Spherically symmetric solution in higher-dimensional teleparallel equivalent of general relativity

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A theory of $(N+1)$-dimensional gravity is developed on the basis of the teleparallel equivalent of general relativity (TEGR). The fundamental gravitational field variables are the $(N+1)$-dimensional vector fields, defined globally on a manifold M, and the gravitational field is attributed to the torsion. The form of Lagrangian density is quadratic in torsion tensor. We then give an exact five-dimensional spherically symmetric solution (Schwarzschild $(4+1)$-dimensions). Finally, we calculate energy and spatial momentum using gravitational energy–momentum tensor and superpotential 2-form.

Keywords: $(N+1)$-dimensional teleparallel equivalent of general relativity, five-dimensional solution, energy and momentum

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1. Introduction

Continuing interest in higher-dimensional black holes over recent years is basically supported for various reasons. Among them, the most direct and important ones come from the motivation of string and membrane theories. In the context of the string theory, which is thought to be the major candidate for the quantum theory of gravity and the unification of all interactions, black holes in higher dimensions are consistent with its mathematical requirement of higher dimensions.[1–3] On the other hand, in braneworld scenarios our universe is supposed to be a slice in higher-dimensional space.[4,5]

On the basis of the above facts, it becomes clear that the study of finding exact solutions in higher dimensional gravity is of great importance. However, it is not an easy task to discover exact solutions in higher dimensional general relativity (GR), compared with the case of four dimensions, where several systematic techniques to generate stationary and axisymmetric solutions have been developed by some authors[6–8] and a lot of interesting solutions have been presented, such as the unique rotating Kerr black hole for a given mass and angular momentum[9] and the charged Kerr–Newman black hole.[10] Up to now, there are still few known exact higher dimensional solutions in general relativity. The first exact rotating one of vacuum Einstein field equations was given by Myers and Perry (MP) in 1986.[1] This solution is the higher dimensional generalization of the classical Kerr black hole and has a spherical horizon topology. Nevertheless, in the case of the five-dimensional pure gravity, MP black hole is not unique. For a given mass and angular momentum, Emparan and Reall discovered a black ring solution in a vacuum, whose horizon topology is $S_1 \times S_2$.[2] Then another general regular black ring solution with two angular momenta was also found.[11] Quite recently, in Ref. [12], the authors presented a new five-dimensional black hole solution with the horizon of distorted lens spaces.

Recently, as a description of gravity equivalent to general relativity (GR), teleparallel gravity has attracted renewed attention owing to many of its salient features.[13–31] Teleparallel gravity can be regarded as a translational gauge theory,[13,30,31] which makes it possible to unify gravity with other kinds of interactions in the gauge theory framework in which the fundamental interactions are described by a connection defined on some principal fiber bundle. Nowadays, there is a trend to analyze the basic solutions of GR with teleparallel theory and compare the results. It is considered an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory[32–61] or metric-affine gravity.[62,63] Physics relevant to geometry may be related to the teleparallel description of gravity.[30,31,64–66] Within the framework of metric-affine gravity, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian by using prolongation techniques (see Refs. [67] and [68] and references therein). A teleparallel approach is used for positive-gravitational-energy proof.[30,31,69–71]

Another advantage for the teleparallel gravity concerns energy–momentum, its representation, positivity, and localization.[17–39] Nester[33–39] succeeded in formulating a pure tensorial proof of the positivity of the total energy for Einstein's theory in terms of the teleparallel geometry. He has found that special gauge features of TEGR, which are usually considered to be problematic, are quite beneficial for this purpose. Because of its simplicity and transparency, the teleparallel gravity seems to be much more appropriate than GR to deal

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with the problem of the gravitation energy–momentum. It is proved that[17–39] in the teleparallel gravity there exists a gravitational energy–momentum tensor which is covariant under general coordinate and global Lorentz transformations.

The localization of gravitational energy–momentum remains a significant problem in GR. Using the standard methods many famous researchers have found their expression. None of these expressions is covariant, they are all reference frame-dependent (referred to as pseudotensors). This feature can be understood in terms of the equivalence principle: gravity cannot be detected at a point, so it cannot have a point-wise-defined energy–momentum density. Now there is another way to address this difficulty. The new idea is quasi-local: energy–momentum is associated with a closed-2 surface surrounding a region. A good quasi-local approach is in terms of the Hamiltonian.[33–39] Then the Hamiltonian boundary term determines the quasi-local quantities. In fact this approach includes all the traditional pseudotensors.[1,33–39] They are each generated by a superpotential which can serve as special type of Hamiltonian boundary term. A good energy–momentum expression for gravitating systems should satisfy a variety of requirements, including giving the standard values for the total quantities for asymptotically flat space and reducing to the material energy–momentum in the proper limit. No entire expression has yet been identified. One of the most restrictive requirements is positivity. A general positivity proof is very difficult.[72]

It is the objective of the present work to find a higher dimensional solution in TEGR. In Section 2, we give a brief review of teleparallel spacetime in $(N+1)$ dimensions. The Lagrangian and the field equations are given in Section 3. In Section 4, we construct the spherically symmetric $(4+1)$-dimensional vector field (pentad) and then apply it to the field equation. A five-dimensional spherically symmetric black hole is also given in Section 4. We calculate the energy and spatial momentum associated with this solution in Section 5 and a divergent result is obtained. In Section 6, we use the regularized approach to obtain a finite result of energy. In Section 7, we use the language of forms to recalculate the energy of the $(4+1)$-dimensional solution. The final section is devoted to our main result.

2. $(N+1)$-dimensional spacetime

The $(N+1)$-dimensional spacetime M is assumed to be a differentiable manifold endowed with the Lorentzian metric $g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ $(\mu,\nu = 0,1,2,\ldots,n)$ related to the fields $\boldsymbol{h}^k = h^k{}_\mu \mathrm{d}x^\mu$, $(k = 0,1,2,\ldots,n)$ through the relation $g_{\mu\nu} = O_{ij}h^i{}_\mu h^j{}_\nu$ with $O_{ij} = (-1,+1,+1,\ldots)$. Here $x^\mu\{\mu = 0,1,2,\ldots,n\}$ is a local coordinate of the spacetime. The fields $\boldsymbol{h}_k = h^\mu{}_k \partial/\partial x^\mu$, which are the dual to $\boldsymbol{h}^k$, are the inverse fields of $\boldsymbol{h}_k$. The field strength of h^k_μ is given by[1)]

$$T^k{}_{\mu\nu} = \Gamma^k{}_{[\mu\nu]}, \quad \text{with} \quad \Gamma^k{}_{\mu\nu} = \partial_\mu h^k{}_\nu. \tag{1}$$

We define the covariant derivative of the Lorentzian vector field V^k by

$$\nabla_l V^k \overset{\text{def.}}{=} h^\mu{}_l \partial_\mu V^k. \tag{2}$$

For the world vector fields $\boldsymbol{V} = V^\mu \partial/\partial x^\mu$, the covariant derivative with respect to the affine connection $\boldsymbol{\Gamma}^\mu{}_{\nu\lambda}$ is defined by

$$\nabla_\nu V^\mu \overset{\text{def.}}{=} \partial_\nu V^\mu + \Gamma^\mu{}_{\lambda\nu} V^\lambda. \tag{3}$$

The requirement

$$\nabla_l V^k = h^\nu{}_l h^k{}_\mu \nabla_\nu V^\mu,$$
$$\text{for} \quad V^\mu = h^\mu{}_k V^k \quad \text{leads to} \quad \Gamma^\mu{}_{\lambda\nu} \overset{\text{def.}}{=} h^\mu{}_k \partial_\nu h^k{}_\lambda, \tag{4}$$

and hence we have the relation[72]

$$T^k{}_{\mu\nu} \overset{\text{def.}}{=} h^k{}_\lambda T^\lambda{}_{\mu\nu},$$
$$R^\mu{}_{\nu\lambda\rho} \overset{\text{def.}}{=} \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\tau\lambda}\Gamma^\tau{}_{\nu\rho} - \Gamma^\mu{}_{\tau\rho}\Gamma^\tau{}_{\nu\lambda} \equiv 0, \tag{5}$$
$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} \equiv 0,$$
$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]}. \tag{6}$$

The components $T^\lambda{}_{\mu\nu}$ and $R^\mu{}_{\nu\lambda\rho}$ are those of the torsion and of the curvature tensor, respectively. Equation (5) implies teleparallelism, and it together with Eq. (6), means that M is the Weitzenböck spacetime. Also from Eq. (6) we get

$$\Gamma^\lambda{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\} + K^\lambda{}_{\mu\nu},$$
$$\text{with} \quad \left\{{\lambda \atop \mu\nu}\right\} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu}\right), \tag{7}$$

where $K^\lambda{}_{\mu\nu}$ stands for the contortion tensor,

$$K^\lambda{}_{\mu\nu} = -\frac{1}{2}\left(T^\lambda{}_{\mu\nu} - T_\mu{}^\lambda{}_\nu - T_\nu{}^\lambda{}_\mu\right) = -K_\mu{}^\lambda{}_\nu. \tag{8}$$

In what follows, the field components $h^k{}_\mu$ and $h^\mu{}_k$ are used to convert Latin and Greek indices, e.g., V^μ and $V^k (= h^k{}_\mu V^\mu)$. Also, raising and lowering the indices $k,l,m,\ldots$ are accomplished with the aid of O^{ij}, $O_{ij} = (O^{ij})^{-1}$ respectively.

3. Lagrangian density and gravitational field equation

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by[17–22,72]

$$\mathscr{L}_\mathrm{G} = hL_\mathrm{G} = -\frac{h}{16\pi}\left(\frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bac}}{2} - T^a T_a\right) - L_\mathrm{m}$$
$$= -\frac{h}{16\pi}\Sigma^{abc}T_{abc} - L_\mathrm{m}, \tag{9}$$

[1)] We will denote the symmetric part by ( ), for example, $A_{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu})$ and the antisymmetric part by the square bracket [ ], $A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu})$.

[2)] Throughout this paper we use the relativistic units, $c = G = 1$.

where $h=\det(h^a{}_\mu)$ and $L_{\rm m}$ is the matter Lagrangian[2]. The tensor Σ^{abc} is defined by

$$\Sigma^{abc} \stackrel{\text{def.}}{=} \frac{1}{4}\left(T^{abc}+T^{bac}-T^{cab}\right)+\frac{1}{2}\left(O^{ac}T^{b}-O^{ab}T^{c}\right). \quad (10)$$

T^{abc} and T^a are the torsion tensor and the basic vector field defined by

$$T^a{}_{\mu\nu} \stackrel{\text{def.}}{=} h^a{}_\lambda T^\lambda{}_{\mu\nu}=\partial_\mu h^a{}_\nu-\partial_\nu h^a{}_\mu,$$
$$T^a{}_{bc} \stackrel{\text{def.}}{=} h_b{}^\mu h_c{}^\nu T^a{}_{\mu\nu},\ T^a \stackrel{\text{def.}}{=} T^b{}_b{}^a. \quad (11)$$

The quadratic combination $\Sigma^{abc}T_{abc}$ is proportional to the scalar curvature $R(h)$, except for a total divergence term.[23–29] $L_{\rm m}$ represents the Lagrangian density for matter fields.

The gravitational field equations for the system described by $L_{\rm G}$ are the following:

$$h_{a\lambda}h_{b\mu}\partial_\nu\left(h\Sigma^{b\lambda\nu}\right)-h\left(\Sigma^{b\nu}{}_a T_{b\nu\mu}-\frac{1}{4}h_{a\mu}T_{bcd}\Sigma^{bcd}\right)$$
$$=\frac{1}{2}\kappa h T_{a\mu}, \quad (12)$$

where

$$\frac{\delta L_{\rm m}}{\delta h^{a\mu}}\equiv hT_{a\mu}.$$

It is possible to prove by explicit calculations that the left-hand side of the symmetric field equations of Eq. (12) is exactly given by

$$\frac{h}{2}\left[R_{a\mu}(h)-\frac{1}{2}h_{a\mu}R(h)\right].$$

4. (4 + 1)-dimensional spherically symmetric solution

In this section we are going to find a solution in $(4+1)$-dimensional spacetime. A solution in $(N+1)$-dimensional spacetime will be studied later. Let us begin with the pentad field which can be written as

$$(h_i{}^\mu)=\begin{pmatrix} A_1(\rho,\phi) & A_2(\rho,\phi) & A_3(\rho,\phi) & A_4(\rho,\phi) & A_5(\rho,\phi) \\ B_1(\rho,\phi)\sin\theta & B_2(\rho,\phi)\sin\theta & B_3(\rho,\phi)\cos\theta & B_4(\rho,\phi)\sin\theta & B_5(\rho,\phi)\cos\theta \\ C_1(\rho,\phi)\sin\theta & C_2(\rho,\phi)\sin\theta & C_3(\rho,\phi)\cos\theta & C_4(\rho,\phi)\sin\theta & C_5(\rho,\phi)\cos\theta \\ F_1(\rho,\phi)\cos\theta & F_2(\rho,\phi)\cos\theta & F_3(\rho,\phi)\sin\theta & F_4(\rho,\phi)\cos\theta & F_5(\rho,\phi)\sin\theta \\ H_1(\rho,\phi)\cos\theta & H_2(\rho,\phi)\cos\theta & H_3(\rho,\phi)\sin\theta & H_4(\rho,\phi)\cos\theta & H_5(\rho,\phi)\cos\theta \end{pmatrix}, \quad (13)$$

where $A_i(\rho,\phi)$, $B_i(\rho,\phi)$, $C_i(\rho,\phi)$, $F_i(\rho,\phi)$, and $H_i(\rho,\phi)$, $i=1,\ldots,5$ are unknown functions of ρ and ϕ.

Applying Eq. (13) to the field equations (12) we get a set of nonlinear partial differential equations. We are interested to find some special solutions to those differential equations. Attributing the following values to the unknown functions given in Eq. (13)

$$A_1=1-\frac{M}{\rho^2},\ A_2=\frac{M}{\Upsilon},\ A_3=0,\ A_4=\frac{Ma\sin^2\theta}{\rho^2},\ A_5=0,$$
$$B_1=\frac{M\cos\phi}{\rho^2},\ B_2=\frac{1}{\Upsilon}\left(\rho\alpha-M\cos\phi\right),\ B_3=\alpha,$$
$$B_4=-\beta,\ B_5=0,\ C_1=\frac{M\sin\phi}{\rho^2},\ C_2=\frac{1}{\Upsilon}\left(\rho\beta-M\sin\phi\right),$$
$$C_3=\beta,\ C_4=\alpha,\ C_5=0,$$
$$F_1=\frac{M}{\rho^2},\ F_2=1+\frac{M}{\Upsilon},\ F_3=-\rho,\ F_4=0,\ F_5=0,$$
$$H_1=0,\ H_2=0,\ H_3=0,\ H_4=0,\ H_5=\rho, \quad (14)$$

where Υ, α, and β are defined by

$$\Upsilon=\rho^2-2M,\quad \alpha=\rho\cos\phi,\quad \beta=\rho\sin\phi, \quad (15)$$

and M is the gravitational mass of the source. The metric associated with the pentad (13) using solution (14), has the form

$$\mathrm{d}s^2=-\left(1-\frac{2M}{\rho^2}\right)\mathrm{d}t^2+\left(1-\frac{2M}{\rho^2}\right)^{-1}\mathrm{d}\rho^2$$
$$+\rho^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2+\cos^2\theta\,\mathrm{d}\psi^2\right), \quad (16)$$

which is exactly the Schwarzschild black hole in five dimensions.[73]

5. Energy, momentum and angular-momentum

The Hamiltonian formulation of TEGR in $(4+1)$-dimensional spacetime is obtained by establishing the phase space variables. The Lagrangian density does not contain the time derivative of the pentad component h_{a0}. Therefore, this quantity will arise as a Lagrange multiplier.[74] The momentum canonically conjugated to h_{ai} is given by

$$\Pi^{ai}=\delta L/\delta\dot{h}_{ai}.$$

The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form $L=p\,\dot{q}-H$, in terms of h_{ai},Π^{ai}, and the Lagrange multipliers. The Legendre transformation can be successfully carried out and the final form of the Hamiltonian density has the form[75]

$$H=h_{a0}C^a+\alpha_{ik}\Gamma^{ik}+\beta_k\Gamma^k, \quad (17)$$

plus a surface term. Here α_{ik} and β_k are Lagrange multipliers that are identified as

$$\alpha_{ik}=\frac{1}{2}(T_{i0k}+T_{k0i})\quad\text{and}\quad\beta_k=T_{00k}, \quad (18)$$

and C^a, Γ^{ik}, and Γ^k are first class constraints. The Poisson brackets between any two field quantities F and G are given by

$$\{F,G\}=\int\mathrm{d}^3x\left(\frac{\delta F}{\delta h_{ai}(x)}\frac{\delta G}{\delta\Pi^{ai}(x)}-\frac{\delta F}{\delta\Pi^{ai}(x)}\frac{\delta G}{\delta h_{ai}(x)}\right). \quad (19)$$

We recall that the Poisson brackets $\{\Gamma^{ij}(x),\Gamma^{kl}(x)\}$ reproduce the angular-momentum algebra.[23–29]

The constraint C^a is written as $C^a=-\partial_i\Pi^{ai}+h^a$, where h^a is an intricate expression of the field variables. The integral

form of the constraint equation $C^a = 0$ motivates the definition of the gravitational energy–momentum P^a five-vector[23–29]

$$P^a = -\int_V \mathrm{d}^4x \partial_i \Pi^{ai}, \tag{20}$$

where V is an arbitrary volume of the four-dimensional space. In the configuration space we have

$$\Pi^{ai} = -4\kappa\sqrt{-g}\Sigma^{a0i},$$
$$\text{with} \quad \partial_\nu(\sqrt{-g}\Sigma^{a\lambda\nu}) = \frac{1}{4\kappa}\sqrt{-g}h^a{}_\mu(t^{\lambda\mu}+T^{\lambda\mu}), \tag{21}$$

where

$$t^{\lambda\mu} = \kappa\left(4\Sigma^{bc\lambda}T_{bc}{}^{\mu} - g^{\lambda\mu}\Sigma^{bcd}T_{bcd}\right).$$

The emergence of total divergences in the form of scalar or vector densities is possible in the framework of theories constructed out of the torsion tensor. Metric theories of gravity do not share this feature. By making $\lambda = 0$ in Eq. (21) and identifying Π^{ai} in the left-hand side of the latter, the integral form of Eq. (20) is written as

$$P^a = \int_V \mathrm{d}^3x\sqrt{-g}h^a{}_\mu\left(t^{0\mu}+T^{0\mu}\right). \tag{22}$$

Equation (20) suggests that P^a is now understood as the gravitational energy–momentum.[23–29] The spatial component $P^{(i)}$ forms a total four-momentum, while temporal component $P^{(0)}$ is the total energy.

It is possible to rewrite the Hamiltonian density of Eq. (17) in the equivalent form

$$H = h_{a0}C^a + \frac{1}{2}\lambda_{ab}\Gamma^{ab}, \qquad \text{with} \qquad \lambda_{ab} = -\lambda_{ba}, \tag{23}$$

being the Lagrangian multipliers such that $\lambda_{ik} = \alpha_{ik}$ and $\lambda_{0k} = -\lambda_{k0} = \beta_k$. The constraints $\Gamma^{ab} = -\Gamma^{ba}$ embodies both constraints Γ^{ik} and Γ^k by means of the relation

$$\Gamma^{ik} = h_a{}^i h_b{}^k \Gamma^{ab}, \quad \text{and} \quad \Gamma^k \equiv \Gamma^{0k} = h_a{}^0 h_b{}^k \Gamma^{ab}. \tag{24}$$

The constraint Γ^{ab} can be read as

$$\Gamma^{ab} = M^{ab} + 4\kappa\sqrt{-g}h_{(0)}{}^0\left(\Sigma^{a(0)b} - \Sigma^{b(0)a}\right). \tag{25}$$

Similar to the definition of P^a, the integral form of the constraint equation $\Gamma^{ab} = 0$ motivates the new definition of the spacetime angular-momentum. The equation $\Gamma^{ab} = 0$ implies

$$M^{ab} = -4\kappa\sqrt{-g}h_c{}^0\left(\Sigma^{acb} - \Sigma^{bca}\right). \tag{26}$$

We use the definition

$$L^{ab} = \int_V \mathrm{d}^4x\, h_\mu{}^a h_\nu{}^b M^{\mu\nu}, \tag{27}$$

as the 5-angular-momentum of the gravitational field for an arbitrary volume V of the four dimensions. Definition (27) is a four-dimensional integral. The quantities P^a and L^{ab} are separately invariant under general coordinate transformations of the four-dimensional space and under time reparametrizations, which is an expected feature since these definitions arise in the Hamiltonian formulation of the theory.

Now we are going to calculate the energy associated with the pentad field (13) after using solution given by Eq. (14). In the context of pentad theories of gravity, asymptotically flat spacetimes may be characterized by the asymptotic boundary condition,

$$h_{a\mu} \cong O_{a\mu} + \frac{1}{2}l_{a\mu}(1/\rho^2), \tag{28}$$

together with the condition $\partial_\mu h^a{}_\mu = O(1/\rho^3)$ in the asymptotic limit $\rho \to \infty$. An important property of pentad fields that satisfy Eq. (28), is that in the flat spacetime limit one has $h^a{}_\mu(t,x,y,z,w) = \delta^a{}_\mu$, and therefore the torsion tensor $T^a{}_{\mu\nu} = 0$.

Now we are going to use the definition (20) in the pentad field (13) using Eqs. (14) and (15), to calculate the energy content. We perform the calculations in the spherical coordinate. Definition (20) implies that the reference space is determined by a set of pentad fields $h^a{}_\mu$ for the flat spacetime such that the condition $T^a{}_{\mu\nu} = 0$ is satisfied. Using Eq. (13) and Eqs. (14) and (15) in Eq. (11), the non-vanishing components of the torsion tensor are given in Appendix A. The only required component of $\Sigma^{\mu\nu\lambda}$ is

$$\Sigma^{(0)01} = -\frac{1}{4\rho^3}\left(2\rho^2 - 6M\right). \tag{29}$$

Using Eq. (29) in Eq. (20) we obtain

$$\begin{aligned} P^{(0)} &= E = -\oint_{V\to\infty} \mathrm{d}V_k \Pi^{(0)k} \\ &= -\frac{1}{4\pi}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{\psi=0}^{2\pi} \mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi e\Sigma^{(0)01} \\ &= \frac{3M\pi}{8} - \frac{\rho^2\pi}{8}, \end{aligned} \tag{30}$$

which is a divergent form.[76] This is due to the fact that some components of the torsion tensor do not vanish identically when the physical quantities vanish, i.e., $M = 0$. This contradicts the flatness condition given by Eq. (28). Therefore, in this case we are going to use the regularized expression for the gravitational energy–momentum.

6. Regularized expression for the gravitational energy–momentum and localization of energy

An important property of the pentad fields that satisfies the condition given by Eq. (28) is that in the flat spacetime limit $h^a{}_\mu(t,x,y,z,w) = \delta^a{}_\mu$, and therefore the torsion $T^\lambda{}_{\mu\nu} = 0$. Hence for the flat spacetime it is normal to consider a set of pentad fields such that $T^\lambda{}_{\mu\nu} = 0$ in any coordinate system. However, in general an arbitrary set of pentad fields that yields the metric tensor for the asymptotically flat spacetime does not satisfy the asymptotic condition given by expression (28). Moreover for such pentad fields the torsion $T^\lambda{}_{\mu\nu} \neq 0$ for the flat spacetime. It might be argued, therefore, that the expression for the gravitational energy–momentum (20) is restricted to a particular class of pentad fields, namely, to the class of frames such that $T^\lambda{}_{\mu\nu} = 0$ if $E_a{}^\mu$ represents the flat spacetime

pentad field. To explain this, let us evaluate Eq. (13) in the case of flat spacetime

$$(E_a{}^{\mu})=\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \cos\phi\sin\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta & 0\\ 0 & \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta & 0\\ 0 & \cos\theta & -\rho\sin\theta & 0 & 0\\ 0 & 0 & 0 & \rho\cos\theta & 0\end{pmatrix}. \quad (31)$$

Expression (31) yields the following non-vanishing torsion components

$$T^{(4)}{}_{14}=-\frac{1}{\rho},\quad T^{(4)}{}_{24}=\cot\theta. \quad (32)$$

In view of the geometric structure of Eq. (32), we see that, Eq. (32) does not display the asymptotic behavior required by Eq. (28). Moreover, in general the pentad field (31) is adapted to accelerated observers. In spite of the above problems discussed for the pentad field of Eq. (13) it yields a satisfactory value for the total gravitational energy–momentum, as we will discuss later.

Definition (20) implies that the reference space is determined by a set of pentad fields $h^a{}_{\mu}$ for flat spacetime such that the condition $T^a{}_{\mu\nu}=0$ is satisfied. However, in general there exist flat spacetime pentad fields for which $T^a{}_{\mu\nu}\neq 0$. In this case Eq. (20) may be generalized[76] by adding a suitable reference space subtraction term, exactly like in the Brown–York formalism.[77,78]

We will denote $T^a{}_{\mu\nu}(E)=\partial_\mu E^a{}_\nu-\partial_\nu E^a{}_\mu$ and $\Pi^{aj}(E)$ as the expression of Π^{aj} constructed out of the flat pentad $E^a{}_\mu$. The regularized form of the gravitational energy–momentum P^a is defined by[76]

$$P^a=-\int_V \mathrm{d}^4x\partial_k\left[\Pi^{ak}(h)-\Pi^{ak}(E)\right]. \quad (33)$$

This condition guarantees that the energy–momentum of the flat spacetime always vanishes. The reference spacetime is determined by pentad fields $E^a{}_\mu$, obtained from $h^a{}_\mu$ by requiring the vanishing of the physical parameters like mass, angular-momentum, etc. Assuming that the spacetime is asymptotically flat then Eq. (33) can have the form[79,80]

$$P^a=-\oint_{V\to\infty}\mathrm{d}V_k\left[\Pi^{ak}(h)-\Pi^{ak}(E)\right], \quad (34)$$

where the volume V is established at spacelike infinity.

We may likewise establish the regularized expression for the gravitational 5-angular momentum. It reads

$$L^{ab}=\int_V \mathrm{d}^3x\left[M^{ab}(h)-M^{ab}(E)\right]. \quad (35)$$

Now we are in a position to prove that the pentad field (13) yields a satisfactory value for the total gravitational energy–momentum. We will integrate Eq. (34) over a volume of constant radius $x^1=r$ and require $r\to\infty$. Therefore, the index k in Eq. (34) takes the value $k=1$. We need to calculate the quantity

$$\Sigma^{(0)01}=h^{(0)}{}_0\Sigma^{001}=\frac{1}{2}h^{(0)}{}_0\left(T^{001}-g^{00}T^1\right).$$

Evaluating the above equation we find

$$\Pi^{(0)1}(h)=\frac{-1}{4\pi}h\Sigma^{(0)01}=-\frac{\sin\theta\cos\theta(\rho^2-3M)}{16\pi}, \quad (36)$$

and the expression of $\Pi^{(0)1}(E)$ is obtained by just making $M=0$, and $a=0$ in Eq. (36), it is given by

$$\Pi^{(0)1}(E)=-\frac{1}{4\pi}\rho^2\sin\theta\cos\theta. \quad (37)$$

Thus the gravitational energy of the pentad field of Eq. (13) is given by

$$\begin{aligned}P^{(0)}&=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{\psi=0}^{2\pi}\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi\frac{3M}{16\pi}\sin\theta\cos\theta\\&=\frac{3M\pi}{8},\end{aligned} \quad (38)$$

which is the desired result.

To make the picture clearer we will use another method to calculate the energy of pentad (13).

7. Brief review of teleparallel theory of gravitation using the language of forms

Notation We use the Latin indices $i,j,\ldots$ for local holonomic spacetime coordinates and the Greek indices $\alpha,\beta,\ldots$ label (co)frame components. Particular frame components are denoted by hats, $\hat{0},\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi\rfloor\Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha\rfloor\vartheta^\beta=\delta^\beta_\alpha$. Using local coordinates x^i, we have $\vartheta^\alpha=h^\alpha_i\,\mathrm{d}x^i$ and $e_\alpha=h^i_\alpha\partial_i$ where h^α_i and h^i_α are the covariant and contravariant components of the tetrad field. We define the volume 4-form by $\eta\overset{\text{def.}}{=}\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}\wedge\vartheta^{\hat{2}}\wedge\vartheta^{\hat{3}}\wedge\vartheta^{\hat{4}}$. Furthermore, with the help of the interior product we define

$$\eta_\alpha\overset{\text{def.}}{=}e_\alpha\rfloor\eta=\frac{1}{4!}\,\varepsilon_{\alpha\beta\gamma\delta\rho}\,\vartheta^\beta\wedge\vartheta^\gamma\wedge\vartheta^\delta\wedge\vartheta^\rho,$$

where $\varepsilon_{\alpha\beta\gamma\delta\rho}$ is a completely antisymmetric tensor with $\varepsilon_{01234}=1$.

$$\eta_{\alpha\beta}\overset{\text{def.}}{=}e_\beta\rfloor\eta_\alpha=\frac{1}{3!}\varepsilon_{\alpha\beta\gamma\delta\rho}\,\vartheta^\gamma\wedge\vartheta^\delta\wedge\vartheta^\rho,$$

$$\eta_{\alpha\beta\gamma}\overset{\text{def.}}{=}e_\gamma\rfloor\eta_{\alpha\beta}=\frac{1}{2!}\varepsilon_{\alpha\beta\gamma\delta\rho}\,\vartheta^\delta\wedge\vartheta^\rho,$$

$$\eta_{\alpha\beta\gamma\delta}\overset{\text{def.}}{=}e_\delta\rfloor\eta_{\alpha\beta\gamma}=\frac{1}{1!}\varepsilon_{\alpha\beta\gamma\delta\rho}\,\vartheta^\rho,$$

which are bases for 4-, 3-, 2-, and 1-forms respectively. Finally,

$$\eta_{\alpha\beta\gamma\delta\rho}\overset{\text{def.}}{=}e_\rho\rfloor\eta_{\alpha\beta\gamma\delta}=e_\rho\rfloor e_\delta\rfloor e_\gamma\rfloor e_\beta\rfloor e_\alpha\rfloor\eta$$

is the Levi–Civita tensor density. The η-forms satisfy the useful identities:

$$\vartheta^\beta\wedge\eta_\alpha\overset{\text{def.}}{=}\delta^\beta_\alpha\eta,\quad \vartheta^\beta\wedge\eta_{\mu\nu}\overset{\text{def.}}{=}\delta^\beta_\nu\eta_\mu-\delta^\beta_\mu\eta_\nu,$$

$$\vartheta^\beta\wedge\eta_{\alpha\mu\nu}\overset{\text{def.}}{=}\delta^\beta_\alpha\eta_{\mu\nu}+\delta^\beta_\mu\eta_{\nu\alpha}+\delta^\beta_\nu\eta_{\alpha\mu},$$

$$\vartheta^\beta\wedge\eta_{\alpha\gamma\mu\nu}\overset{\text{def.}}{=}\delta^\beta_\nu\eta_{\alpha\gamma\mu}-\delta^\beta_\mu\eta_{\alpha\gamma\nu}+\delta^\beta_\gamma\eta_{\alpha\mu\nu}-\delta^\beta_\alpha\eta_{\gamma\mu\nu},$$

$$\vartheta^\beta\wedge\eta_{\alpha\gamma\mu\nu\rho}\overset{\text{def.}}{=}\delta^\beta_\rho\eta_{\alpha\gamma\mu\nu}-\delta^\beta_\nu\eta_{\alpha\gamma\mu\rho}+\delta^\beta_\mu\eta_{\alpha\gamma\nu\rho}$$

$$-\delta^{\beta}_{\gamma}\eta_{\alpha\mu\nu\rho}+\delta^{\beta}_{\alpha}\eta_{\gamma\mu\nu\rho}. \tag{39}$$

The line element $\mathrm{d}s^2 \overset{\text{def.}}{=} g_{\alpha\beta}\vartheta^{\alpha}\otimes\vartheta^{\beta}$ is defined by the space-time metric $g_{\alpha\beta}$.

Teleparallel geometry can be viewed as a gauge theory of translation.[47,66,79–89] In this geometry the co-frame ϑ^{α} plays the role of the gauge translational potential of the gravitational field. GR can be reformulated as the teleparallel theory. Geometrically, teleparallel gravity can be considered as a special case of the metric-affine gravity in which the co-frame 1-form ϑ^{α} and the local Lorentz connection are subject to the distant parallelism constraint $R_{\alpha}{}^{\beta}=0$.[90–95] In this geometry the torsion 2-form

$$\begin{aligned} T^{\alpha} &= \mathrm{D}\vartheta^{\alpha}=\mathrm{d}\vartheta^{\alpha}+\Gamma_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}=\frac{1}{2}T_{\mu\nu}{}^{\alpha}\vartheta^{\mu}\wedge\vartheta^{\nu}\\ &= \frac{1}{2}T_{ij}{}^{\alpha}\mathrm{d}x^{i}\wedge\mathrm{d}x^{j}, \end{aligned} \tag{40}$$

arises as the gravitational gauge field strength, $\Gamma_{\alpha}{}^{\beta}$ being the Weitzenböck connection 1-form, d is the exterior derivative and D is the exterior covariant derivative. The torsion T^{α} can be decomposed into three irreducible pieces: the tensor part, the trace, and the axial trace, given respectively by,[47,92] for example

$$\begin{aligned} &{}^{(1)}T^{\alpha}\overset{\text{def.}}{=}T^{\alpha}-{}^{(2)}T^{\alpha}-{}^{(3)}T^{\alpha}, \quad \text{with}\\ &{}^{(2)}T^{\alpha}\overset{\text{def.}}{=}\frac{1}{3}\vartheta^{\alpha}\wedge T, \quad \text{where} \quad T=\left(e_{\beta}\rfloor T^{\beta}\right),\\ &\quad e_{\alpha}\rfloor T=T_{\mu\alpha}{}^{\mu}, \quad \text{vectors of trace of torsion},\\ &{}^{(3)}T^{\alpha}\overset{\text{def.}}{=}\frac{1}{3}e^{\alpha}\rfloor P, \quad \text{with} \quad P=\left(\vartheta^{\beta}\wedge T_{\beta}\right),\\ &\quad e_{\alpha}\rfloor P=T^{\mu\nu\lambda}\eta_{\mu\nu\lambda\alpha}, \quad \text{axial of trace of torsion}. \end{aligned} \tag{41}$$

The Lagrangian of the teleparallel equivalent using the language of forms has the form

$$V=-\frac{1}{2\kappa}T^{\alpha}\wedge{}^{*}\left({}^{(1)}T_{\alpha}-2{}^{(2)}T_{\alpha}-\frac{1}{2}{}^{(3)}T_{\alpha}\right). \tag{42}$$

$\kappa=8\pi G/c^3$, G is the Newton gravitational constant, c is the speed of light and $*$ denotes the Hodge duality in the metric $g_{\alpha\beta}$ which is assumed to be flat Minkowski metric $g_{\alpha\beta}=O_{\alpha\beta}=\mathrm{diag}(-1,+1,+1,+1,+1)$, that is used to raise and lower local frame (Greek) indices.

The variation of the total action with respect to the co-frame gives the field equations in the form[92]

$$\mathrm{D}H_{\alpha}-E_{\alpha}=\Sigma_{\alpha}, \quad \text{where} \quad \Sigma_{\alpha}\overset{\text{def.}}{=}\frac{\delta L_{\text{mattter}}}{\delta\vartheta^{\alpha}}, \tag{43}$$

is the canonical energy–momentum current 3-form of matter which is considered as the source. In accordance with the general Lagrange–Noether scheme[66,88] one derives from Eq. (42) the translational momentum 2-form and the canonical energy–momentum 3-form

$$\begin{aligned} &H_{\alpha}\overset{\text{def.}}{=}-\frac{\partial V}{\partial T^{\alpha}}=\frac{1}{\kappa}*\left({}^{(1)}T_{\alpha}-2{}^{(2)}T_{\alpha}-\frac{1}{2}{}^{(3)}T_{\alpha}\right),\\ &E_{\alpha}\overset{\text{def.}}{=}\frac{\partial V}{\partial\vartheta^{\alpha}}=e_{\alpha}\rfloor V+\left(e_{\alpha}\rfloor T^{\beta}\right)\wedge H_{\beta}. \end{aligned} \tag{44}$$

Due to geometric identities,[91] the Lagrangian (42) can be recast as

$$V=-\frac{1}{2}T^{\alpha}\wedge H_{\alpha}. \tag{45}$$

The presence of the connection field $\Gamma^{\alpha}{}_{\beta}$ plays an important regularizing role due to the following statements.

(i) The theory becomes explicitly covariant under the local Lorentz transformations of the co-frame, i.e., the Lagrangian (42) is invariant under the change of variables

$$\begin{aligned} &\vartheta'^{\alpha}=\Lambda^{\alpha}{}_{\beta}\vartheta^{\beta},\\ &\Gamma'^{\beta}_{\alpha}=\Lambda^{\mu}{}_{\alpha}\Gamma_{\mu}{}^{\nu}(\Lambda^{-1})^{\beta}{}_{\nu}-(\Lambda^{-1})^{\beta}{}_{\gamma}\mathrm{d}\Lambda^{\gamma}{}_{\alpha}. \end{aligned} \tag{46}$$

Due to the non-covariant transformation law of $\Gamma_{\alpha}{}^{\beta}$ as shown by Eq. (46), if a connection vanishes in a given frame, it will not vanish in any other frame related to the first by a local Lorentz transformation.

(ii) $\Gamma_{\alpha}{}^{\beta}$ plays an important role in the teleparallel framework. This role represents the inertial effects which arise from the choice of the reference system.[92] The contributions of this inertia in many cases lead to non-physical results for the total energy of the system. Therefore, the role of the teleparallel connection, is to separate the inertial contribution from the truly gravitational one. Since the teleparallel curvature is zero, the connection is a pure gauge, that is

$$\Gamma_{\alpha}{}^{\beta}=(\Lambda^{-1})^{\beta}{}_{\gamma}\mathrm{d}\Lambda^{\gamma}{}_{\alpha}. \tag{47}$$

The Weitzenböck connection always has the form (47). The translational momentum has the form[92]

$$\widetilde{H}_{\alpha}=\frac{1}{2\kappa}\widetilde{\Gamma}^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma}, \quad \Gamma_{\alpha}{}^{\beta}\overset{\text{def.}}{=}\widetilde{\Gamma}^{\beta}_{\alpha}-K_{\alpha}{}^{\beta}, \tag{48}$$

where $\widetilde{\Gamma}^{\beta}_{\alpha}$ is the purely Riemannian connection and $K^{\mu\nu}$ is the contorsion 1-form which is related to the torsion through the relation

$$T^{\alpha}\overset{\text{def.}}{=}K^{\alpha}{}_{\beta}\wedge\vartheta^{\beta}. \tag{49}$$

We now express the energy using the language of forms. Using the spherical local coordinates (t,r,θ,ϕ), the $(4+1)$-dimensional Schwarzschild, using Eq. (14), frame is described by the co-frame components

$$\begin{aligned} \vartheta^{\hat{0}} &= \left(1-\frac{M}{\rho^2}\right)\mathrm{d}t+\frac{M\,\mathrm{d}\rho}{\rho^2-2M},\\ \vartheta^{\hat{1}} &= -\frac{\sin\theta\cos\phi\,(M-\rho^2)\,\mathrm{d}\rho}{\rho^2-2M}+\rho\cos\theta\cos\phi\,\mathrm{d}\theta\\ &\quad -\rho\sin\theta\sin\phi\,\mathrm{d}\phi+\frac{M\sin\theta\cos\phi}{\rho^2}\mathrm{d}t,\\ \vartheta^{\hat{2}} &= -\frac{\sin\theta\sin\phi\,(M-\rho^2)\,\mathrm{d}\rho}{\rho^2-2M}+\rho\cos\theta\sin\phi\,\mathrm{d}\theta\\ &\quad +\rho\sin\theta\cos\phi\,\mathrm{d}\phi+\frac{M\sin\theta\sin\phi}{\rho^2}\mathrm{d}t,\\ \vartheta^{\hat{3}} &= -\frac{\cos\theta\,(M-\rho^2)\,\mathrm{d}\rho}{\rho^2-2M}-\rho\sin\theta\,\mathrm{d}\theta+\frac{M\cos\theta}{\rho^2}\mathrm{d}t,\\ \vartheta^{\hat{4}} &= \rho\cos\theta\,\mathrm{d}\psi. \end{aligned} \tag{50}$$

If we take co-frame (50), as well as the Riemannian connection $\tilde{\Gamma}_\alpha^\beta$ and substitute into Eq. (48) we finally get[3)]

$$\widetilde{H}_{\hat{0}} = \frac{\sin\theta\cos\theta(3M-2\rho^2)}{32\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi)+\cdots. \quad (51)$$

Using Eq. (51) to compute the total energy at a fixed time in the 4-space with a spatial 3-dimensional boundary surface $\partial V=\{\rho=R,\theta,\phi\,\psi\}$ we finally obtain

$$\widetilde{E}=\int_{\partial V}\widetilde{H}_{\hat{0}}=\frac{3M\pi}{8}-\frac{3R^2\pi}{4}\to\infty \text{ when } R\to\infty! \quad (52)$$

Now let us associate with the co-frame (50) a nontrivial teleparallel connection

$$\begin{aligned}&\Gamma_{\hat{1}}^{\ \hat{2}}=\mathrm{d}\theta,\quad \Gamma_{\hat{1}}^{\ \hat{3}}=\sin\theta\,\mathrm{d}\phi,\quad \Gamma_{\hat{1}}^{\ \hat{4}}=\cos\theta\,\mathrm{d}\psi,\\ &\Gamma_{\hat{2}}^{\ \hat{3}}=\cos\theta\,\mathrm{d}\phi,\quad \Gamma_{\hat{2}}^{\ \hat{4}}=-\sin\theta\,\mathrm{d}\psi.\end{aligned} \quad (53)$$

The curvature obviously vanishes for this connection, but torsion is nontrivial. Using Eqs. (50) and (53) we then find

$$\begin{aligned}H_{\hat{0}} &= \frac{1}{2\kappa}(\tilde{\Gamma}^{\beta\gamma}-\Gamma^{\beta\gamma})\wedge\eta_{\alpha\beta\gamma}\\ &= \frac{3\sin\theta\cos\theta M}{32\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi)+\cdots.\end{aligned} \quad (54)$$

The integral of Eq. (54) over the spatial boundary yields

$$\widetilde{E}=\frac{3M\pi}{8}, \quad (55)$$

which is identical with Eq. (38).

8. Summary

We have constructed $(N+1)$-dimensional theory of gravitation. The basic entity of this theory is the field $h_a{}^\mu$. A Lagrangian which is quadratic in torsion in $(N+1)$-dimensional spacetime is constructed. By making a variational principle to the Lagrangian we get the field equation (12).

We have found a new, static, spherically symmetric solution to the field equations of $(4+1)$-dimensional TEGR in a vacuum. The solution contains one parameter M. Solution of the TEGR with electromagnetism has not yet been obtained. This will be done in a future work.

The energy of the obtained solution is calculated using two methods:

1) The gravitational energy–momentum tensor, which is coordinate-independent. We have obtained a form of energy which is divergent! Therefore, we use the regularized form and obtained a consistence result.

2) The Riemannian connection $\tilde{\Gamma}_\alpha^\beta$ in which the language of forms is used. When the superpotential 2-form is used, we also got a divergent result of energy. Therefore, we associate a Riemannian connection in which its curvature components are vanishing and some of the torsion components are not vanishing. Using this Riemannian connection as a regularized tool we get a consistent result of energy.

Appindex A

Here, we are going to write the non-vanishing components of the torsion $T^\mu{}_{\nu\sigma}$ and the vector T^μ needed for the calculations of energy.

The non-vanishing components of the torsion have the form

$$\begin{aligned}&T^0{}_{01}=\frac{2M}{\rho\Upsilon},\quad T^1{}_{10}=T^2{}_{20}=\frac{2M}{\rho},\quad T^2{}_{12}=T^3{}_{13}=\frac{M}{\rho\Upsilon},\\ &T^4{}_{14}=\frac{-1}{\rho},\quad T^4{}_{24}=\cot\theta,\end{aligned} \quad (A1)$$

while, the vector T^μ have the form

$$T^1=\frac{\Upsilon}{\rho^3},\quad T^2=\frac{2\tan\theta}{\rho^2}. \quad (A2)$$

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[3)] In Eq. (51), $\cdots$ means terms which are multiplied by $\mathrm{d}\rho\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi$, $\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}t$, $\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi$ etc.

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