Analytic solution for magnetohydrodynamic boundary layer flow of Casson fluid over a stretching/shrinking sheet with wall mass transfer

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In this analysis, the magnetohydrodynamic boundary layer flow of Casson fluid over a permeable stretching/shrinking sheet in the presence of wall mass transfer is studied. Using similarity transformations, the governing equations are converted to an ordinary differential equation and then solved analytically. The introduction of a magnetic field changes the behavior of the entire flow dynamics in the shrinking sheet case and also has a major impact in the stretching sheet case. The similarity solution is always unique in the stretching case, and in the shrinking case the solution shows dual nature for certain values of the parameters. For stronger magnetic field, the similarity solution for the shrinking sheet case becomes unique.

Keywords: magnetohydrodynamic boundary layer, Casson fluid, stretching/shrinking sheet, wall mass transfer, analytic solution

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1. Introduction

The study of magnetohydrodynamic (MHD) flow is very important because the influence of a magnetic field on the viscous flow of electrically conducting fluid is applicable in many industrial processes, such as in magnetic materials processing, purification of crude oil, magnetohydrodynamic electrical power generation, glass manufacturing, geophysics, and paper production, etc. Pavlov[1] investigated the effect of a magnetic field on the viscous flow of an electrically conducting fluid past a stretching sheet. Some important aspects of MHD flow over a stretching sheet can be found in Refs. [2]–[14]. A few other important investigations of the stretching sheet flow have been made recently.[15–20] A new type of flow introduced by Wang,[21] the flow due to a shrinking sheet, is very interesting, and it is quite different from forward stretching sheet flow because the generated vorticity due to shrinking does not remain inside the boundary layer. Later, Miklavčič and Wang[22] established that for steady flow due to a shrinking sheet, a certain amount of wall mass suction is required to restrain the generated vorticity. Fang and his co-authors[23,24] discussed the magnetohydrodynamic viscous flow of Newtonian fluid over a shrinking sheet with wall mass transfer, taking no-slip as well as slip boundary conditions. Many more important features of shrinking sheet flow have already been explored in Refs. [25]–[47].

In all the investigations mentioned above, classical Newtonian fluid flows are discussed. But, in modern engineering, many materials are used whose flow characteristics are not explainable with the Newtonian fluid model. Hence non-Newtonian fluid theory has become essential, but a single constitutive equation cannot cover all physical properties of such non-Newtonian fluids and hence many non-Newtonian fluid models[48–55] have been proposed to clarify those physical behaviors. The governing equations for non-Newtonian fluid flows are highly nonlinear. Also, the industrial applications of non-Newtonian fluid flow are increasing day by day. Among the many industrial non-Newtonian fluids some fluids behave like elastic solids, and for those fluids, a yield shear stress exists in the constitutive equations. Casson fluid is one of such non-Newtonian fluids. So if the shear stress magnitude is greater than yield shear stress, then flow occurs. Fredrickson[56] discussed the steady-flow behavior of a Casson fluid in a tube. The unsteady boundary layer flow and heat transfer of a Casson fluid over a moving flat plate with a parallel free stream were investigated by Mustafa *et al.*,[57] and they solved it analytically using the homotopy analysis method (HAM). The flows for various non-Newtonian fluids over a stretching/shrinking sheet were demonstrated by Liao,[58] Hayat *et al.*[59–61] and Ishak *et al.*[62–66] Recently, the stagnation-point flow and heat transfer of a Casson fluid towards a stretching sheet was studied by Mustafa *et al.*[67]

In the present paper, the steady MHD boundary layer flow of electrically conducting Casson fluid past a stretching/shrinking sheet is investigated. The closed form analytic

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solutions of a transformed similarity equation are found for both the stretching sheet and shrinking sheet cases. The solution curves are presented and detailed discussions are presented.

2. Flow analysis

Let us consider the steady two-dimensional MHD flow of an electrically conducting non-Newtonian Casson fluid above a stretching/shrinking sheet situated at $y = 0$, the flow being confined in $y > 0$. It is also assumed that the rheological equation of state for an isotropic and incompressible flow of Casson fluid can be written as (see Nakamura and Sawada[68] and Mustafa *et al.*[67])

$$\tau_{ij} = \begin{cases} (\mu_{\rm B} + p_y/\sqrt{2\pi})2e_{ij}, & \pi > \pi_{\rm c}, \\ (\mu_{\rm B} + p_y/\sqrt{2\pi_{\rm c}})2e_{ij}, & \pi < \pi_{\rm c}, \end{cases} \tag{1}$$

where $\mu_{\rm B}$ is plastic dynamic viscosity of the non-Newtonian fluid, p_y is the yield stress of fluid, π is the product of the component of deformation rate with itself, namely, $\pi = e_{ij}e_{ij}$, e_{ij} is the (i, j)-th component of the deformation rate, and $\pi_{\rm c}$ is the critical value of this product based on the non-Newtonian model.

Under these conditions the MHD boundary layer equations for the steady flow of Casson fluid over a stretching/shrinking sheet may be written in usual notation as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon(1+1/\beta)\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}u, \tag{3}$$

where u and v are the velocity components in x and y directions respectively, x is the distance along the sheet, y is the distance perpendicular to the sheet; υ is the kinematic fluid viscosity, ρ is the fluid density, $\beta = \mu_{\rm B}\sqrt{2\pi_{\rm c}}/p_y$ is the non-Newtonian (Casson) parameter, σ is the electrical conductivity of the fluid, and B_0 is the strength of magnetic field applied in the y direction, the induced magnetic field being neglected.

The boundary conditions are

$$u = U_w,\ v = -v_w \text{ at } y = 0;\ u \to 0 \text{ as } y \to \infty, \tag{4}$$

where U_w is stretching/shrinking velocity of the sheet with $U_w = cx$ for stretching sheet case and $U_w = -cx$ for shrinking sheet case with $c(>0)$ being the stretching/shrinking constant. Here v_w is the wall mass transfer velocity with $v_w > 0$ for mass suction and $v_w < 0$ for mass injection.

The following transformations are introduced

$$\psi = \sqrt{c\upsilon}xf(\eta) \text{ and } \eta = y\sqrt{c/\upsilon}, \tag{5}$$

where ψ is the stream function defined in the usual notation as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ and η is the similarity variable.

Now, equation (2) is identically satisfied and equation (3) reduces to the following nonlinear self-similar equation

$$(1+1/\beta)f''' + ff'' - f'^2 - Mf' = 0, \tag{6}$$

where primes denote differentiation with respect to η and $M = \sigma B_0^2/(\rho c)$ is the magnetic parameter, f is a dimensionless stream function and f' is the dimensionless velocity.

The boundary conditions are described as follows. For stretching sheet

$$f(\eta) = S,\ f'(\eta) = 1 \text{ at } \eta = 0; \quad f'(\eta) \to 0 \text{ as } \eta \to \infty; \tag{7}$$

for shrinking sheet

$$f(\eta) = S,\ f'(\eta) = -1 \text{ at } \eta = 0; \quad f'(\eta) \to 0 \text{ as } \eta \to \infty, \tag{8}$$

where $S = v_w/(c\upsilon)^{1/2}$ is wall mass transfer parameter with $S > 0$ (i.e. $v_w > 0$) for wall mass suction and $S < 0$ (i.e. $v_w < 0$) for wall mass injection.

In this investigation, a closed form analytic solution of the self-similar equation (7) will be obtained in both stretching and shrinking sheet cases.

2.1. Stretching sheet case

For MHD flow of Newtonian fluid, the exact solution is obtained by Chakrabarti and Gupta[2] (extended by Vajravelu and Rollins,[5] Pop and Na,[8] and Fang *et al.*[12]). Considering Ref. [7], we assume the solution here in the form:

$$f(\eta) = a + b\,\mathrm{e}^{-\lambda\eta}, \tag{9}$$

where a, b, and λ are constants with $\lambda > 0$.

Substituting the relation (9) into Eqs. (6) and (7) we obtain

$$b = -\frac{1}{\lambda},\ a = S + \frac{1}{\lambda} \text{ and } \lambda = \frac{S + \sqrt{S^2 + 4(1+1/\beta)(1+M)}}{2(1+1/\beta)}. \tag{10}$$

So, the analytic solution reduces to

$$\begin{aligned} f(\eta) \ = \ & S + \frac{2(1+1/\beta)}{S + \sqrt{S^2 + 4(1+1/\beta)(1+M)}} \\ & - \frac{2(1+1/\beta)}{S + \sqrt{S^2 + 4(1+1/\beta)(1+M)}} \\ & \times \exp\left[-\frac{S + \sqrt{S^2 + 4(1+1/\beta)(1+M)}}{2(1+1/\beta)}\eta\right]. \end{aligned} \tag{11}$$

Hence

$$f'(\eta) = \exp\left[-\frac{S + \sqrt{S^2 + 4(1+1/\beta)(1+M)}}{2(1+1/\beta)}\eta\right]$$

and

$$f''(0) = -\lambda = -\frac{S + \sqrt{S^2 + 4(1+1/\beta)(1+M)}}{2(1+1/\beta)}.$$

The similarity solution for MHD Casson fluid flow over a permeable stretching sheet is unique for any values of M, β, and S. The values of $f''(0)$, which is related to the skin-friction

coefficient, are plotted against S in Figs. 1 and 2 for various values of M and β, respectively. It is seen that for increasing values of M and S the magnitude of $f''(0)$ increases and for a decrease in Casson parameter β, this value decreases. The effects of the magnetic parameter on the dimensionless velocity profile are presented in Figs. 3 and 4. It is important to note that both the velocity and the boundary layer thickness decrease due to an increase in the magnetic parameter. This reduction is caused by the Lorentz force, a mechanical force arising due to the interaction of magnetic and electric fields for the motion of an electrically conducting fluid. The Lorentz force increases when M increases and consequently boundary layer thickness decreases.

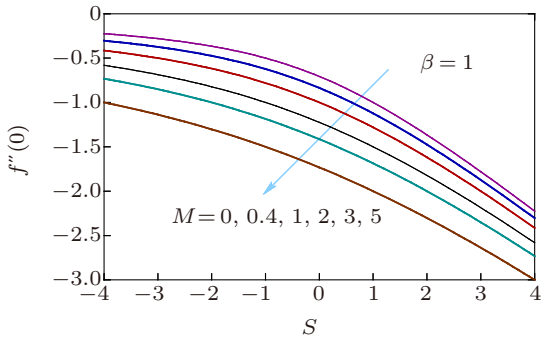


Fig. 1. (color online) Skin friction coefficient $f''(0)$ versus S for several values of M (stretching sheet case).

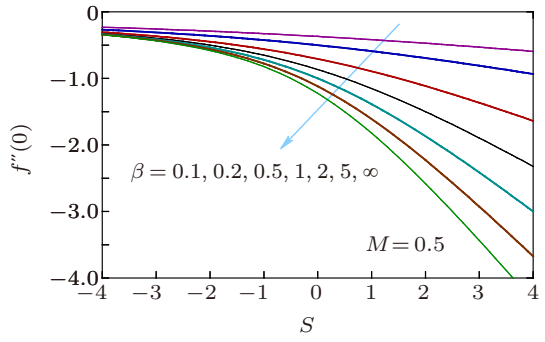


Fig. 2. (color online) Skin friction coefficient $f''(0)$ versus S for several values of β (stretching sheet case).

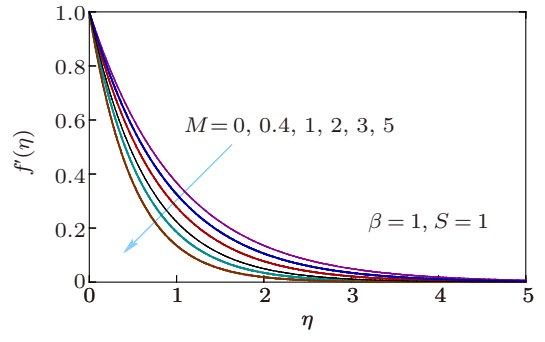


Fig. 3. (color online) Velocity profiles $f'(\eta)$ for several values of M with mass suction (stretching sheet case).

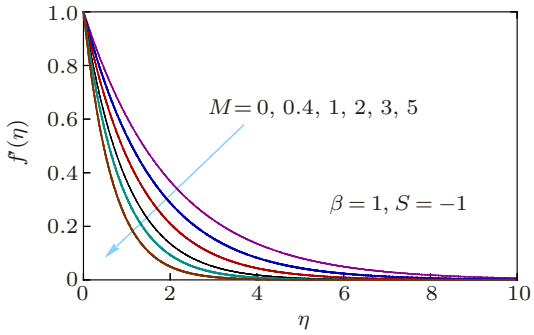


Fig. 4. (color online) Velocity profiles $f'(\eta)$ for several values of M with mass injection (stretching sheet case).

2.2. Shrinking sheet case

The shrinking sheet flow is more interesting than the stretching sheet flow. The Newtonian fluid case of the steady MHD flow due to the shrinking sheet was solved analytically by Fang and Zhang,[23] and they established that the existence of the similarity solution strongly depends upon the magnetic parameter. Presently, the effect of magnetic field on the flow of Casson fluid over a shrinking sheet is obtained. The similarity solution is assumed in the form[7,23] $f(\eta) = a + b\mathrm{e}^{-\lambda\eta}$, where a, b, and λ are constants with $\lambda > 0$.

Putting in Eqs. (6) and (8), the following relations are found

$$b = \frac{1}{\lambda},\ a = S - \frac{1}{\lambda},\ \text{and}\ \lambda = \frac{S \pm \sqrt{S^2 - 4(1+1/\beta)(1-M)}}{2(1+1/\beta)}. \quad (12)$$

Now the exact analytical solution reduces to

$$\begin{aligned} f(\eta) &= S - \frac{2(1+1/\beta)}{S \pm \sqrt{S^2 - 4(1+1/\beta)(1-M)}} \\ &\quad + \frac{2(1+1/\beta)}{S \pm \sqrt{S^2 - 4(1+1/\beta)(1-M)}} \\ &\quad \times \exp\left[-\frac{S \pm \sqrt{S^2 - 4(1+1/\beta)(1-M)}}{2(1+1/\beta)}\eta\right]. \end{aligned} \quad (13)$$

Hence,

$$f'(\eta) = -\exp\left[-\frac{S \pm \sqrt{S^2 - 4(1+1/\beta)(1-M)}}{2(1+1/\beta)}\eta\right]$$

and

$$f''(0) = \lambda = \frac{S \pm \sqrt{S^2 - 4(1+1/\beta)(1-M)}}{2(1+1/\beta)}.$$

For a physical solution of steady flow, the value λ must be positive. Similar to the Newtonian case, for MHD flow of Casson fluid the existence and uniqueness of the similarity solution strongly depends on magnetic parameter M. Three cases arise to be considered, which are $M < 1$, $M = 1$, $M > 1$. This parameter M in fact describes the importance of magnetic field on the flow.

(i) Case 1: $M < 1$

The steady flow is possible when the mass suction parameter $S(> 0)$ satisfies the following condition

$$S^2 \geq 4(1+1/\beta)(1-M). \quad (14)$$

In this case, it is important to note that the similarity solution is unique for $S^2 = 4(1+1/\beta)(1-M)$ and dual similarity solutions are observed for $S^2 > 4(1+1/\beta)(1-M)$. These facts can be spotted graphically in Figs. 5 and 6.

(ii) Case 2: $M = 1$

The steady flow of Casson fluid occurs for mass suction only, i.e., for $S > 0$, and in this case the solution is always unique.

(iii) Case 3: $M > 1$

There is no restriction on the steady flow of Casson fluid over a shrinking sheet, i.e., the similarity solution exists for any values of the mass transfer parameter and Casson parameter and the solution is unique (See Fig. 7). Also, it is important to note that in addition to mass suction the steady flow occurs for mass injection [Figs. 5 and 7].

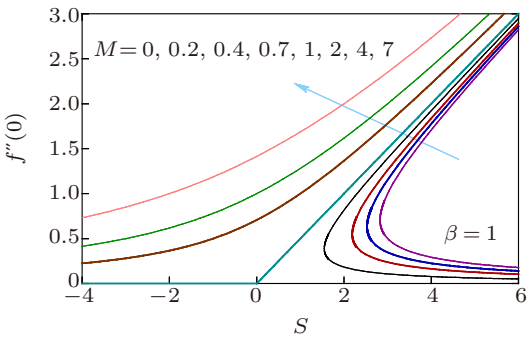


Fig. 5. (color online) Skin friction coefficient $f''(0)$ versus S for several values of M (shrinking sheet case) for $\beta = 1$.

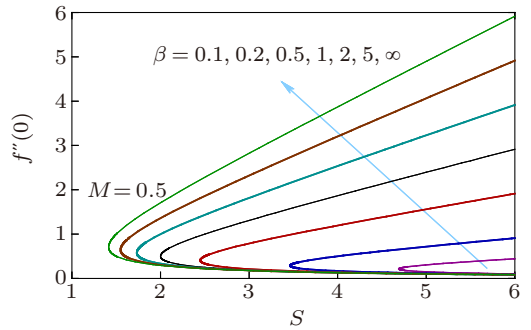


Fig. 6. (color online) Skin friction coefficient $f''(0)$ versus S for several values of β (shrinking sheet case) for $M = 0.5$.

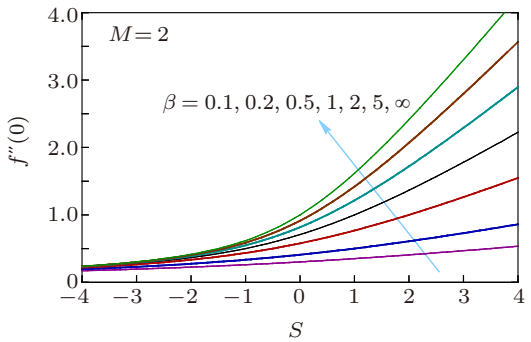


Fig. 7. (color online) Skin friction coefficient $f''(0)$ versus S for several values of β (shrinking sheet case) for $M = 2$.

Hence, it is found that, similar to the Newtonian case, the magnetic field controls the existence and uniqueness of the similarity solution for the flow of Casson fluid over a shrinking sheet. For increasing value of magnetic parameter ($M < 1$), the range of the mass suction parameter where a solution exists increases, and dual solutions are also found. For further increment of M, the solution becomes unique and independent of the value of mass transfer parameter S ($M > 1$). Actually, the Lorentz force stabilizes the flow dynamics due to its resisting nature. Normally, the steady boundary layer flow of Casson fluid over a shrinking sheet (without magnetic field) has dual similarity solutions for sufficiently strong mass suction and due to a decrease of the Casson parameter the requirement of mass suction for steady flow increases. But, for the introduction of magnetic field, the Lorentz force reduces the requirement of mass suction by suppressing the generated vorticity due to shrinking velocity, though dual solutions exist (weaker magnetic field, $M < 1$) and for stronger magnetic field ($M > 1$), that is, for stronger Lorentz force, the similarity solution becomes unique regardless of mass suction/injection and the Casson parameter. So, the effect of stronger magnetic field is very important in removing the uncertainty in the flow dynamics, that is, it makes the similarity solution unique.

The effects of all physical parameters on velocity field for the flow of Casson fluid are also significant. Therefore, the dimensionless velocity profiles for various values of magnetic parameter M and Casson parameter β are plotted in Figs. 8–10. Examining the dual velocity profiles [Fig. 8] for various $M(< 1)$, it reveals that due to the magnetic field the boundary layer thickness reduces for the first solution and increases for the second solution. Moreover, in Fig. 9 the unique velocity profiles ($M \geq 1$) exhibit behavior similar to that of the first solution in the dual solutions case. Similar to the magnetic parameter, when the Casson parameter is incremented the boundary layer thickness decreases for the first solution and it becomes larger for the second solution. Due to an increase of β, the reduction of boundary layer thickness in the first solution is because the introduction of tensile stress due to elasticity leads to contraction in the boundary layer thickness.

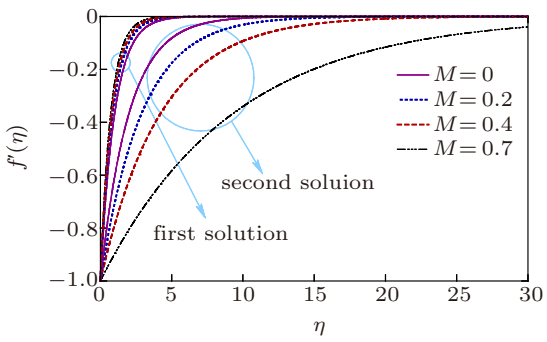


Fig. 8. (color online) Dual velocity profiles $f'(\eta)$ for several values of M (shrinking sheet case).

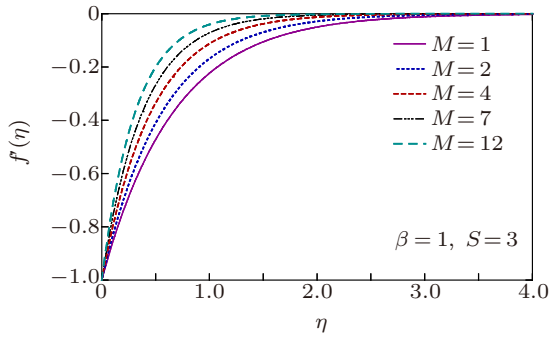


Fig. 9. (color online) Unique velocity profiles $f'(\eta)$ for several values of M (shrinking sheet case).

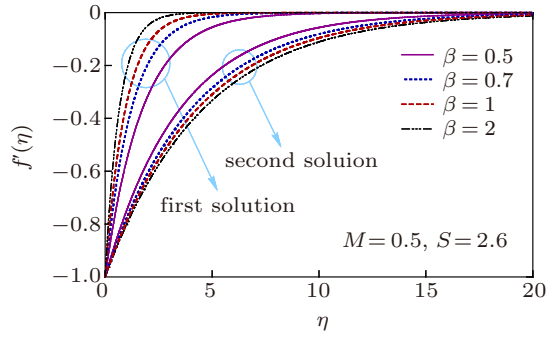


Fig. 10. (color online) Dual velocity profiles $f'(\eta)$ for several values of β (shrinking sheet case).

3. Conclusions

The MHD boundary layer flow of Casson fluid over a porous stretching/shrinking sheet is investigated. The governing equations are transformed and solved analytically. A unique closed form analytic solution is obtained in the case of flow over a stretching sheet. For a shrinking case, it is found that for Casson fluid, the increase of the magnetic parameter causes an increase of the range of the mass transfer parameter for which steady flow is possible. Unique solution and dual solutions for the flow field are observed depending on the domains of physical parameters. For stronger magnetic field, the similarity solution of the flow over a shrinking sheet is always unique which is independent of other physical parameters.

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