic perturbation method can be a continuous analytic operation. Thus, we can further study relevant physical quantities in a thorough way.

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