Mei symmetry and conservation laws of discrete nonholonomic dynamical systems with regular and irregular lattices*

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In this paper, Noether symmetry and Mei symmetry of discrete nonholonomic dynamical systems with regular and the irregular lattices are investigated. Firstly, the equations of motion of discrete nonholonomic systems are introduced for regular and irregular lattices. Secondly, for cases of the two lattices, based on the invariance of the Hamiltomian functional under the infinitesimal transformation of time and generalized coordinates, we present the quasi-extremal equation, the discrete analogues of Noether identity, Noether theorems, and the Noether conservation laws of the systems. Thirdly, in cases of the two lattices, we study the Mei symmetry in which we give the discrete analogues of the criterion, the theorem, and the conservative laws of Mei symmetry for the systems. Finally, an example is discussed for the application of the results.

Keywords: regular and irregular lattice, nonholonomic system, Noether symmetry, Mei symmetry

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1. Introduction

The applications of Lie group theory have been described in many papers and books.[1–3] Lie group theory has been developed into a powerful tool to solve differential equations, construct properties of their solution space, and to establish the symmetric theory.[4,5] Symmetric theories are extremely useful tools in mathematics, mechanics, and physics. These symmetric theories have primarily established the relationships between symmetry and conserved quantity in the dynamical system. Noether symmetry theory and Lie symmetry theory are built by the infinitesimal transformations with respect to time and the generalized coordinates.[6–12] Non-Noether theory is a special type of symmetry theory, which keeps the equations of motion invariant under infinitesimal transformations, rather than Hamilton action.[13,14] In recent years, these symmetry approaches have been applied to discrete dynamical systems. The Lie symmetry theory of difference equation, Noether-type symmetry theory of a discrete system, and non-Noether symmetry theory of a discrete electromechanical system have been investigated in many studies.[15–28]

The form invariance (i.e., Mei symmetry) is a new symmetry, which was proposed by Mei.[29–31] The form invariance of mechanical systems are the invariances of differential equations when we first take infinitesimal transformations of a group of physical quantities such as Lagrange, non-potential forces, and generalized constraint forces. Recently, the study on Mei symmetry has become an active topic in many systems, such as the Lagrange system,[32–34] Chaplygin system,[35] rotational relativistic system,[36,37] generalized classical mechanics,[38–40] Birkhoff system,[41] Tzénoff equations of holonomic system,[42] nonholonomic mechanical system,[43,44] and general discrete and continuum holonomic system.[45,46] The relationship between Mei symmetry and Mei conserved quantity is also given for the above systems. However, the studies on Mei symmetry are mostly considered in continuous dynamical systems. The purpose of this paper is to extend these studies to Noether symmetry and the Mei conservation law of discrete nonholonomic dynamical systems.

The present paper is organized as follows. Section 2 defines the transformation operators in the space of discrete variables, and the equations of motion of discrete nonholonomic system are introduced through two types of discrete variables. In Section 3, we give the discrete analogue of a generalized variational formula of the systems, from which we derive the discrete analogue of generalized Noether-type identity, and then we obtain the discrete analogue of Noether theorems and the discrete analogue of Noether conservation laws of the system. In Section 4, the definition and criterion of Mei symmetry of discrete nonholonomic dynamical systems are obtained by an infinitesimal transformation group, and we study a new type of conserved quantity which is directly induced by the theorem of Mei symmetry of this system. In Section 5, we

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also discuss an example to illustrate the above results. Finally some conclusions are drawn in Section 6.

2. Discrete transformation operators and equations of motion of discrete systems

Let us consider the space of discrete variables. Using the simplest invariant mesh h in the t direction, we write the operator of discrete transformation to the right and left as

$$\underset{+h}{S} = \mathrm{e}^{hD} = \sum_{i=0}^{\infty} \frac{h^i}{i!} D^i, \quad \underset{-h}{S} = \mathrm{e}^{-hD} = \sum_{i=0}^{\infty} \frac{(-h)^i}{i!} D^i, \tag{1}$$

where

$$D = \frac{\partial}{\partial x} + \dot{q}^s \frac{\partial}{\partial q^s} + \ddot{q}^s \frac{\partial}{\partial \dot{q}^s} + \cdots, \tag{2}$$

which is a full differentiation operator of continuous variables, the operators $\underset{+h}{S}$ and $\underset{-h}{S}$ commute with each other, and satisfy $\underset{+h}{S} \cdot \underset{-h}{S} = \underset{-h}{S} \cdot \underset{+h}{S} = 1$. It is also convenient to use the following total shift and discrete differentiation operators:

$$\underset{\pm h}{S} f(q^s) = f(q^s_{\pm}), \underset{\pm h}{D} = \frac{\underset{\pm h}{S} - 1}{\pm h_{\pm}}, \tag{3}$$

where $h_+ = t_+ - t$ and $h_- = t - t_-$. Two of the operators $\underset{+h}{S}$, $\underset{-h}{S}$, $\underset{+h}{D}$, and $\underset{-h}{D}$ commute with each other, and $\underset{\pm h}{D} = \underset{\mp h}{D}\,\underset{\pm h}{S}$ obey the corresponding finite-difference Leibniz rule[24]

$$\begin{aligned} \underset{+h}{D}(FG) &= \underset{+h}{D}(F)\,\underset{+h}{S}(G) + F\,\underset{+h}{D}(G) \\ &= \underset{+h}{D}(F)G + \underset{+h}{S}(F)\,\underset{+h}{D}(G), \\ \underset{-h}{D}(FG) &= \underset{-h}{D}(F)\,\underset{-h}{S}(G) + F\,\underset{-h}{D}(G) \\ &= \underset{-h}{D}(F)G + \underset{-h}{S}(F)\,\underset{-h}{D}(G). \end{aligned} \tag{4}$$

We now introduce the infinitesimal transformations with respect to time and the generalized coordinates:

$$t^* = t + \Delta t, \quad q^{s*} = q^s + \Delta q^s, \quad (s = 1,2,...,n), \tag{5}$$

and their expanded form is

$$t^* = t + \varepsilon \xi^0(t,q^s), \quad q^{s*} = q^s + \varepsilon \xi^s(t,q^s), \quad (s = 1,2,...,n), \tag{6}$$

which constitutes the single-parameter Lie group of infinitesimal transformation, where $\xi^0(t,q^s)$ and $\xi^s(t,q^s)$ are infinitesimal transformation generators, and ε is the infinitesimal parameter. The group G_1 is completely defined by the operator

$$X = \xi^0 \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial q^s}. \tag{7}$$

We can extend the operator (7) into the space of discrete variables $(t, q^s, q^s_k, q^s_{k\bar{k}}, ..., h_+, h_-)$ as

$$\begin{aligned} prX &= \xi^0 \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial q^s} + \varsigma_1 \frac{\partial}{\partial q^s_k} + \varsigma_2 \frac{\partial}{\partial q^s_{k\bar{k}}} + \cdots \\ &\quad + h_+ \underset{+h}{D}(\xi^0) \frac{\partial}{\partial h_+} + h_- \underset{+h}{D}(\xi^0) \frac{\partial}{\partial h_-}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} q^s_k &= \underset{+h}{D}(q^s) = \frac{q^s_+ - q^s}{h_+}, \quad q^s_{k\bar{k}} = \underset{-h}{D}\,\underset{+h}{D}(q^s), \\ \varsigma_1 &= \underset{+h}{D}(\xi^s) - q^s_k \underset{+h}{D}(\xi^0), \quad \varsigma_2 = \underset{-h}{D}(\varsigma_1) - q^s_{k\bar{k}} \underset{+h}{D}(\xi^0). \end{aligned} \tag{9}$$

The discrete mechanical system with Lagrangian $L_{\mathrm{d}}(t, q^s, q^s_k)$, subjected to the discrete nonholonomic constrains of the form

$$f^{\mathrm{d}}_\beta(t, q^s, q^s_k) = 0, \quad (\beta = 1,2,\ldots,g), \tag{10}$$

satisfies the Appell–Chetaev conditions

$$\frac{\partial f^{\mathrm{d}}_\beta}{\partial q^s_k} \delta q^s = 0, \quad (s = 1,2,...,n), \tag{11}$$

where δq^s is the discrete virtual placement which satisfies

$$\delta q^s = \Delta q^s - q^s_{\bar{k}} \Delta t = \varepsilon(\xi^s - q^s_{\bar{k}} \xi^0). \tag{12}$$

The equations of motion of a discrete nonholonomic dynamical system in the space of discrete variables $(t, q^s, q^s_k, q^s_{k\bar{k}}, ..., h_+, h_-)$ are[24,26]

$$\frac{\partial L_{\mathrm{d}}}{\partial q^s} - \underset{-h}{D}\left(\frac{\partial L_{\mathrm{d}}}{\partial q^s_k}\right) = -\Lambda^s_{\mathrm{d}}, \tag{13}$$

$$\frac{\partial L_{\mathrm{d}}}{\partial t} + \underset{-h}{D}\left(q^s_k \frac{\partial L_{\mathrm{d}}}{\partial q^s_k} - L_{\mathrm{d}}\right) = \Lambda^s_{\mathrm{d}} q^s_{\bar{k}}, \tag{14}$$

where

$$\Lambda^s_{\mathrm{d}} = \lambda_\beta \frac{\partial f^{\mathrm{d}}_\beta}{\partial q^s_k}$$

is the force corresponding to discrete nonholonomic constrains, with λ_β being the discrete constraint multiplication.

Discrete variables in the form $(t_-, t, t_+, q^s_-, q^s, q^s_+)$ correspond to the following operator:

$$\begin{aligned} \tilde{X} &= \xi^0_- \frac{\partial}{\partial t_-} + \xi^0 \frac{\partial}{\partial t} + \xi^0_+ \frac{\partial}{\partial t_+} + \xi^s_- \frac{\partial}{\partial q^s_-} \\ &\quad + \xi^s \frac{\partial}{\partial q^s} + \xi^s_+ \frac{\partial}{\partial q^s_+}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \xi^0_- &= \underset{-h}{S}(\xi^0), \quad \xi^0_+ = \underset{+h}{S}(\xi^0), \\ \xi^s_- &= \underset{-h}{S}(\xi^s), \quad \xi^s_+ = \underset{+h}{S}(\xi^s). \end{aligned} \tag{16}$$

The discrete nonholonomic constraint is given by

$$f^{\mathrm{d}}_\beta(t, t_+, q^s, q^s_+) = 0, \quad (\beta = 1,2,\ldots,g), \tag{17}$$

which satisfies the Appell–Chetaev condition

$$\frac{\partial f^{\mathrm{d}}_\beta}{\partial \underset{+h}{D}(q^s)} \delta q^s = 0, \quad (s = 1,2,\ldots,n), \tag{18}$$

where δq^s is the discrete virtual placement which satisfies[25]

$$\delta q^s = \Delta q^s - q^s_{\bar{k}}\Delta t = \varepsilon\left(\xi^s - \frac{q^s - q^s_-}{h_-}\xi^0\right). \tag{19}$$

The equations of motion of a discrete nonholonomic dynamical system with the space of discrete variables $(t_-, t, t_+, q^s_-, q^s, q^s_+)$ are given by[24,26]

$$\frac{\partial L_\mathrm{d}}{\partial q^s} + \frac{h_-}{h_+}\frac{\partial L^-_\mathrm{d}}{\partial q^s} = -(\Lambda^s_\mathrm{d} + \frac{h_-}{h_+}\Lambda^{s-}_\mathrm{d}), \tag{20}$$

$$\frac{\partial L_\mathrm{d}}{\partial t} + \frac{\partial L^-_\mathrm{d}}{\partial t}\frac{h_-}{h_+} - \underset{-h}{D}(L_\mathrm{d}) = \left(\Lambda^s_\mathrm{d} + \frac{h_-}{h_+}\Lambda^{s-}_\mathrm{d}\right)\frac{q^s - q^s_-}{h_-}, \tag{21}$$

where $L^-_\mathrm{d} = \underset{-h}{S}[L_\mathrm{d}(t, t_+, q^s, q^s_+)] = L^-_\mathrm{d}(t_-, t, q^s_-, q^s)$, and $\Lambda^{s-}_\mathrm{d}(t_-, t, q^s_-, q^s) = \underset{-h}{S}(\Lambda^s_\mathrm{d})$.

3. Noether symmetry of a discrete nonholonomic dynamical system

The discrete Hamilton action on a regular lattice

$$M_\mathrm{d} = \sum_{\underset{h}{\Omega}} L_\mathrm{d}(t, q^s, q^s_k)h_+, \quad (h_+ = h_-), \tag{22}$$

which is defined on a domain $\underset{h}{\Omega}$ of a one-dimensional lattice.

The discrete Hamilton action M_d in the group G_1 of infinitesimal transformation can be derived by

$$M^*_\mathrm{d} = \sum_{\underset{h}{\Omega^*}} L_\mathrm{d}(t^*, q^{s*}, q^{s*}_k)h^*_+, \quad (h^*_+ = h^*_-), \tag{23}$$

where $h_+ \neq h^*_+, h_- \neq h^*_-$, and $M_\mathrm{d} \neq M^*_\mathrm{d}$.

Definition 1 If the quasi-invariant Hamilton action of the discrete nonholonomic dynamical system with a regular lattice is generalized under the infinitesimal transformation (6), and equation (11) holds, i.e., the infinitesimal transformations satisfies

$$\begin{aligned}\Delta M_\mathrm{d} &= M^*_\mathrm{d} - M_\mathrm{d} \\ &= -\sum_{s=1}^{n}(\underset{+h}{D}(G) + \Lambda^s_\mathrm{d}\delta q^s)h_+, \quad (h_+ = h_-),\end{aligned} \tag{24}$$

where $G_\mathrm{d} = G_\mathrm{d}(t, q^s, q^s_k)$is the discrete gauge function, then we call the infinitesimal transformation (6) the discrete analogue of the generalized Noether quasi-symmetrical transformation.

According to the method used in Refs. [22] and [24], the invariance condition of a discrete functional on the regular lattice is expressed as follows:

$$\begin{aligned}&\xi^0\frac{\partial L_\mathrm{d}}{\partial t} + \xi^s\frac{\partial L_\mathrm{d}}{\partial q^s} + \underset{+h}{D}(\xi^s)\frac{\partial L_\mathrm{d}}{\partial q^s_k} + \left(L_\mathrm{d} - q^s_k\frac{\partial L_\mathrm{d}}{\partial q^s_k}\right)\underset{+h}{D}(\xi^0) \\ &+ \Lambda^s_\mathrm{d}(\xi^s - q^s_{\bar{k}}\xi^0) + \underset{+h}{D}(G_\mathrm{d}) \\ &= \xi^0\left[\frac{\partial L_\mathrm{d}}{\partial t} + \underset{-h}{D}\left(q^s_k\frac{\partial L_\mathrm{d}}{\partial q^s_k} - L_\mathrm{d}\right) - \Lambda^s_\mathrm{d}q^s_{\bar{k}}\right] \\ &= \xi^s\left[\frac{\partial L_\mathrm{d}}{\partial q^s} - \underset{-h}{D}\left(\frac{\partial L_\mathrm{d}}{\partial q^s_k}\right) + \Lambda^s_\mathrm{d}\right] \\ &+ \underset{+h}{D}\left[\xi^0 L^-_\mathrm{d} + (\xi^s - q^s_{\bar{k}}\xi^0)\left(\frac{\partial L^-_\mathrm{d}}{\partial q^s_{\bar{k}}}\right) + G_\mathrm{d}\right], \\ &\underset{-h}{D}\underset{+h}{D}(\xi^0) = 0.\end{aligned} \tag{25}$$

If the left-hand side of Eq. (25) equals 0, i.e.,

$$\begin{aligned}&\xi^0\frac{\partial L_\mathrm{d}}{\partial t} + \xi^s\frac{\partial L_\mathrm{d}}{\partial q^s} + \underset{+h}{D}(\xi^s)\frac{\partial L_\mathrm{d}}{\partial q^s_k} + \left(L_\mathrm{d} - q^s_k\frac{\partial L_\mathrm{d}}{\partial q^s_k}\right)\underset{+h}{D}(\xi^0) \\ &+ \Lambda^s_\mathrm{d}(\xi^s - q^s_{\bar{k}}\xi^0) + \underset{+h}{D}(G_\mathrm{d}) = 0,\end{aligned} \tag{26}$$

it is called the discrete analogue of generalized Noether-type identity for the discrete nonholonomic dynamical system. On the right-hand side of Eq. (25), when

$$\begin{aligned}&\xi^0\left[\frac{\partial L_\mathrm{d}}{\partial t} + \underset{-h}{D}\left(q^s_k\frac{\partial L_\mathrm{d}}{\partial q^s_k} - L_\mathrm{d}\right) - \Lambda^s_\mathrm{d}q^s_{\bar{k}}\right] \\ &+ \xi^s\left[\frac{\partial L_\mathrm{d}}{\partial q^s} - \underset{-h}{D}\left(\frac{\partial L_\mathrm{d}}{\partial q^s_k}\right) + \Lambda^s_\mathrm{d}\right] = 0,\end{aligned} \tag{27}$$

it is called the quasi-extremal equations of the discrete nonholonomic dynamical system. From Eqs. (25)–(27), this system possesses the discrete analogy of conserved quantity

$$I^\mathrm{d}_N = \xi^0 L^-_\mathrm{d} + (\xi^s - q^s_{\bar{k}}\xi^0)\left(\frac{\partial L^-_\mathrm{d}}{\partial q_{\bar{k}}}\right) + G_\mathrm{d} = \mathrm{const}. \tag{28}$$

Now we present the following theorem.

Theorem 1 If the Lie group (8) of the infinitesimal transformation (6) for the discrete system (13) and (14) with regular lattice and a discrete gauge functionG_d satisfy the discrete analogue of generalized Noether-type identity (26) and the discrete analogue of Appell–Chetaev condition (11), then the nonholonomic dynamical system possesses the discrete analogue of Noether conserved quantity (28).

If discrete variables have the form $(t_-, t, t_+, q^s_-, q^s, q^s_+)$, the lattice in not regular, and we have the following definition.

Definition 2 If the quasi-invariant Hamilton action of the discrete nonholonomic dynamical system with an irregular lattice is generalized under the infinitesimal transformation (6), and equation (18) holds, i.e., the infinitesimal transformations satisfy

$$\begin{aligned}&\Delta M_\mathrm{d} = M^*_\mathrm{d} - M_\mathrm{d} = -\sum_{s=1}^{n}[\underset{+h}{D}(G) + \Lambda^s_\mathrm{d} + \frac{h_-}{h_+}\Lambda^{s-}_\mathrm{d}]\delta q^s h_+, \\ &h_+ = \varphi(t, t_+, q^s, q^s_+),\end{aligned} \tag{29}$$

where $G_\mathrm{d} = G_\mathrm{d}(t, t_+, q^s, q^s_+)$ is the discrete gauge function, then we call the infinitesimal transformation (6) the discrete analogue of the generalized Noether quasi-symmetrical transformation with variable form $(t_-, t, t_+, q^s_-, q^s, q^s_+)$.

If the Lagrangian has the form $L_\mathrm{d}(t, t_+, q^s, q^s_+)$, under the discrete nonholonomic constraint (17), according to the

method in Refs. [24] and [26], the invariance condition of a discrete functional on the irregular lattice is given by

$$\begin{aligned}
&\xi^0\frac{\partial L_{\rm d}}{\partial t}+\xi^0_+\frac{\partial L_{\rm d}}{\partial t_+}+\xi^s\frac{\partial L_{\rm d}}{\partial q^s}+\xi^s_+\frac{\partial L_{\rm d}}{\partial q^s_+}+L_{\rm d}\underset{+h}{D}(\xi^0)\\
&+\left(\Lambda^s_{\rm d}+\frac{h_-}{h_+}\Lambda^{s-}_{\rm d}\right)\left(\xi^s-\frac{q^s-q^s_-}{h_-}\xi^0\right)+\underset{+h}{D}(G_{\rm d})\\
=&\ \xi^0\left[\frac{\partial L_{\rm d}}{\partial t}+\frac{\partial L^-_{\rm d}}{\partial t}\frac{h_-}{h_+}-\underset{-h}{D}(L_{\rm d})-\left(\Lambda^s_{\rm d}+\frac{h_-}{h_+}\Lambda^{s-}_{\rm d}\right)\frac{q^s-q^s_-}{h_-}\right]\\
&+\xi^s\left[\frac{\partial L_{\rm d}}{\partial q^s}+\frac{\partial L^-_{\rm d}}{\partial q^s}\frac{h_-}{h_+}+\left(\Lambda^s_{\rm d}+\frac{h_-}{h_+}\Lambda^{s-}_{\rm d}\right)\right]\\
&+\underset{+h}{D}\left[\xi^0L^-_{\rm d}+\xi^0h_-\frac{\partial L^-_{\rm d}}{\partial t}+\xi^sh_-\frac{\partial L^-_{\rm d}}{\partial q^s}+G_{\rm d}\right], \qquad (30)\\
&\xi^0_+\left(1-\frac{\partial\varphi}{\partial t_+}\right)-\xi^0\left(1+\frac{\partial\varphi}{\partial t}\right)-\xi^s\frac{\partial\varphi}{\partial q^s}-\xi^s_+\frac{\partial\varphi}{\partial q^s_+}\\
=&\ 0. \qquad (31)
\end{aligned}$$

If the left-hand side of Eq. (30) equals 0, i.e.,

$$\begin{aligned}
&\xi^0\frac{\partial L_{\rm d}}{\partial t}+\xi^0_+\frac{\partial L_{\rm d}}{\partial t_+}+\xi^s\frac{\partial L_{\rm d}}{\partial q^s}+\xi^s_+\frac{\partial L_{\rm d}}{\partial q^s_+}+L_{\rm d}\underset{+h}{D}(\xi^0)\\
&+\left(\Lambda^s_{\rm d}+\frac{h_-}{h_+}\Lambda^{s-}_{\rm d}\right)\left(\xi^s-\frac{q^s-q^s_-}{h_-}\xi^0\right)+\underset{+h}{D}(G_{\rm d})\\
=&\ 0, \qquad (32)
\end{aligned}$$

it is called the discrete analogue of a generalized Noether-type identity for the discrete nonholonomic dynamical system with an irregular lattice. If the right-hand side of Eq. (30) equals 0, i.e.,

$$\begin{aligned}
&\xi^0\left[\frac{\partial L_{\rm d}}{\partial t}+\frac{\partial L^-_{\rm d}}{\partial t}\frac{h_-}{h_+}-\underset{-h}{D}(L_{\rm d})-\left(\Lambda^s_{\rm d}+\frac{h_-}{h_+}\Lambda^{s-}_{\rm d}\right)\frac{q^s-q^s_-}{h_-}\right]\\
&+\xi^s\left[\frac{\partial L_{\rm d}}{\partial q^s}+\frac{\partial L^-_{\rm d}}{\partial q^s}\frac{h_-}{h_+}+\left(\Lambda^s_{\rm d}+\frac{h_-}{h_+}\Lambda^{s-}_{\rm d}\right)\right]=0, \qquad (33)
\end{aligned}$$

it is called the quasi-extremal equations corresponding to the equations of motion (20) and (21), then this system possesses the conserved quantity

$$I^{\rm d}_N=\xi^0L^-_{\rm d}+h_-\xi^0\frac{\partial L^-_{\rm d}}{\partial t}+h_-\xi^s\frac{\partial L^-_{\rm d}}{\partial q^s}+G_{\rm d}=\text{const}. \qquad (34)$$

Theorem 2 If the Lie group (15) of the infinitesimal transformations (6) for the discrete system (20) and (21) with an irregular lattice, where there is a discrete gauge function $G_{\rm d}$, satisfy the discrete analogue of the generalized Noether-type identity (32), and the discrete analogue of the Appell–Chetaev condition (18) holds, then the discrete nonholonomic dynamical system possesses the discrete analogue of a Noether conserved quantity (34).

4. Mei symmetry of the discrete nonholonomic dynamical system

In the following, we study the Mei symmetry of a discrete nonholonomic dynamical system with a regular lattice. Under the infinitesimal transformations (6), $L_{\rm d}(t,q^s,q^s_k)$, $f^{\rm d}_\beta$, h_+, h_-, and $\Lambda^s_{\rm d}$ become $L^*_{\rm d}(t^*,q^{s*},q^{s*}_k)$, f^{d*}_β, h^*_+, h^*_-, and $\Lambda^{s*}_{\rm d}$, and we obtain the following equations:

$$f^{d*}_\beta(t^*,q^{s^*},q^{s^*}_k)=0, \qquad (35)$$

$$\frac{\partial L^*_{\rm d}}{\partial q^s}-\underset{-h}{D}\left(\frac{\partial L^*_{\rm d}}{\partial q^s_k}\right)=-\Lambda^{s^*}_{\rm d}, \qquad (36)$$

$$\frac{\partial L^*_{\rm d}}{\partial t}+\underset{-h}{D}\left(q^s_k\frac{\partial L^*_{\rm d}}{\partial q^s_k}-L^*_{\rm d}\right)=\Lambda^{s*}_{\rm d}q^s_{\bar{k}}, \qquad (37)$$

$$h^*_+=h^*_-. \qquad (38)$$

Expanding $L^*_{\rm d}$, $\Lambda^{s*}_{\rm d}$, f^{d*}_β, and $h^*_+-h^*_-$, one obtains

$$\begin{aligned}
&L^*_{\rm d}=L^*_{\rm d}(t^*,q^{s*},q^{s*}_k)=L_{\rm d}(t,q^s,q^s_k)+\varepsilon prX(L_{\rm d})+O(\varepsilon^2),\\
&\Lambda^{s*}_{\rm d}=\Lambda^{s*}_{\rm d}(t^*,q^{s*},q^{s*}_k)=\Lambda^s_{\rm d}+\varepsilon prX(\Lambda^s_{\rm d})+O(\varepsilon^2),\\
&f^{d*}_\beta=f^{d*}_\beta(t^*,q^{s*},q^{s*}_k)=f^{\rm d}_\beta(t,q^s,q^s_k)+\varepsilon prX(f^{\rm d}_\beta)+O(\varepsilon^2),\\
&h^*_+-h^*_-=h_+-h_-+\varepsilon prX(h_+-h_-)+O(\varepsilon^2). \qquad (39)
\end{aligned}$$

Substituting Eq. (39) into Eqs. (35)–(38), ignoring ε^2 and higher-order infinitesimal terms, we have the following criterion.

Criterion 1 If the infinitesimal generators ξ^0 and ξ^s satisfy the following relationship:

$$prX(f^{\rm d}_\beta(t,q^s,q^s_k))=0, \qquad (40)$$

$$\frac{\partial prX(L_{\rm d})}{\partial q^s}-\underset{-h}{D}\frac{\partial prX(L_{\rm d})}{\partial q^s_k}=-prX\left(\Lambda^s_{\rm d}\right), \qquad (41)$$

$$\frac{\partial pr(L_{\rm d})}{\partial t}+\underset{-h}{D}\left(q^s_k\frac{\partial pr(L_{\rm d})}{\partial q^s_k}-pr(L_{\rm d})\right)=pr(\Lambda^s_{\rm d})q^s_{\bar{k}}, \qquad (42)$$

$$prX(h_+-h_-)=0, \qquad (43)$$

then the form invariance is called a form invariance of the system with a regular lattice, i.e., the Mei symmetry of the discrete nonholonomic system with a regular lattice. Equations (40)–(43) are called the determining equations of Mei symmetry of the discrete nonholonomic system with a regular lattice.

Theorem 3 For the discrete nonholonomic dynamical system with a Lie group (8), if the infinitesimal generators ξ^0 and ξ^s of Mei symmetry and a discrete gauge function $G_{\rm d}=G_{\rm d}(t,q^s,q^s_k)$ satisfy the structure equation

$$\begin{aligned}
&prX(L_{\rm d})\underset{+h}{D}(\xi^0)+prX[prX(L_{\rm d})]\\
&+prX(\Lambda^s_{\rm d})(\xi^s-q^s_{\bar{k}}\xi^0)+\underset{+h}{D}(G_{\rm d})=0, \qquad (44)
\end{aligned}$$

then the Mei symmetry of the system with a regular lattice will lead to the following discrete conserved quantity:

$$\begin{aligned}
I^{\rm d}_M=&\ prX(L^-_{\rm d})\xi^0+\frac{\partial prX(L^-_{\rm d})}{\partial q^s_{\bar{k}}}(\xi^s-q^s_{\bar{k}}\xi^0)+G_{\rm d}\\
=&\ \text{const}. \qquad (45)
\end{aligned}$$

We prove it below.

$$\underset{+h}{D}(I^{\rm d}_M)=\underset{+h}{D}\left[prX(L^-_{\rm d})\xi^0+\frac{\partial prX(L^-_{\rm d})}{\partial q^s_{\bar{k}}}(\xi^s-q^s_{\bar{k}}\xi^0)+G_{\rm d}\right]$$

$$
\begin{aligned}
&= \frac{prX(L_{\rm d})\xi^0_+ - prX(L^-_{\rm d})\xi^0}{h_+} + \frac{\partial prX(L_{\rm d})}{\partial q^s_k} \\
&\quad \times \underset{+h}{D}(\xi^s - q^s_k\xi^0) - prX(L_{\rm d})\underset{+h}{D}(\xi^0) \\
&\quad + (\xi^s - q^s_k\xi^0)\underset{-h}{D}\left(\frac{\partial prX(L_{\rm d})}{\partial q^s_k}\right) - prX[prX(L_{\rm d})] \\
&\quad - prX(\Lambda^s_{\rm d})(\xi^s - q^s_k\xi^0) \\
&= \frac{prX(L_{\rm d})\xi^0_+ - prX(L^-_{\rm d})\xi^0}{h_+} + \frac{\partial prX(L_{\rm d})}{\partial q^s_k} \\
&\quad \times [\underset{+h}{D}(\xi^s) - q^s_k\underset{+h}{D}(\xi^0)] - prX(L_{\rm d})\underset{+h}{D}(\xi^0) \\
&\quad - q^s_{k\bar{k}}\xi^0\frac{\partial prX(L_{\rm d})}{\partial q^s_k} + (\xi^s - q^s_k\xi^0)\underset{-h}{D}\left(\frac{\partial prX(L_{\rm d})}{\partial q^s_k}\right) \\
&\quad - \xi^0\frac{\partial prX(L_{\rm d})}{\partial t} - \xi^s\frac{\partial prX(L_{\rm d})}{\partial q^s_k} \\
&\quad - [\underset{+h}{D}(\xi^s) - q^s_k\underset{+h}{D}(\xi^0)]\frac{\partial prX(L_{\rm d})}{\partial q^s_k} \\
&\quad - prX(\Lambda^s_{\rm d})(\xi^s - q^s_k\xi^0) \\
&= -\xi^0\left\{\frac{\partial prX(L_{\rm d})}{\partial t} + \underset{-h}{D}\left[q^s_k\frac{\partial prX(L_{\rm d})}{\partial q^s_k}\right.\right. \\
&\quad \left.\left. - prX(L_{\rm d})\right] - prX(\Lambda^s_{\rm d})q^s_k\right\} \\
&\quad - \xi^s\left\{\frac{\partial prX(L_{\rm d})}{\partial q^s} - \underset{-h}{D}\left(\frac{\partial prX(L_{\rm d})}{\partial q^s_k}\right) + prX(\Lambda^s_{\rm d})\right\} \\
&= 0. \qquad (46)
\end{aligned}
$$

In a similar way, if the Lagrange function has the form $L_{\rm d}(t,t_+,q^s,q^s_+)$, and generally the lattice is irregular, under the infinitesimal transformations (6), we obtain

$$f^{\rm d^*}_\beta(t^*,t^*_+,q^{s^*},q^{s^*}_k) = 0, \qquad (47)$$

$$\frac{\partial L^*_{\rm d}}{\partial q^s} + \frac{h_-}{h_+}\frac{\partial L^{-*}_{\rm d}}{\partial q^s_k} = -\left(\Lambda^{s^*}_{\rm d} + \frac{h_-}{h_+}\Lambda^{s-*}_{\rm d}\right), \qquad (48)$$

$$\frac{\partial L^*_{\rm d}}{\partial t} + \frac{\partial L^{-*}_{\rm d}}{\partial t}\frac{h_-}{h_+} - \underset{-h}{D}(L^*_{\rm d}) = (\Lambda^{s*}_{\rm d} + \frac{h_-}{h_+}\Lambda^{s-*}_{\rm d})\frac{q^s - q^s_-}{h_-}, \qquad (49)$$

$$h^*_+ = \varphi^*(t^*,t^*_+,q^{s^*},q^{s^*}_+). \qquad (50)$$

Expanding $L^*_{\rm d}$, $\Lambda^{s*}_{\rm d} + \frac{h_-}{h_+}\Lambda^{s-*}_{\rm d}$, $f^{\rm d*}_\beta$, and h^*_+, we have

$$
\begin{aligned}
L^*_{\rm d} &= L^*_{\rm d}(t^*,t^*_+,q^{s^*},q^{s^*}_+) \\
&= L_{\rm d}(t^*,t^*_+,q^{s^*},q^{s^*}_+) + \varepsilon prX(L_{\rm d}) + O(\varepsilon^2), \\
\Lambda^{s*}_{\rm d} + \frac{h_-}{h_+}\Lambda^{s-*}_{\rm d} &= \Lambda^s_{\rm d} + \frac{h_-}{h_+}\Lambda^{s-}_{\rm d} + \varepsilon\left[\tilde{X}(\Lambda^{s-}_{\rm d})\right]O(\varepsilon^2), \\
f^{d*}_\beta &= f^{d*}_\beta(t^*,t^*_+,q^{s^*},q^{s^*}_+) \\
&= h_+ - \varphi(t^*,t^*_+,q^{s^*},q^{s^*}_+) \\
&= \varepsilon\tilde{X}(h_+ - \varphi(t^*,t^*_+,q^{s^*},q^{s^*}_+)) + O(\varepsilon^2). \qquad (51)
\end{aligned}
$$

Substituting Eq. (51) into Eqs. (47)–(50), ignoring ε^2 and higher-order infinitesimal terms, we have the following criterion.

Criterion 2 If the infinitesimal generators ξ^0 and ξ^s satisfy the following relationship:

$$\tilde{X}(f^{\rm d}_\beta(t,t_+,q^s,q^s_+)) = 0, \qquad (52)$$

$$\frac{\partial\tilde{X}(L_{\rm d})}{\partial q^s} + \frac{h_-}{h_+}\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial q^s} = -\tilde{X}(\Lambda^s_{\rm d}) - \frac{h_-}{h_+}\tilde{X}(\Lambda^{s-}_{\rm d}), \qquad (53)$$

$$
\begin{aligned}
&\frac{\partial\tilde{X}(L_{\rm d})}{\partial t} + \frac{h_-}{h_+}\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial t} - \underset{-h}{D}[\tilde{X}(L_{\rm d})] \\
&\quad = \left[\tilde{X}(\Lambda^s_{\rm d}) + \frac{h_-}{h_+}\tilde{X}(\Lambda^{s-}_{\rm d})\right]\frac{q^s - q^s_-}{h_-}, \qquad (54)
\end{aligned}
$$

$$\tilde{X}(h_+ - \varphi(t,t_+,q^s,q^s_+)) = 0, \qquad (55)$$

then the form invariance is called a form invariance of the system with an irregular lattice, i.e., the Mei symmetry of the discrete nonholonomic system with an irregular lattice. Equations (52)–(55) are called the determining equations of Mei symmetry of the discrete nonholonomic system with an irregular lattice.

Theorem 4 For a discrete nonholonomic dynamical system with a Lie group (15), if the infinitesimal generators ξ^0 and ξ^s of Mei symmetry and a discrete gauge function$G_{\rm d} = G_{\rm d}(t,t_+,q^s,q^s_+)$ satisfy the structural equation

$$
\begin{aligned}
&\tilde{X}(L_{\rm d})\underset{+h}{D}(\xi^0) + \tilde{X}[\tilde{X}(L_{\rm d})] + \left[\tilde{X}(\Lambda^s_{\rm d}) + \frac{h_-}{h_+}\tilde{X}(\Lambda^{s-}_{\rm d})\right] \\
&\times \left(\xi^s - \frac{q^s - q^s_-}{h_-}\xi^0\right) + \underset{+h}{D}(G_{\rm d}) = 0, \qquad (56)
\end{aligned}
$$

then the Mei symmetry of the system with an irregular lattice will lead to the following discrete conserved quantity:

$$
\begin{aligned}
I^{\rm d}_M &= \xi^0\tilde{X}(L^-_{\rm d}) + \xi^0 h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial t} + \xi^s h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial q^s} + G_{\rm d} \\
&= \text{const.} \qquad (57)
\end{aligned}
$$

This can be proved as follows:

$$
\begin{aligned}
&\underset{+h}{D}(I^{\rm d}_M) \\
&= \underset{+h}{D}\left[\xi^0\tilde{X}(L^-_{\rm d}) + \xi^0 h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial t} + \xi^s h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial q^s} + G_{\rm d}\right] \\
&= \tilde{X}(L_{\rm d})\underset{+h}{D}(\xi^0) + \xi^0\underset{-h}{D}[\tilde{X}(L_{\rm d})] \\
&\quad + \frac{1}{h_+}\left[\xi^0_+ h_+\frac{\partial\tilde{X}(L_{\rm d})}{\partial t_+} - \xi^0 h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial t}\right] \\
&\quad + \frac{1}{h_+}\left[\xi^s_+ h_+\frac{\partial\tilde{X}(L_{\rm d})}{\partial t_+} - \xi^s h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial q^s}\right] + \underset{+h}{D}(G_{\rm d}) \\
&= \tilde{X}(L_{\rm d})\underset{+h}{D}(\xi^0) + \xi^0\underset{-h}{D}[\tilde{X}(L_{\rm d})] \\
&\quad + \frac{1}{h_+}\left[\xi^0_+ h_+\frac{\partial\tilde{X}(L_{\rm d})}{\partial t_+} - \xi^0 h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial t}\right] \\
&\quad + \frac{1}{h_+}\left[\xi^s_+ h_+\frac{\partial\tilde{X}(L_{\rm d})}{\partial q^s_+} - \xi^s h_-\frac{\partial\tilde{X}(L^-_{\rm d})}{\partial q^s}\right] \\
&\quad + \tilde{X}(L_{\rm d})\underset{+h}{D}(\xi^0) - \xi^0\frac{\partial\tilde{X}(L_{\rm d})}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
&-\xi_+^0\frac{\partial\tilde{X}(L_{\rm d})}{\partial t_+}-\xi^s\frac{\partial\tilde{X}(L_{\rm d})}{\partial q^s}-\xi_+^s\frac{\partial\tilde{X}(L_{\rm d})}{\partial q_+^s}\\
&-\left[\tilde{X}(\Lambda_{\rm d}^s)+\frac{h_-}{h_+}\tilde{X}(\Lambda_{\rm d}^{s-})\right]\left(\xi^s-\frac{q^s-q_-^s}{h_-}\xi^0\right)\\
=&-\xi^0\left\{\frac{\partial\tilde{X}(L_{\rm d})}{\partial t}+\frac{h_-}{h_+}\frac{\partial\tilde{X}(L_{\rm d}^-)}{\partial t}-\underset{-h}{D}[\tilde{X}(L_{\rm d})]\right.\\
&\left.-[\tilde{X}(\Lambda_{\rm d}^s)+\frac{h_-}{h_+}\tilde{X}(\Lambda_{\rm d}^{s-})]\frac{q^s-q_-^s}{h_-}\right\}\\
&-\xi^s\left\{\frac{\partial\tilde{X}(L_{\rm d})}{\partial q^s}+\frac{h_-}{h_+}\frac{\partial\tilde{X}(L_{\rm d}^-)}{\partial q^s}+\left[\tilde{X}(\Lambda_{\rm d}^s)+\frac{h_-}{h_+}\tilde{X}(\Lambda_{\rm d}^{s-})\right]\right\}\\
=&\ 0. \qquad (58)
\end{aligned}
$$

5. Example

We consider a discrete nonholonomic system with regular lattice. The discrete Lagrange function in the form

$$
\begin{aligned}
&L_{\rm d}=\frac{1}{2}[(q_k^1)^2+(q_k^2)^2],\\
&h_+=h_-,
\end{aligned}
\qquad (59)
$$

is limited by the discrete nonholonomic constraint

$$
f^{\rm d}=q_k^1+btq_k^2-bq^2+t=0, \qquad (60)
$$

where b is constant.

The equations of motion of the discrete nonholonomic system with a regular lattice are given by

$$
\begin{aligned}
&-q_{k\bar{k}}^1+\lambda\frac{\partial f^{\rm d}}{\partial q_k^1}=0,\ \ -q_{k\bar{k}}^2+\lambda\frac{\partial f^{\rm d}}{\partial q_k^2}=0,\\
&q_k^1q_{k\bar{k}}^1+q_k^2q_{k\bar{k}}^2=\lambda\frac{\partial f^{\rm d}}{\partial q_k^1}q_k^1+\lambda\frac{\partial f^{\rm d}}{\partial q_k^2}q_k^2.
\end{aligned}
\qquad (61)
$$

The discrete constraint multiplication λ is solved based on Eqs. (60) and (61), which equals

$$
\lambda=-\frac{1}{1+b^2t^2}. \qquad (62)
$$

From Eqs. (61) and (62), we can obtain

$$
q_{k\bar{k}}^1=-\frac{1}{1+b^2t^2}=\Lambda_{\rm d}^1,\ \ q_{k\bar{k}}^2=-\frac{bt}{1+b^2t^2}=\Lambda_{\rm d}^2. \qquad (63)
$$

Using Noether-type identity (26), we obtain

$$
\begin{aligned}
&\underset{+h}{D}(\xi^1)q_k^1+\underset{+h}{D}(\xi^2)q_k^2-\frac{1}{2}[(q_k^1)^2+(q_k^2)^2]\underset{+h}{D}(\xi^0)\\
&+\Lambda_{\rm d}^1(\xi^1-q_k^1\xi^0)+\Lambda_{\rm d}^2(\xi^2-q_k^2\xi^0)+\underset{+h}{D}(G_{\rm d})=0. \qquad (64)
\end{aligned}
$$

Equation (64) has the following Noether symmetry generators:

$$
\xi^0=0,\ \ \xi^1=1,\ \ \xi^2=0,\ \ G_{\rm d}=\frac{1}{b}\arg\tan(bt), \qquad (65)
$$

$$
\xi^0=0,\ \ \xi^1=0,\ \ \xi^2=1,\ \ G_{\rm d}=\frac{1}{2b}\ln(1+b^2t^2), \qquad (66)
$$

$$
\xi^0=0,\ \ \xi^1=-bt,\ \ \xi^2=1,\ \ G_{\rm d}=bq^1. \qquad (67)
$$

Substituting Eqs. (65)–(67) into Eq. (28) leads to the discrete analogue of Noether conserved quantities as follows:

$$
I_N^1=q_{\bar{k}}^1+\frac{1}{b}\arg\tan(bt)=\text{const}, \qquad (68)
$$

$$
I_N^2=q_{\bar{k}}^2+\frac{1}{2b}\ln(1+b^2t^2)=\text{const}, \qquad (69)
$$

$$
I_N^3=-btq_{\bar{k}}^1+q_{\bar{k}}^2+bq^1=\text{const}. \qquad (70)
$$

The Appell–Chetaev condition (11) is expressed as

$$
\xi^1-q_{\bar{k}}^1\xi^0+bt(\xi^2-q_{\bar{k}}^2\xi^0)=0. \qquad (71)
$$

Substituting Eqs. (65)–(67) into Eq. (71), we obtain the generalized Noether quasi-symmetries of discrete nonholonomic dynamical systems (59) and (60).

As for the Mei conservation law, we can have the following relationship under the operator (8):

$$
\begin{aligned}
prX(L_{\rm d})\ =&\ q_k^1[\underset{+h}{D}(\xi^1)-q_k^1\underset{+h}{D}(\xi^0)]\\
&+q_k^2[\underset{+h}{D}(\xi^2)-q_k^2\underset{+h}{D}(\xi^0)],\\
prX(\Lambda_{\rm d}^1)\ =&\ \frac{2bt}{1+b^2t^2}\xi^0,\ \ prX(\Lambda_{\rm d}^2)=-\frac{1-b^2t^2}{1+b^2t^2}\xi^0. \qquad (72)
\end{aligned}
$$

Using determining equations in expressions (40)–(43), one obtains

$$
\begin{aligned}
&\underset{+h}{D}(\xi^1)-q_k^1\underset{+h}{D}(\xi^0)+b\xi^0q_k^2+bt[\underset{+h}{D}(\xi^2)-q_k^2\underset{+h}{D}(\xi^0)]\\
&-b\xi^2+\xi^0=0, \qquad (73)\\
&\underset{-h}{D}\underset{+h}{D}(\xi^1)-2q_k^1\underset{-h}{D}\underset{+h}{D}(\xi^0)-2q_{\bar{k}}^1\underset{-h}{D}\underset{+h}{D}(\xi^0)\\
=&\ \frac{2bt}{1+b^2t^2}\xi^0,\\
&\underset{-h}{D}\underset{+h}{D}(\xi^2)-2q_{\bar{k}}^2\underset{-h}{D}\underset{+h}{D}(\xi^0)-2q_{\bar{k}}^2\underset{-h}{D}\underset{+h}{D}(\xi^0)\\
=&-\frac{1-b^2t^2}{1+b^2t^2}\xi^0, \qquad (74)\\
&\underset{-h}{D}[q_k^1\underset{+h}{D}(\xi^0)+q_{\bar{k}}^2\underset{+h}{D}(\xi^0)]\\
=&\ \frac{1-b^2t^2}{1+b^2t^2}\xi^0q_k^2-\frac{2bt}{1+b^2t^2}\xi^0q_k^1, \qquad (75)\\
&h_+\underset{+h}{D}(\xi^0)-h_-\underset{-h}{D}(\xi^0)=0. \qquad (76)
\end{aligned}
$$

The solutions of Eqs. (73)–(76) can be expressed as

$$
\xi^0=0,\ \ \xi^1=q_{\bar{k}}^2-btq_{\bar{k}}^1+bq^1+bt,\ \ \xi^2=1. \qquad (77)
$$

Substituting Eq. (77) into Eq. (44) yields the discrete gauge function

$$
G_{\rm d}=-b^2t. \qquad (78)
$$

According to Theorem 3, we obtain the conserved quantity of the Mei symmetry of the discrete nonholonomic systems (59) and (60) as

$$
I_M^{\rm d}=bq_{\bar{k}}^2-b^2tq_{\bar{k}}^1+b^2q^1. \qquad (79)
$$

6. Conclusion

In this work, we have extended the theories of Mei symmetry and Noether symmetry of discrete nonholonomic dynamical systems with regular and irregular lattices. The results open a significant approach to seeking conservation laws for these discrete nonconservative and nonholonomic systems. We will also develop this method for use with electromechanical dynamical systems.

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