Bound state solutions of d-dimensional Schrödinger equation with Eckart potential plus modified deformed Hylleraas potential

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We study the d-dimensional Schrödinger equation for Eckart plus modified deformed Hylleraas potentials using the generalized parametric form of Nikiforov–Uvarov method. We obtain energy eigenvalues and the corresponding wave function expressed in terms of a Jacobi polynomial. We also discuss two special cases of this potential comprised of the Hulthen potential and the Rosen–Morse potential in three dimensions. Numerical results are also computed for the energy spectrum and the potentials.

Keywords: parametric Nikiforov–Uvarov method, new approximation scheme, Eckart plus Hylleraas potential

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1. Introduction

The exact or approximate solution of the Schrödinger equation has been solved analytically in three dimensions for various potentials.[1–9] However, only a few of these potentials are exactly solvable, such as the Coulomb and harmonic potentials.[1,2] However, various approximation methods have been developed to solve bound state solutions of the Schrödinger equation in recent years.[3–7] Recently, the study of exponential-like type potentials have attracted much attention.[10] The potentials under investigation include the Hulthen potential,[11] Maninng–Rosen potential,[12] and Eckart-like potential.[13]

The purpose of this paper is to use a new approximation scheme proposed in Ref. [14] to study the Schrödinger equation for Eckart plus modified Hylleraas potentials in d dimensions using the Nikiforov–Uvarov method.[15–23] The Eckart potential which has been studied by many researchers[8,9] is one of the most important exponential-type potentials in physics and chemical physics whereas the Hylleraas potential can be used to study diatomic molecules.[24,25]

Recently, researchers have shown interest in the solutions of the Schrödinger equation in d dimensions, out of the desire to generalize the solution to multi-dimensional space. Such problems have been solved for relativistic and non-relativistic cases with different potentials.[26–33] The rest of the present paper is organized as follows. In Section 2, we briefly review the Nikiforov–Uvarov method in its parametric form. Section 3 is devoted to the Schrödinger equation in d space. In Section 4, we present the bound state solution of the Schrödinger equation in d dimensions. Finally, we give a brief conclusion in Section 5.

2. Review of Nikiforov–Uvarov method and its parametric form

The Nikiforov–Uvarov (NU) method is based on the solution of a generalized second order linear differential equation with special orthogonal function.[34,35] The Schrödinger equation

$$\psi''(r)+[E-V(r)]\psi(r)=0 \tag{1}$$

can be solved by transforming it into a hypergeometric-type equation through using the transformation, $s=s(x)$ and its resulting equation is expressed as

$$\psi''(s)+\frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s)=0. \tag{2}$$

In order to find the solution to Eq. (2) we set the wave function as

$$\psi(s)=\varphi(s)\chi_n(s). \tag{3}$$

Substituting Eq. (3) into Eq. (2) reduces Eq. (2) into a hypergeometric-type equation

$$\sigma(s)\chi_n''(s)+\tau(s)\chi_n'(s)+\lambda\chi_n(s)=0, \tag{4}$$

where the wave function $\varphi(s)$ is a logarithmic function

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)}, \tag{5}$$

and the other wave function $\chi(s)$ is the hypergeometric-type function, whose polynomial satisfies the Rodrigues relation

$$\chi_n(s)=\frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)], \tag{6}$$

where B_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition

$$[\sigma(s)\rho(s)]'=\tau(s)\rho(s). \tag{7}$$

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The required $\pi(s)$ and λ for the NU method are defined as

$$\pi(s)=\frac{\sigma'-\bar{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'-\bar{\tau}}{2}\right)^2-\bar{\sigma}(s)+k\sigma(s)} \tag{8}$$

and

$$\lambda=k+\pi'(s) \tag{9}$$

respectively. It is necessary that the term under the square root sign in Eq. (8) is the square of a polynomial. To calculate k from Eq. (8), the discriminant of the quadratic term must vanish. The eigenvlues in Eq. (9) take the form

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s),\ n=0,1,2,\ldots, \tag{10}$$

where

$$\tau(s)=\bar{\tau}(s)+2\pi(s) \tag{11}$$

and its derivative is less than zero, which is the necessary condition for bound state solutions. The energy eigenvalues are obtained by comparing Eq. (9) with Eq. (10).

The parametric generalization of the NU method that is valid for central and non-central exponential-type potential[36] can be derived by comparing the generalized hypergeometric-type equation

$$\psi''(s)+\frac{c_1-c_2s}{s(1-c_3s)}\psi'(s)+\frac{1}{s^2(1-c_3s)^2}[-\xi_1s^2+\xi_2s-\xi_3]\psi(s)=0 \tag{12}$$

with Eq. (2), and we obtain the following parametric polynomials

$$\bar{\tau}(s)=(c_1-c_2s), \tag{13}$$

$$\bar{\sigma}(s)=-\xi_1s^2+\xi_2s-\xi_3, \tag{14}$$

$$\sigma(s)=s(1-c_3s). \tag{15}$$

Substituting Eqs. (13) –(15) into Eq. (8), we obtain

$$\pi(s)=c_4-c_5s\pm\sqrt{[(c_6-c_3k_\pm)s^2(c_7+k_\pm)s+c_8]}, \tag{16}$$

where

$$c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),\ c_6=c_5^2+\xi_1,$$
$$c_7=2c_4c_5-\xi_2,\ c_8=c_4^2+\xi_3. \tag{17}$$

We obtain parameters $k_\pm$ from the condition that the function under the square root should be the square of a polynomial, and their expressions

$$k_\pm=-(c_7+2c_3c_8)\pm2\sqrt{c_8c_9}, \tag{18}$$

where

$$c_9=c_3c_7+c_3^2c_8+c_6. \tag{19}$$

Hence function $\pi(s)$ in Eq. (16) becomes

$$\pi(s)=c_4+c_5s-[(\sqrt{c_9}+c_3\sqrt{c_8})s-\sqrt{c_8}]. \tag{20}$$

The negative k_- value is obtained as

$$k_-=-(c_7+2c_3c_8)-2\sqrt{c_8c_9}. \tag{21}$$

Thus, from relation Eq. (11), we have

$$\tau(s)=c_1+2c_4-(c_2-2c_5)s-2[(\sqrt{c_9}+c_3\sqrt{c_8})s-\sqrt{c_8}], \tag{22}$$

whose derivative must be negative, that is,

$$\tau'(s)=-2c_3-2[(\sqrt{c_9}+c_3\sqrt{c_8})<0]. \tag{23}$$

From Eqs. (9) and (10), we obtain the parametric energy equation as

$$(c_2-c_3)n+c_3n^2-(2n+1)c_5+(2n+1)[\sqrt{c_9}+c_3\sqrt{c_8}]+c_7+2c_3c_8+2\sqrt{c_8c_9}=0. \tag{24}$$

The weight function is obtained as

$$\rho(s)=s^{c_{10}}(1-c_3s)^{c_{11}}. \tag{25}$$

Together with Eq. (6), we obtain

$$\chi_n(s)=\mathrm{P}_n^{(c_{10},\,c_{11})}(1-2c_3s), \tag{26}$$

where $\mathrm{P}_n^{(c_{10},\,c_{11})}$ are the Jacobi polynomials. For the special case where $c_3=0$, the Jacobi polynomial reduces to the Laguerre polynomial, where

$$c_{10}=c_1+2c_4+2\sqrt{c_8}, \tag{27}$$

$$c_{11}=1-c_1-2c_4+\frac{2}{c_3}\sqrt{c_9}. \tag{28}$$

The other part of the wave function is obtained from Eq. (5) as

$$\varphi(s)=s^{c_{12}}(1-c_3s)^{c_{13}}, \tag{29}$$

where

$$c_{12}=c_4+\sqrt{c_8}, \tag{30}$$

$$c_{13}=-c_4+\frac{1}{c_3}(\sqrt{c_9}-c_5). \tag{31}$$

Thus, the total wave function becomes

$$\psi_n(s)=N_ns^{c_{12}}(1-c_3s)^{c_{13}}\rho_n^{(c_{10}-c_{11})}(1-2c_3s), \tag{32}$$

where N_n is the normalization constant.

3. Schrödinger equation in *d* dimensions

The Schrödinger equation presented in Eq. (1) is in three dimensions. For d-dimensional space, the Schrödinger equation is [37]

$$-\frac{\hbar^2}{2\mu}\left[\nabla_D^2+V(r)\right]\psi_{nlm}(r,\Omega_D)=E_{n,l}\psi_{nlm}, \tag{33}$$

where the Laplacian operator is

$$\nabla_D^2=\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right)-\frac{\Lambda_D^2}{r^2}(\Omega_D). \tag{34}$$

The second term on the right-hand side of Eq. (34) is the multi-dimensional space centrifugal term, and Ω_D represents the angular coordinates. In this case, the operator Λ_D^2 yields a hyperspherical harmonic as its eigenfunction. This helps us to write

$$\psi_{nlm}(r,\Omega_D)=R_{n,l}(r)\mathrm{Y}_l^m(\Omega_D), \tag{35}$$

where $R_{n,l}$ is the radial part of the equation and $\mathrm{Y}_l^m(\Omega_D)$ is the angular part called hyperspherical harmonics. The $\mathrm{Y}_l^m(\Omega_D)$ obeys the eigenvalue equation

$$\Lambda_D^2\mathrm{Y}_l^m(\Omega_D)=l(l+D-2)\mathrm{Y}_l^m(\Omega_D). \tag{36}$$

Substituting Eqs. (34) and (35) into Eq. (33) and making use of Eq. (36) we obtain

$$\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial R(r)}{\partial r}\right) + \frac{2\mu}{\hbar^2}[E_{n,l} - \frac{l(l+D-2)}{r^2} - V(r)]R(r) = 0. \quad (37)$$

Equation (37) is the Schrödinger equation in d-dimensional space.

4. Bound state solutions of the Schrödinger equation

The potential Eckart plus Hylleraas is defined as[8,9,24,25]

$$V(r) = \frac{V_0}{b}\left(\frac{a-\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right) - V_1\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}} + V_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}, \quad (38)$$

where V_0, V_1, and V_2 are the depths of the potential well, a, and b Hylleraas parameters and α is the inverse of the range of the potential. If $V_1 = V_2 = 0$, equation (38) reduces to a deformed Hylleraas potential,[24,25] and when $V_0 = 0$ or $b \to \infty$ equation (37) reduces to the Eckart potential.[8,9] We display in Figs. 1–3 the plots of the deformed Hylleraas potential, Eckart potential, and the combined potentials $V(r)$ of Eq. (38) respectively.

The form of Eq. (37) is not suitable for bound state solutions, so we choose

$$R(r) = r^{-(D-1)/2}U(r). \quad (39)$$

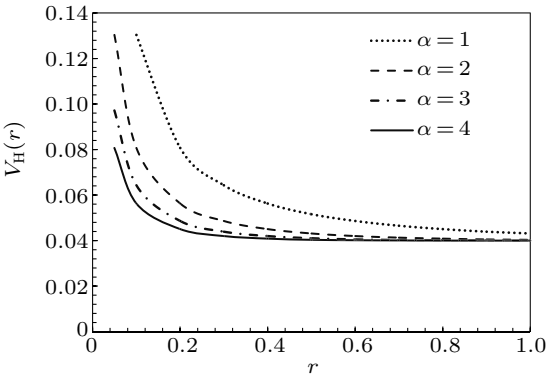


Fig. 1. Modified deformed Hylleraas potentials versus r with $a = 2.0$, $b = 50.0$, and $V_0 = 1.0$ MeV for parameter $\alpha = 1$, 2, 3, and 4.

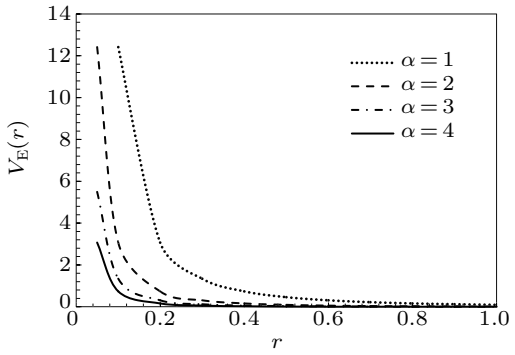


Fig. 2. Eckart potentials versus r with $V_1 = 0.01$ MeV and $V_2 = 0.5$ MeV for parameter $\alpha = 1$, 2, 3, and 4.

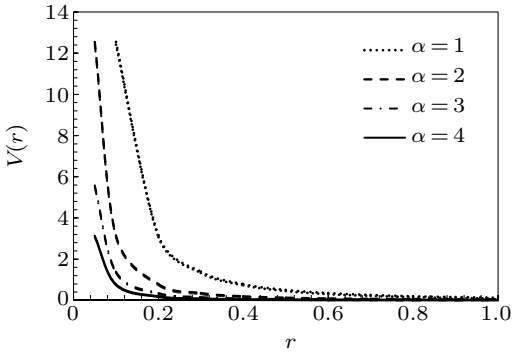


Fig. 3. Deformed Hylleraas plus Eckart potentials versus r with $V_0 = 1.0$ MeV, $V_1 = 0.01$ MeV, $V_2 = 0.5$ MeV, $a = 2.0$, and $b = 50.0$ for parameter $\alpha = 1$, 2, 3, and 4.

Inserting Eqs. (39) and (38) into Eq. (37), we obtain

$$U''(r) + \left\{\frac{2\mu}{\hbar^2}\left[E - \frac{V_0}{b}\left(\frac{a-\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right) + V_1\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}} - V_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}\right] - \left(\frac{(D-1)(D-3)}{4} + l(l+D-2)\right)\frac{1}{r^2}\right\}U(r) = 0. \quad (40)$$

Equation (40) can be solved exactly only in the s-wave for this potential. In order to find the approximate solutions of the radial Schrödinger equation of Eq. (40), we invoke using an approximation for the centrifugal term[5,6]

$$\frac{1}{r^2} = \frac{\alpha^2\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}. \quad (41)$$

This approximation is good for small values of the parameter α but inapplicable for large values.[11,38] To accommodate large values of α, we use a newly improved approximation scheme,[14] that is,

$$\frac{1}{r^2} = \frac{\omega\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}} + \frac{\lambda\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}, \quad (42)$$

where ω and λ are adjustable dimensionless parameters. In order to show that equation (42) is a better approximation than Eq. (41), we compare the effective potentials V_{eff} for $1/r^2$, V_{eff1} for the approximation in Eq. (41), and V_{eff2} for the approximation in Eq. (42) at different values of α as shown in Figs. 4–6.

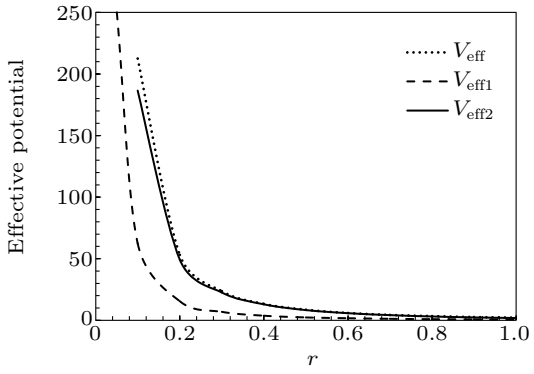


Fig. 4. Effective potentials ($V_{\rm eff}$ for $1/r^2$, $V_{\rm eff1}$ for the approximation in Eq. (41), and $V_{\rm eff2}$ for the approximation in Eq. (42)) versus r with $V_0 = 1.0$ MeV, $V_1 = 0.01$ MeV, $V_2 = 0.5$ MeV, $a = 2.0$, $b = 50.0$, $l = 1$, $\alpha = 1$, $\lambda = 3.2$, and $\omega = 1.6$.

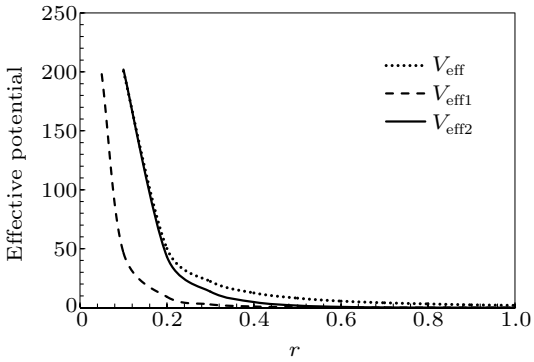


Fig. 5. Effective potentials ($V_{\rm eff}$ for $1/r^2$, $V_{\rm eff1}$ for the approximation in Eq. (41), and $V_{\rm eff2}$ for the approximation in Eq. (42)) versus r with $V_0 = 1.0$ MeV, $V_1 = 0.01$ MeV, $V_2 = 0.5$ MeV, $a = 2.0$, $b = 50.0$, $l = 1$, $\alpha = 5$, $\lambda = 3.3$, and $\omega = 1.7$.

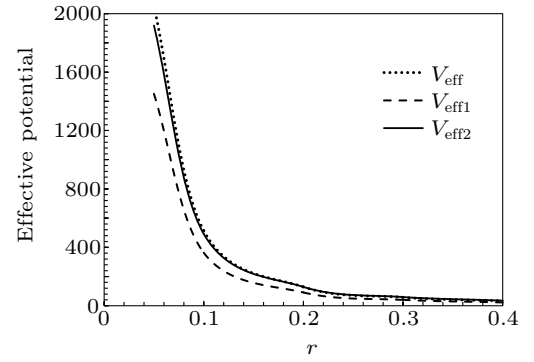


Fig. 6. Effective potentials ($V_{\rm eff}$ for $1/r^2$, $V_{\rm eff1}$ for the approximation in Eq. (41), and $V_{\rm eff2}$ for the approximation in Eq. (42)) versus r with $V_0 = 1.0$ MeV, $V_1 = 0.01$ MeV, $V_2 = 0.5$ MeV, $a = 2.0$, $b = 50.0$, $l = 1$, $\alpha = 0.2$, $\lambda = 3.1$, and $\omega = 12$.

Substituting Eq. (42) into Eq. (40) and using the transformation

$$s = \mathrm{e}^{-2\alpha r}, \tag{43}$$

equation (42) becomes

$$\frac{\mathrm{d}^2U}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}U}{\mathrm{d}s} + \frac{1}{s^2}\left[-\varepsilon^2 - \gamma\frac{(a-s)}{(1-s)} + \frac{\theta s}{(1-s)} - \frac{\Phi s}{(1-s)^2}\right]U = 0, \tag{44}$$

where

$$-\varepsilon^2 = \frac{\mu E}{2\hbar^2\alpha^2}, \tag{45}$$

$$\gamma = \frac{\mu V_0}{2\hbar^2\alpha^2 b}, \tag{46}$$

$$\theta = \frac{1}{4\alpha^2}\left[\frac{2\mu V_1}{\hbar^2} - \omega\left(\frac{(D-1)(D-3)}{4} + l(l+D-2)\right)\right], \tag{47}$$

$$\phi = \frac{1}{4\alpha^2}\left[\frac{2\mu V_2}{\hbar^2} + \lambda\left(\frac{(D-1)(D-3)}{4} + l(l+D-2)\right)\right]. \tag{48}$$

Equation (44) can be reduced further to the following form

$$\frac{\mathrm{d}^2U}{\mathrm{d}s^2} + \frac{(1-s)}{s(1-s)}\frac{\mathrm{d}U}{\mathrm{d}s} + \frac{1}{s^2(1-s)^2}\left[-(\varepsilon^2+A)s^2 + (B+2\varepsilon^2)s - (\varepsilon^2+C)\right]U = 0, \tag{49}$$

where

$$A = \gamma + \theta, \tag{50}$$

$$B = (a+1)\gamma + \theta - \phi, \tag{51}$$

$$C = \gamma a. \tag{52a}$$

Comparing Eq. (49) and Eq. (12) and performing further calculations, we obtain

$$c_1 = c_2 = c_3 = 1,\ \xi_1 = \varepsilon^2 + A,\ \xi_2 = B + 2\varepsilon^2,\ \xi_3 = C + \varepsilon^2,$$
$$c_4 = 0,\ c_5 = -\frac{1}{2},\ c_6 = \varepsilon^2 + A + \frac{1}{4},\ c_7 = -B - 2\varepsilon^2,$$
$$c_8 = \varepsilon^2 + C,\ c_9 = \sqrt{\phi + \frac{1}{4}},\ c_{10} = 1 + 2\sqrt{\varepsilon^2 + \gamma a},$$
$$c_{11} = -2\sqrt{\phi + \frac{1}{4}},\ c_{12} = \sqrt{\varepsilon^2 + \gamma a},$$
$$c_{13} = \frac{1}{2}(1 - \sqrt{1+4\phi}). \tag{52b}$$

Using Eqs. (24) and (52b) we obtain the energy equation as,

$$\varepsilon^2 = \left[\frac{(1-a)\gamma + \theta - \phi - (n^2 + \frac{(2n+1)}{2}[1+\sqrt{1+4\Phi}])}{n + [1+\sqrt{1+4\phi}]}\right]^2. \tag{53}$$

From Eq. (45) we obtain the energy eigenvalues as

$$E_{n,l} = \frac{-2\hbar^2\alpha^2}{\mu}\left[\frac{(1-a)\gamma + \theta - \phi - (n^2 + (2n+1)\sigma)}{2(n+\sigma)}\right]^2 + \frac{aV_0}{b}, \tag{54}$$

where

$$\sigma = \frac{1}{2}\left[1 + \sqrt{1+4\phi}\right]. \tag{55}$$

5. Discussion

In this section, we consider some special cases of the potential under consideration.

(i) If we set $V_0 = V_2 = 0$, $a = 0$, and $b = 1$ the potential in Eq. (38) reduces to[10]

$$V(r) = \frac{-V_1\,\mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}}, \tag{56}$$

which is the Hulthen potential. Furthermore, if we set $D = 3$, $\omega = 0$, and $\lambda = 4\alpha^2$ in Eqs. (46)–(48) and using the resulting values in eigenvalue equation (54) we obtain

$$E_{n,l} = \frac{-2\hbar^2\alpha^2}{\mu}\left[\frac{\mu V_0}{4\hbar^2\alpha^2(n+l+1)} - \frac{(n+l+1)}{2}\right]^2. \quad (57)$$

This is the eigenvalue of the Hulthen potential. The result is consistent with that of Eq. (36) in Ref. [14] if we make a simple substitution $\alpha = \delta/2$.

(ii) If we set $V_{01} = V_2 = 0$, $b = 1$, and $a = -1$, the potential in Eq. (38) is reduced to Rosen–Morse potential[40]

$$V(r) = \frac{-V_0(1+\mathrm{e}^{-2\alpha r})}{(1-\mathrm{e}^{-2\alpha r})} = \coth(\alpha r). \quad (58)$$

Again, if we set $D = 3$, $\omega = 0$, and $\lambda = 4\alpha^2$ in Eqs. (46)–(48) and using the resulting values in Eq. (54), we obtain the eigenvalues of the Rosen–Morse potential as

$$E_{n,l} = \frac{-2\hbar^2\alpha^2}{\mu}\left[\frac{\mu V_0}{2\hbar^2\alpha^2(n+l+1)} - \frac{(n+l+1)}{2}\right]^2 - V_0. \quad (59)$$

Now using Eq. (29) we obtain the first part of the wave function as

$$\varphi(s) = s^{\mu}(1+s)^{(-v+1)/2}, \quad (60)$$

where $\mu = \sqrt{\varepsilon^2 + \gamma a}$ and $v = \sqrt{1+4\phi}$. From Eq. (25) the weight function is obtained as

$$\rho(s) = s^{1+2\mu}(1+s)^{-v}, \quad (61)$$

and from Eq. (26) the second part of the wave function is

$$\chi_n(s) = P_n^{1+2\mu,2v}(1+2s). \quad (62)$$

The Jacobi polynomials that we documented can be found in Refs. [41]– [43]. The radial wave function is thus

$$U(r) = N_{n,l}(\mathrm{e}^{-2\alpha r})^{\mu}(1-\mathrm{e}^{-2\alpha r})^{(v+1)} \times P_n^{1+2\mu,-v}(1-2\mathrm{e}^{-2\alpha r}), \quad (63)$$

where $N_{n,l}$ is the normalization constant.

6. Numerical results

The numerical results of the approximation schemes we employed are displayed in Figs. 4–6 , showing that this approximation is good for all values of parameter α since one can always adjust the parameters ω and λ unlike the approximation in Eq. (41) which is only good for small values of α. We make appropriate choices for the parameters α, ω, λ, V_0, V_1, b, and a and computed the energy eigenvalues for the system for different values of ω and λ as shown in Tables 1–3 for different values of n and l.

Table 1. Energy eigenvalues of the potential for $\omega = 1.6$ and $\lambda = 3.2$. Here, $\alpha = 1$, $\omega = 1.6$, $\lambda = 3.2$, $a = 2.0$, $b = 50$, $\mu = 1$, $V_0 = 1.0$ MeV, $V_1 = 0.01$ MeV, and $V_2 = 0.5$ MeV.

n	L	Eigenvalues		
		$D=3$	$D=4$	$D=5$
0	0	−0.693561969	−1.483794256	−2.749854638
1	0	−2.400662738	−3.520372978	−5.150960406
2	0	−5.107768168	−6.565715334	−8.583619961
	1	−8.583619961	−11.07635272	−14.02084003
3	0	−8.81487444	−10.61307885	−13.02521012
	1	−13.02521012	−15.92716134	−19.28351018
	2	−19.28351018	−23.08296764	−27.32202956
4	0	−13.52198098	−15.66115049	−18.47027444
	1	−18.47027444	−21.78663214	−25.56228156
	2	−25.56228156	−29.78098662	−34.43704278
	3	−34.43704278	−39.52876901	−45.05614563
5	0	−19.22908762	−21.7095332	−24.91696686
	1	−24.91696686	−28.65040322	−32.84945913
	2	−32.84945913	−37.4928681	−42.57256298
	3	−42.57256298	−48.08566295	−54.0315293
	4	−54.0315293	−60.41053758	−67.22351809

Table 2. Energy eigenvalues of the potential for $\omega = 1.7$, and $\lambda = 3.3$. Here, $\alpha = 5$, $\omega = 1.7$, $\lambda = 3.3$, $a = 2.0$, $b = 50$, $\mu = 1$, $V_0 = 1.0$ MeV, $V_1 = 0.01$ MeV, and $V_2 = 0.5$ MeV.

n	L	Eigenvalues		
		$D=3$	$D=4$	$D=5$
0	0	−113.1097402	−560.9727952	−1316.065556
1	0	−201.8384501	−694.656059	−1491.095494
2	0	−315.5719709	−856.6308917	−1697.976766
	1	−1697.976766	−2836.531982	−4271.56472
3	0	−454.3074065	−1045.392669	−1934.208287
	1	−1934.208287	−3118.419739	−4597.997338
	2	−4597.997338	−6373.079767	−8443.782235
4	0	−618.0437527	−1260.207686	−2198.334215
	1	−2198.334215	−3429.966834	−4955.615228
	2	−4955.615228	−6775.822056	−8890.971435
	3	−8890.971435	−11301.32424	−14007.05962
5	0	−806.7805866	−1500.683512	−2489.460427
	1	−2489.460427	−3769.946194	−5342.999737
	2	−5342.999737	−7209.520922	−9370.16336
	3	−9370.16336	−11825.38215	−14575.49657
	4	−14575.49657	−17620.73484	−20961.26357

Table 3. Energy eigenvalues of the potential for $\omega = 12$ and $\lambda = 3.1$. Here, $\alpha = 5$, $\omega = 12$, $\lambda = 3.1$, $a = 2.0$, $b = 50$, $\mu = 1$, $V_0 = 1.0$ MeV, $V_1 = 0.01$ MeV, and $V_2 = 0.5$ MeV.

n	L	Eigenvalues		
		$D=3$	$D=4$	$D=5$
0	0	–	–	−0.006591882
1	0	−0.051095051	−0.056748905	−0.066201555
2	0	−0.151319951	−0.1577294	−0.168286976
	1	−0.168286976	−0.18286666	−0.201369665
3	0	−0.291655888	−0.298908764	−0.310765078
	1	−0.310765078	−0.326970774	−0.347291388
	2	−0.347291388	−0.371548105	−0.399625631
4	0	−0.472024074	−0.480146418	−0.493359927
	1	−0.493359927	−0.51130219	−0.533626149
	2	−0.533626149	−0.560049273	−0.590366077
	3	−0.590366077	−0.62444112	−0.662195555
5	0	−0.692404982	−0.701407183	−0.71600143
	1	−0.71600143	−0.735725353	−0.760129567
	2	−0.760129567	−0.78883907	−0.821569698
	3	−0.821569698	−0.858120539	−0.89835829
	4	−0.89835829	−0.94220149	−0.989607378

7. Conclusions

In this paper, we investigated the solutions of the Schrödinger equation for the Eckart plus Hylleraas potentials in d-dimensional space, using a new approximation method for the centrifugal term via the parametric Nikiforov–Uvarov method. The eigenvalues of the potential reduce to those of well-known potentials viz Hulthen potential in Eq. (57) and Rose–Morse potential in Eq. (59), when we make appropriate choices of the parameters α, ω, λ, V_0, V_1, b, and a. Finally, we also obtained the wave function that is expressed in terms of the Jacobi polynomials.

References

[1] Gomul O and Kocak M 2001 arxiv: Quant-Ph/0106144
[2] Oyewumi K J, Akinpelu F O and Agboola A.D 2008 *Int. J. Theor. Phys.* **47** 1039
[3] Ikot A N and Akpabio L E 2010 *Appl. Phys. Res.* **2** 202
[4] Ikot A N, Akpabio L E and Umoren E B 2011 *J. Sci. Res.* **3** 25
[5] Hassanabadi H, Zurrinkamar S and Rahimov H 2011 *Commun. Theor. Phys.* **56** 423
[6] Greene R L and Aldrich C 1976 *Phys. Rev.* A **14** 2363
[7] Ikot A N 2011 *Afr. Phys. Rev.* **6** 221
[8] Zhang A P, Qiang W C and Ling Y W 2006 *Chin. Phys. Lett.* **26** 100302
[9] Wei G F, Long C Y, Duan X Y and Dong S H 2008 *Phys. Scr.* **77** 035001
[10] Qiang W C and Dong S H 2009 *Phys. Scr.* **79** 045004
[11] Bayrak O, Kocak G and Boztosun I 2006 *J. Phys* A: *Math. Gen.* **39** 11521
[12] Dong S H and Garcia-Ravelo J 2007 *Phys. Scr.* **75** 307
[13] Jia C S, Guo P and Peng X L 2006 *J. Phys.* A: *Math. Gen.* **39** 7737
[14] Taskin F and Kocak G 2010 *Chin. Phys.* B **19** 090314
[15] Ikot A N, Akpabio L E and Obu J A 2011 *J. Vect. Relat.* **6** 1
[16] Ikot A N, Awoga O A , Akpabio L E and Antia A D 2011 *Elixir. Vib. Spec.* **33** 2303
[17] Ikot A N, Antia A D, Akpabio L E and Obu J A 2011 *J. Vect . Relat.* **6** 65
[18] Yasuk F, Durwins A and Boztsun I 2006 *J. Math. Phys.* **47** 083202
[19] Falaye B J and Oyewumi K J 2011 *Afri. Phys. Rev.* **6** 0025
[20] Ikhadair S M and Sever R 2007 *J. Math. Chem.* **42** 461
[21] Ikhadair S M and Sever R 2007 *Int. J. Theor. Phys.* **46** 1643
[22] Ikhadair S M 2008 *Chin. J. Phys.* **46** 291
[23] Ikhadair S M and Sever R 2008 *Cent. Eur. J. Phys.* **6** 685
[24] Varshni Y P 1957 *Rev. Mod. Phys.* **29** 664
[25] Hylleraas E A 1935 *J. Chem. Phys.* **3** 595
[26] Cardose J L and Alvarez–Nodarse R 2003 *J. Phys.* A: *Math. Gen.* **36** 2055
[27] Dong S H and Sun G H 2003 *Phys. Lett.* A **314** 261
[28] Dong S H, Chen C Y and Cassou M L 2005 *J. Phys.* B **38** 2211
[29] Oyewumi K J and Ogunsola A W 2004 *J. Pure Appl. Sci.* **10** 343
[30] Paz G 2011 *Eur. J. Phys.* **22** 337
[31] Dong S H, Sun G H and Popov D 2003 *J. Math. Phys.* **44** 4467
[32] Ikhadair S M and Sever R 2008 *Mol. Struc: THEOCHEM* **855** 13
[33] Dong S H and Ma Z Q 2002 *Phys. Rev.* A **65** 042717
[34] Nikiforov A F and Uvarov V B 1998 *Special Functions of Mathematical Physics* (Basel: Birkhauser)
[35] Tezcan C and Sever R 2009 *Int. J. Theor. Phys.* **48** 337
[36] Arda A, Sever R and Tezcan C 2010 *Chin. J. Phys.* **48** 27
[37] Hassanabadi H, Zarrinkamar S and Rajabi A A 2011 *Commun. Theor. Phys.* **55** 541
[38] Dong S H, Qiang W C, Sun G H and Bezerra V B 2007 *J. Phys.* A: *Math. Theor.* **40** 10535
[39] Ikhadair S M 2009 *Eur. Phys. J.* A **39** 307
[40] Debnath S and Biswas B 2012 *EJTP* **26** 191
[41] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (New York: Dover)
[42] Polyanin A D and Manzhirov A V 1998 *Handbook of Integral Equations* (Boca Raton: CRC Press)
[43] Gradshteyn I S and Rhyzhik I M 2007 *Table of Integrals, Series and Product* (Burlington: Elsevier)