Two new discrete integrable systems

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In this paper, we focus on the construction of new (1+1)-dimensional discrete integrable systems according to a subalgebra of loop algebra $\tilde{A}_1$. By designing two new (1+1)-dimensional discrete spectral problems, two new discrete integrable systems are obtained, namely, a 2-field lattice hierarchy and a 3-field lattice hierarchy. When deriving the two new discrete integrable systems, we find the generalized relativistic Toda lattice hierarchy and the generalized modified Toda lattice hierarchy. Moreover, we also obtain the Hamiltonian structures of the two lattice hierarchies by means of the discrete trace identity.

Keywords: discrete integrable system, Hamiltonian structure, loop algebra

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1. Introduction

It is well known that an important and attractive topic in the soliton theory is the search for as many new integrable systems as possible. The discrete integrable systems have many applications in various scientific contexts ranging from mathematical physics, numerical analysis to quantum physics, as well as statistical physics.[1–4] Searching for new discrete integrable systems is of course very meaningful. Many discrete integrable systems have been presented and studied, such as the Toda lattice,[5–7] which possesses many beautiful integrable properties, the Volterra lattice,[1,8] and the discrete Kadomtsev–Petviashvili (KP) lattice.[9] The Tu scheme provides a powerful method for generating discrete integrable Hamiltonian systems of soliton equations in the frame of the discrete zero curvature equation.[5] Following this technique, some discrete integrable systems with Significance in physics have been established and investigated,[10–19] and the Hamiltonian structures of the discrete integrable systems can be constructed by the discrete trace identity. The key step of using Tu scheme is to design proper spectral problems. Constructing subalgebras of loop algebras, which is presented by Guo and Zhang,[15] is a simple and efficient method for designing proper spectral problems and generating integrable systems.

In this paper, we consider two (1+1)-dimensional discrete spectral problems:

$$\begin{aligned} &E\varphi_n = \boldsymbol{U}_n(u_n,\lambda)\varphi_n,\ \lambda_t = 0,\\ &\boldsymbol{U}_n(u_n,\lambda) = \begin{pmatrix} 0 & \lambda \\ -r_n\lambda & \lambda^2 - p_n \end{pmatrix}, \end{aligned} \tag{1}$$

and

$$E\varphi_n = U_n(u_n,\lambda)\varphi_n,\ \lambda_t = 0,$$

$$U_n(u_n,\lambda) = \begin{pmatrix} \lambda^2 + p_n & q_n\lambda \\ r_n\lambda & 1 \end{pmatrix}, \tag{2}$$

where u_n is a potential function, λ is a spectral parameter, and E is a shift operator defined as follows:

$$\begin{aligned} &Ef(n,t) = f(n+1,t) = f_{n+1},\\ &E^{-1}f(n,t) = f(n-1,t) = f_{n-1},\ \ (n\in\mathbb{Z}, t\in\mathbb{R}). \end{aligned}$$

To the best of our knowledge, the above discrete spectral problems are new. By using the Tu scheme, we obtain the new (1+1)-dimensional discrete integrable systems and the Hamiltonian structures.

2. A new discrete integrable system

Loop algebra $\tilde{A}_1$ with the following basis is frequently used to construct the discrete spectral problems by the Tu scheme.

$$\begin{aligned} &\boldsymbol{e}_1(n) = \begin{pmatrix} \lambda^n & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{e}_2(n) = \begin{pmatrix} 0 & \lambda^n \\ 0 & 0 \end{pmatrix},\\ &\boldsymbol{e}_3(n) = \begin{pmatrix} 0 & 0 \\ \lambda^n & 0 \end{pmatrix}, \quad \boldsymbol{e}_4(n) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^n \end{pmatrix},\\ &[\boldsymbol{e}_1(m),\boldsymbol{e}_4(n)] = 0, \quad [\boldsymbol{e}_3(m),\boldsymbol{e}_4(n)] = -\boldsymbol{e}_3(m+n),\\ &[\boldsymbol{e}_1(m),\boldsymbol{e}_2(n)] = \boldsymbol{e}_2(m+n),\\ &[\boldsymbol{e}_1(m),\boldsymbol{e}_3(n)] = -\boldsymbol{e}_3(m+n),\\ &[\boldsymbol{e}_2(m),\boldsymbol{e}_3(n)] = \boldsymbol{e}_1(m+n) - \boldsymbol{e}_4(m+n),\\ &[\boldsymbol{e}_2(m),\boldsymbol{e}_4(n)] = \boldsymbol{e}_4(m+n),\\ &\deg[\boldsymbol{e}_i(n)] = n, \quad (i = 1,2,3,4). \end{aligned}$$

In this paper, we choose the subalgebra of $\tilde{A}_1$ as

$$\begin{aligned} &\boldsymbol{e}_1(n) = \begin{pmatrix} \lambda^{2n} & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{e}_2(n) = \begin{pmatrix} 0 & \lambda^{2n+1} \\ 0 & 0 \end{pmatrix},\\ &\boldsymbol{e}_3(n) = \begin{pmatrix} 0 & 0 \\ \lambda^{2n+1} & 0 \end{pmatrix}, \quad \boldsymbol{e}_4(n) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^{2n} \end{pmatrix}, \end{aligned}$$

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$$
\begin{aligned}
&[\boldsymbol{e}_1(m),\boldsymbol{e}_4(n)]=0,\quad [\boldsymbol{e}_3(m),\boldsymbol{e}_4(n)]=-\boldsymbol{e}_3(m+n),\\
&[\boldsymbol{e}_1(m),\boldsymbol{e}_2(n)]=\boldsymbol{e}_2(m+n),\\
&[\boldsymbol{e}_1(m),\boldsymbol{e}_3(n)]=-\boldsymbol{e}_3(m+n),\\
&[\boldsymbol{e}_2(m),\boldsymbol{e}_3(n)]=\boldsymbol{e}_1(m+n+1)-\boldsymbol{e}_4(m+n+1),\\
&[\boldsymbol{e}_2(m),\boldsymbol{e}_4(n)]=\boldsymbol{e}_4(m+n),\\
&\deg[\boldsymbol{e}_i(n)]=2n,\ \ (i=1,4);\\
&\deg[\boldsymbol{e}_i(n)]=2n+1,\ \ (i=2,3).
\end{aligned}
$$

Consider the discrete spectral problem (1)

$$
\begin{aligned}
&E\varphi_n=\boldsymbol{U}_n(u_n,\lambda)\varphi_n,\ \ \lambda_t=0,\\
&\boldsymbol{U}_n(u_n,\lambda)=\begin{pmatrix}0 & \lambda\\ -r_n\lambda & \lambda^2-p_n\end{pmatrix}\\
&=\boldsymbol{e}_4(1)+\boldsymbol{e}_2(0)-r_n\boldsymbol{e}_3(0)-p_n\boldsymbol{e}_4(0).
\end{aligned}
$$

Let

$$
\begin{aligned}
\boldsymbol{\Gamma}&=\begin{pmatrix}a_n & b_n\lambda\\ c_n\lambda & -a_n\end{pmatrix}\\
&=a_n\boldsymbol{e}_1(0)+b_n\boldsymbol{e}_2(0)+c_n\boldsymbol{e}_3(0)-a_n\boldsymbol{e}_4(0),
\end{aligned}
$$

where

$$
a_n=\sum_{j=0}^{\infty}a_n^{(j)}\lambda^{-2j},\ \ b_n=\sum_{j=0}^{\infty}b_n^{(j)}\lambda^{-2j},\ \ c_n=\sum_{j=0}^{\infty}c_n^{(j)}\lambda^{-2j}.
$$

Solving the discrete stationary zero-curvature equation

$$
E\boldsymbol{\Gamma}\boldsymbol{U}_n-\boldsymbol{U}_n\boldsymbol{\Gamma}=0, \tag{3}
$$

we obtain the recursive relationship

$$
\begin{aligned}
&\lambda^2c_n=-r_n\lambda^2b_{n+1},\\
&\lambda(a_{n+1}+a_n)+\lambda^3b_{n+1}-p_n\lambda b_{n+1}=0,\\
&\lambda r_n(a_{n+1}+a_n)-\lambda^3c_n+p_n\lambda c_n=0,\\
&\lambda^2c_{n+1}-\lambda^2(a_{n+1}-a_n)+p_n(a_{n+1}-a_n)\\
&+\lambda^2r_nb_n=0.
\end{aligned} \tag{4}
$$

Then, according to Eq. (4), we have

$$
\begin{aligned}
&a_{n+1}^{(j+1)}-a_n^{(j+1)}=p_n(a_{n+1}^{(j)}-a_n^{(j)})+r_nb_n^{(n+1)}-r_{n+1}b_{n+2}^{(j+1)},\\
&b_{n+1}^{(j+1)}=p_nb_{n+1}^{(j)}-(a_{n+1}^{(j)}+a_n^{(j)}),\\
&a_n^{(0)}=\frac{1}{2},\quad b_n^{(0)}=0,\quad a_n^{(1)}=r_n,\\
&b_{n+1}^{(1)}=-1,\quad b_{n+1}^{(2)}=-p_n-(r_{n+1}+r_n),\\
&a_n^{(2)}=p_nr_n+p_{n-1}r_n+r_n^2+r_nr_{n+1}+r_nr_{n-1},\ldots.
\end{aligned} \tag{5}
$$

Note that

$$
\begin{aligned}
\boldsymbol{\Gamma}_+^{(m)}&=\sum_{j=0}^{m}\begin{pmatrix}a_n^{(j)} & b_n^{(j)}\lambda\\ b_n^{(j)}\lambda & -a_n^{(j)}\end{pmatrix}\lambda^{-2j},\\
\boldsymbol{\Gamma}_-^{(m)}&=\lambda^{2m}\boldsymbol{\Gamma}-\boldsymbol{\Gamma}_+^{(m)},
\end{aligned}
$$

then equation (3) can be written as

$$
E\boldsymbol{\Gamma}_+^{(m)}\boldsymbol{U}_n-\boldsymbol{U}_n\boldsymbol{\Gamma}_+^{(m)}=-E\boldsymbol{\Gamma}_-^{(m)}\boldsymbol{U}_n+\boldsymbol{U}_n\boldsymbol{\Gamma}_-^{(m)}.
$$

As a result,

$$
\begin{aligned}
&E\boldsymbol{\Gamma}_+^{(m)}\boldsymbol{U}_n-\boldsymbol{U}_n\boldsymbol{\Gamma}_+^{(m)}\\
&=-b_{n+1}^{(m+1)}\boldsymbol{e}_2(0)+c_n^{m+1}\boldsymbol{e}_3(0)+p_n(a_{n+1}^{(m)}-a_n^{(m)})\boldsymbol{e}_4(0).
\end{aligned} \tag{6}
$$

Taking

$$
\boldsymbol{V}_n^{(m)}=\boldsymbol{\Gamma}_n^{(m)}+\boldsymbol{\Delta}_n^{(m)},\quad \boldsymbol{\Delta}_n^{(m)}=b_n^{(m+1)}\boldsymbol{e}_1(0),
$$

then we obtain

$$
\begin{aligned}
&E\boldsymbol{V}_n^{(m)}\boldsymbol{U}_n-\boldsymbol{U}_n\boldsymbol{V}_n^{(m)}\\
&=(c_n^{m+1}+r_nb_n^{(m+1)})\boldsymbol{e}_3(0)+p_n(a_n^{(m)}-a_{n+1}^{(m)})\boldsymbol{e}_4(0).
\end{aligned}
$$

Let the time evolution of the eigenfunction φ_n of Eq. (1) satisfy

$$
\varphi_{n,t_m}=\boldsymbol{V}_n^{(m)}\varphi_n. \tag{7}
$$

The compatibility conditions (1) and (7) give the following discrete zero curvature equation:

$$
\boldsymbol{U}_{n,t_m}=E\boldsymbol{V}_n^{(m)}\boldsymbol{U}_n-\boldsymbol{U}_n\boldsymbol{V}_n^{(m)}. \tag{8}
$$

Then, equation (8) determines the Lax integrable system as follows:

$$
\begin{aligned}
\begin{pmatrix}p_{n,t_m}\\ r_{n,t_m}\end{pmatrix}&=\begin{pmatrix}p_n(a_n^{(m)}-a_{n+1}^{(m)})\\ r_n(b_{n+1}^{(m+1)}-b_n^{(m+1)})\end{pmatrix}=\boldsymbol{J}\begin{pmatrix}-b_{n+1}^{(m)}\\ \dfrac{a_n^{(m)}}{r_n}\end{pmatrix}\\
&=\boldsymbol{J}\boldsymbol{L}\begin{pmatrix}-b_{n+1}^{(m-1)}\\ \dfrac{a_n^{(m-1)}}{r_n}\end{pmatrix}=\boldsymbol{J}\boldsymbol{L}^{m-1}\begin{pmatrix}-b_{n+1}^{(1)}\\ \dfrac{a_n^{(1)}}{r_n}\end{pmatrix},
\end{aligned} \tag{9}
$$

where

$$
\boldsymbol{J}=\begin{pmatrix}0 & p_n(1-E)r_n\\ r_n(E^{-1}-1)p_n & r_n(E^{-1}-1)(1+E)r_n\end{pmatrix},
$$

$$
\boldsymbol{L}=\begin{pmatrix}p_n & (E+1)r_n\\ \dfrac{1}{r_n}(E-1)^{-1}(r_{n+1}E-r_nE^{-1})p_n & \dfrac{1}{r_n}(E-1)^{-1}(p_n(E-1)-(r_nE^{-1}-r_{n+1}E)(E+1))r_n\end{pmatrix}.
$$

It can be verified that $\boldsymbol{JL} = \boldsymbol{L}^*\boldsymbol{J}$, thus the system (9) is Liouville integrable. Taking $m = 1$ in Eq. (9) gives rise to

$$\begin{aligned} p_{n,t_1} &= p_n(r_n - r_{n+1}),\\ r_{n,t_1} &= r_n(p_{n-1} - p_n + r_{n-1} - r_{n+1}), \end{aligned}$$

which is similar to the relativistic Toda lattice equation,[7] so we call Eq. (9) the generalized relativistic Toda lattice hierarchy.

In the following, we construct the Hamiltonian structure of Eq. (9). A direct calculation reads

$$\begin{aligned} \boldsymbol{V}_n &= \boldsymbol{\Gamma}_n \boldsymbol{U}_n^{-1} = \begin{pmatrix} \dfrac{a_n(\lambda^2 - p_n)}{r_n\lambda^2} + b_n & -\dfrac{a_n}{r_n\lambda}\\ \dfrac{\lambda c_n(\lambda^2 - p_n)}{r_n\lambda^2} - \dfrac{a_n}{r_n} & -\dfrac{c_n}{r_n} \end{pmatrix},\\ <\boldsymbol{V}_n, \frac{\partial \boldsymbol{U}_n}{\partial p_n}> &= -\frac{c_n}{r_n} = -b_{n+1},\\ <\boldsymbol{V}_n, \frac{\partial \boldsymbol{U}_n}{\partial r_n}> &= \frac{a_n}{r_n},\\ <\boldsymbol{V}_n, \frac{\partial \boldsymbol{U}_n}{\partial \lambda}> &= -\frac{c_n\lambda}{r_n} - \frac{c_n p_n}{r_n\lambda}. \end{aligned}$$

Substituting the above formula into discrete trace identity and comparing the coefficients of λ^{-2m-1} yields

$$\begin{pmatrix} \dfrac{\delta}{\delta p_n}\\ \dfrac{\delta}{\delta r_n} \end{pmatrix} \sum_{n\in\mathbb{Z}} \frac{-c_n^{(m+1)} - p_n c_n^{(m)}}{r_n} = (\gamma - 2m) \begin{pmatrix} -b_{n+1}^{(m)}\\ \dfrac{a_n^{(m)}}{r_n} \end{pmatrix}.$$

Setting $m = 0$ leads to $\gamma = 0$. Therefore, we obtain the Hamiltonian structure of Eq. (9) as

$$\begin{aligned} \boldsymbol{u}_{n,t_m} &= \begin{pmatrix} p_n\\ r_n \end{pmatrix}_{t_m} = \boldsymbol{J}\frac{\delta H_n^{(m)}}{\delta u},\\ H_n^{(m)} &= \frac{-c_n^{(m+1)} - p_n c_n^{(m)}}{2mr_n}. \end{aligned} \tag{10}$$

3. Another new discrete integrable system

Consider the spectral problem (2)

$$\begin{aligned} &E\varphi_n = \boldsymbol{U}_n(u,\lambda)\varphi_n, \quad \lambda_t = 0,\\ &\boldsymbol{U}_n(u,\lambda) = \begin{pmatrix} \lambda^2 + p_n & q_n\lambda\\ r_n\lambda & 1 \end{pmatrix}\\ &= \boldsymbol{e}_1(1) + p_n\boldsymbol{e}_1(0) + q_n\boldsymbol{e}_2(0) + r_n\boldsymbol{e}_3(0) + \boldsymbol{e}_4(0). \end{aligned}$$

Let

$$\begin{aligned} \boldsymbol{\Gamma} &= \begin{pmatrix} a_n & b_n\lambda\\ c_n\lambda & -a_n \end{pmatrix}\\ &= a_n\boldsymbol{e}_1(0) + b_n\boldsymbol{e}_2(0) + c_n\boldsymbol{e}_3(0) - a_n\boldsymbol{e}_4(0), \end{aligned}$$

where

$$a_n = \sum_{j=0}^{\infty} a_n^{(j)}\lambda^{-2j}, \quad b_n = \sum_{j=0}^{\infty} b_n^{(j)}\lambda^{-2j}, \quad c_n = \sum_{j=0}^{\infty} c_n^{(j)}\lambda^{-2j}.$$

Solving the stationary discrete zero curvature equation (3), we obtain the recursive relationship

$$\begin{aligned} &(\lambda^2 + p_n)(a_{n+1} - a_n) + \lambda^2 r_n b_{n+1} - \lambda^2 q_n c_n = 0,\\ &\lambda q_n a_{n+1} + \lambda b_{n+1} - (\lambda^2 + p_n)\lambda b_n + \lambda q_n a_n = 0,\\ &(\lambda^2 + p_n)\lambda c_{n+1} - \lambda r_n(a_{n+1} + a_n) - \lambda c_n = 0,\\ &\lambda^2 q_n c_{n+1} - \lambda^2 r_n b_n - (a_{n+1} - a_n) = 0. \end{aligned} \tag{11}$$

It follows from Eq. (11) that

$$\begin{aligned} &a_{n+1}^{(j+1)} - a_n^{(j+1)} = q_n c_n^{(j+1)} - p_n(a_{n+1}^{(j)} - a_n^{(j)}) - r_n b_{n+1}^{(j+1)},\\ &b_n^{(j+1)} = q_n(a_{n+1}^{(j)} + a_n^{(j)}) - p_n b_n^{(j)} + b_{n+1}^{(j)},\\ &c_{n+1}^{(j+1)} = r_n(a_{n+1}^{(j)} + a_n^{(j)}) - p_n c_{n+1}^{(j)} + c_n^{(j)},\\ &q_n c_{n+1}^{(j+1)} - r_n b_n^{(j+1)} = a_{n+1}^{(j)} - a_n^{(j)},\\ &a_{n+1}^{(0)} - a_n^{(0)} = q_n c_n^{(0)} - r_n b_{n+1}^{(0)},\\ &b_n^{(0)} = c_n^{(0)} = 0, \ \ a_n^{(0)} = \frac{1}{2}, \ \ b_n^{(1)} = q_n,\\ &c_n^{(1)} = r_{n-1}, \quad a_n^{(1)} = -r_{n-1}q_n,\\ &b_n^{(2)} = q_n(-r_n q_{n+1} - r_{n-1}q_n) - p_n q_n - q_{n+1},\\ &c_{n+1}^{(2)} = r_n(q_n r_{n-1} - r_n q_{n+1}) - p_n r_n + r_{n-1}, \ldots. \end{aligned}$$

Note that

$$\begin{aligned} \boldsymbol{\Gamma}_+^{(m)} &= \sum_{j=0}^{m} \begin{pmatrix} a_n^{(j)} & b_n^{(j)}\lambda\\ b_n^{(j)}\lambda & -a_n^{(j)} \end{pmatrix} \lambda^{-2j},\\ \boldsymbol{\Gamma}_-^{(m)} &= \lambda^{2m}\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_+^{(m)}. \end{aligned}$$

According to

$$E\boldsymbol{\Gamma}_+^{(m)}\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{\Gamma}_+^{(m)} = -E\boldsymbol{\Gamma}_-^{(m)}\boldsymbol{U}_n + \boldsymbol{U}_n\boldsymbol{\Gamma}_-^{(m)},$$

we obtain

$$\begin{aligned} &E\boldsymbol{\Gamma}_+^{(m)}\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{\Gamma}_+^{(m)}\\ &= p_n(a_{n+1}^{(m)} - a_n^{(m)})\boldsymbol{e}_1(0) + b_n^{(m+1)}\boldsymbol{e}_2(0)\\ &\quad - c_{n+1}^{(m+1)}\boldsymbol{e}_3(0) + (a_n^{(m)} - a_{n+1}^{(m)})\boldsymbol{e}_4(0). \end{aligned}$$

Taking

$$\boldsymbol{V}_n^{(m)} = \boldsymbol{\Gamma}_n^{(m)} + \boldsymbol{\Delta}_n^{(m)}, \quad \boldsymbol{\Delta}_n^{(m)} = a_n^{(m)}\boldsymbol{e}_4(0),$$

then we obtain

$$\begin{aligned} &E\boldsymbol{V}_n^{(m)}\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{V}_n^{(m)}\\ &= p_n(a_{n+1}^{(m)} - a_n^{(m)})\boldsymbol{e}_1(0) + (-q_n a_n^{(m)} + b_n^{(m+1)})\boldsymbol{e}_2(0)\\ &\quad + (r_n a_{n+1}^{(m)} - c_{n+1}^{(m+1)})\boldsymbol{e}_3(0). \end{aligned}$$

Let the time evolution of the eigenfunction φ_n of Eq. (2) satisfy Eq. (7). The compatibility conditions (2) and (7) give the discrete zero curvature equation (8) and determine the Lax integrable system as follows:

$$\begin{pmatrix} p_{n,t_m}\\ q_{n,t_m}\\ r_{n,t_m} \end{pmatrix} = \begin{pmatrix} p_n(a_{n+1}^{(m)} - a_n^{(m)})\\ -q_n a_n^{(m)} + b_n^{(m+1)}\\ r_n a_{n+1}^{(m)} - c_{n+1}^{(m+1)} \end{pmatrix}. \tag{12}$$

Setting $m=1$ in Eq. (12) gives rise to

$$\begin{aligned} p_{n,t_1} &= p_n(r_{n-1}q_n - r_nq_{n+1}),\\ q_{n,t_1} &= q_n(-r_nq_{n+1} - 2r_{n-1}q_n) - p_nq_n + q_{n+1},\\ r_{n,t_1} &= -r_nq_nr_{n-1} + p_nr_n - r_{n-1}. \end{aligned} \tag{13}$$

Let $q_nr_n = 1$, we can show that the consistency condition $\mathrm{d}(q_nr_n)/\mathrm{d}t = 0$ holds. It follows from Eq. (12) that

$$\begin{aligned} \frac{\mathrm{d}(q_nr_n)}{\mathrm{d}t} &= r_nq_{n,tm} + q_nr_{n,tm}\\ &= -q_nr_na_n^{(m)} + r_nb_n^{(m+1)} + q_nr_na_{n+1}^{(m)} - q_nc_{n+1}^{(m+1)}, \end{aligned}$$

and then it follows from the last of the recursive relationships (11) that $\mathrm{d}(q_nr_n)/\mathrm{d}t = 0$. Thus, we see that equation (12) reduces to

$$\begin{pmatrix} p_{n,t_m}\\ q_{n,t_m} \end{pmatrix} = \begin{pmatrix} p_n(a_{n+1}^{(m)} - a_n^{(m)})\\ -q_na_n^{(m)} + b_n^{(m+1)} \end{pmatrix}. \tag{14}$$

When $m=1$, $q_n \to \mathrm{e}^{q_n}$, $r_n \to \mathrm{e}^{-q_n}$, and $p_n \to \mathrm{e}^{p_n}$, equation (14) is rewritten as

$$p_{n,t_1} = \mathrm{e}^{q_n-q_{n-1}} - \mathrm{e}^{q_{n+1}-q_n},\quad q_{n,t_1} = -\mathrm{e}^{p_n} - 2\mathrm{e}^{q_n-q_{n-1}},$$

which is similar to the modified Toda lattice equation,[7] so we call Eq. (14) the generalized modified Toda lattice hierarchy.

In the following, we construct the Hamiltonian structure of Eq. (14). A direct calculation reads

$$\begin{aligned} \boldsymbol{V}_n &= \boldsymbol{\Gamma}_n\boldsymbol{U}_n^{-1}\\ &= \frac{1}{p_n}\begin{pmatrix} a_n - r_n\lambda^2b_n & -q_n\lambda a_n + \lambda^3b_n + p_n\lambda b_n\\ \lambda c_n + r_n\lambda a_n & -q_n\lambda^2c_n - \lambda^2a_n - p_na_n \end{pmatrix},\\ &< V_n, \frac{\partial U_n}{\partial p_n} >= \frac{a_n}{p_n} - \frac{\lambda^2b_n}{p_nq_n},\\ &< V_n, \frac{\partial U_n}{\partial q_n} >= \frac{\lambda^2c_n}{p_n} + \frac{\lambda^2a_n}{p_nq_n},\\ &< V_n, \frac{\partial U_n}{\partial \lambda} >= a_{n+1} - a_n + \frac{2\lambda^2b_n}{q_n}. \end{aligned}$$

Substituting the above formula into discrete trace identity and comparing the coefficients of λ^{-2m-1} gives

$$\begin{pmatrix} \dfrac{\delta}{\delta p_n}\\ \dfrac{\delta}{\delta q_n} \end{pmatrix} \sum_{n\in\mathbb{Z}} \left(a_{n+1}^{(m)} - a_n^{(m)} + \frac{2b_n^{(m+1)}}{q_n} \right)$$

$$= (\gamma - 2m)\begin{pmatrix} \dfrac{a_n^{(m)}}{p_n} - \dfrac{b_n^{(m+1)}}{p_nq_n}\\ \dfrac{c_n^{(m+1)}}{p_n} + \dfrac{a_n^{(m+1)}}{p_nq_n} \end{pmatrix}.$$

Setting $m=0$ leads to $\gamma = 0$. Therefore, we obtain the Hamiltonian structure of Eq. (14) as

$$\begin{aligned} \boldsymbol{u}_{n,t_m} &= \begin{pmatrix} p_n\\ q_n \end{pmatrix}_{t_m} = \boldsymbol{J}\frac{\delta H_n^{(m)}}{\delta u},\\ H_n^{(m)} &= \frac{-a_{n+1}^{(m)} + a_n^{(m)}}{2m} + \frac{-b_n^{(m+1)}}{mq_n}. \end{aligned}$$

4. Conclusions

In this paper, we have constructed two new discrete spectral problems by the subalgebra of loop algebra $\tilde{A}_1$ and two new discrete integrable systems, namely a 2-field lattice hierarchy and a 3-field lattice hierarchy. We have considered reductions of our discrete integrable systems, and found the generalized relativistic Toda lattice hierarchy and the generalized modified Toda lattice hierarchy. Furthermore, we have also obtained the Hamiltonian structures of the two lattice hierarchies by means of the discrete trace identity.

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