New optical field operator expansion in number state representation*

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By virtue of the Weyl ordering method, we find a new formalism of optical field operator expansion in number state representation. Miscellaneous optical fields' (coherent state, squeezed field, Wigner operator, etc.) new expansions are therefore exhibited. Some new generating functions of special polynomials are derived herewith.

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1. Introduction

In quantum optics theory researchers often convert operator master equations into classical function differential equations. The way they usually do this is by using the Wigner function of density operators, the P-representation (coherent state representation) or Q-representation (average values of operators).[1–5] The Q-representation of an operator A is obtained through its particle number representation, i.e., expanding operator A as

$$A=\sum_{m,n}|m\rangle\langle m|A|n\rangle\langle n|=\sum_{m,n}A_{m,n}|m\rangle\langle n|, \tag{1}$$

where

$$|n\rangle=\frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$$

is the number state (Fock state), $a^{\dagger}$ (a) are creation (annihilation) Bose operator, $\left[a,a^{\dagger}\right]=1$, $|n\rangle$ span the complete space

$$\sum_{n=0}^{\infty}|n\rangle\langle n|=1. \tag{2}$$

In previous literature, Eq. (1) does not provide much information, since $|m\rangle\langle n|$ seems to be an ultimate expression. Nevertheless, in this work we shall go a step further to convert Eq. (1) to its Weyl ordered expansion[6] with the help of the Weyl ordered form of $|m\rangle\langle n|$. The Weyl ordered form of operators has the advantage that the ordering is invariant under unitary transformations.[7,8] Using this property we not only obtain new information about optical fields but also derive many useful new integration formulas and generating function formulas about special polynomials.

2. The Weyl ordered form of $|m\rangle\langle n|$

We start with putting $|m\rangle\langle n|$ into its Weyl ordering. Recall in Ref. [9] using the technique of integration within ordered product (IWOP) of operators we have derived a formula which can convert an operator ρ into its Weyl ordered form

$$\rho=2\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\ \langle-\beta|\rho|\beta\rangle\exp\left[2\left(\beta^*a-a^{\dagger}\beta+a^{\dagger}a\right)\right]\vdots, \tag{3}$$

where $\vdots\ \vdots$ denotes Weyl ordering, and $|\beta\rangle$ is a coherent state

$$|\beta\rangle=\mathrm{e}^{-|\beta^2|/2}\mathrm{e}^{\beta a^{\dagger}}|0\rangle. \tag{4}$$

In particular, when $\rho=1$, the identity operator, then from $\langle-\beta|\,\beta\rangle=\mathrm{e}^{-2|\beta|^2}$, the right-hand side of Eq. (3) reduces to

$$2\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\ \exp\left[-2\left(\beta^*+a^{\dagger}\right)(\beta-a)\right]\vdots=1, \tag{5}$$

which is true. Then from

$$\langle n\,|\beta\rangle=\mathrm{e}^{-\frac{1}{2}|\beta|^2}\frac{(\beta)^m}{\sqrt{m!}} \tag{6}$$

and using the integration formula about the two-variable Hermite polynomial

$$\begin{aligned}H_{m,n}(\xi,\eta)&=(-1)^n\mathrm{e}^{\xi\eta}\int\frac{\mathrm{d}^2z}{\pi}z^nz^{*m}\exp\left[-|z|^2+\xi z-\eta z^*\right]\\&=(-1)^m\mathrm{e}^{\xi\eta}\int\frac{\mathrm{d}^2z}{\pi}z^nz^{*m}\\&\quad\times\exp\left[-|z|^2-\xi z+\eta z^*\right],\end{aligned} \tag{7}$$

here the definition of $\mathrm{H}_{m,n}$ is

$$\mathrm{H}_{m,n}(\xi,\eta)=\sum_{l=0}^{\min(m,n)}l!\binom{m}{l}\binom{n}{l}(-1)^l\xi^{m-l}\eta^{n-l}, \tag{8}$$

and we can derive the Weyl ordered form of $|m\rangle\langle n|$

$$\begin{aligned}|m\rangle\langle n|&=2\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\ \langle-\beta|\,m\rangle\langle n\,|\beta\rangle\\&\quad\times\exp\left[2\left(\beta^*a-a^{\dagger}\beta+a^{\dagger}a\right)\right]\vdots\\&=2\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\ \frac{\beta^n(-\beta^*)^m}{\sqrt{n!m!}}\end{aligned}$$

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$$\times \exp\left[-|\beta|^2+2\left(a\beta^*-a^\dagger\beta+a^\dagger a\right)\right]\vdots$$
$$=\frac{2}{\sqrt{n!m!}}\vdots \mathrm{H}_{m,n}\left(2a^\dagger,2a\right)\exp\left[-2a^\dagger a\right]\vdots . \quad (9)$$

3. Miscellaneous optical fields' new expansions

The Weyl ordered form (9) leads Eq. (1) to the new expansion of any operators in Fock space

$$A=\sum_{m,n}A_{m,n}|m\rangle\langle n|$$
$$=\sum_{m,n}A_{m,n}\frac{2}{\sqrt{n!m!}}\vdots \mathrm{H}_{m,n}\left(2a^\dagger,2a\right)\exp\left[-2a^\dagger a\right]\vdots . \quad (10)$$

As its applications, we present new expansions of miscellaneous optical fields (coherent state, squeezed field, Wigner operator, etc).

When $A=|z\rangle\langle z|$, the pure density operator of coherent state, from Ref. [10] we know its Weyl ordered form is

$$|z\rangle\langle z|=2\vdots\ \mathrm{e}^{-2\left(a^\dagger-z^*\right)(a-z)}\vdots , \quad (11)$$

then from

$$\langle z|\,n\rangle=\mathrm{e}^{-|z|^2/2}\frac{z^{*n}}{\sqrt{n!}}, \quad (12)$$

we see

$$|z\rangle\langle z|=2\mathrm{e}^{-|z|^2}\sum_{m,n}\frac{z^m z^{*n}}{n!m!}\vdots \mathrm{H}_{m,n}\left(2a^\dagger,2a\right)\exp\left[-2a^\dagger a\right]\vdots$$
$$=2\vdots\ \mathrm{e}^{-2\left(a^\dagger-z^*\right)(a-z)}\vdots , \quad (13)$$

since within the Weyl ordered symbol $\vdots\ \vdots$, $a^\dagger$ and a are commutable, so Eq. (13) implies

$$\sum_{m,n}\frac{z^m z^{*n}}{n!m!}\mathrm{H}_{m,n}\left(2\lambda^*,2\lambda\right)=\mathrm{e}^{-|z|^2-2z^*\lambda-2z\lambda^*}. \quad (14)$$

This actually gives the generating function formula of the two-variable Hermite polynomial.

When $A=D(\alpha)=\mathrm{e}^{\alpha a^\dagger-a^*a}$, the displacement operator, due to $D(\alpha)=\vdots\ \mathrm{e}^{\alpha a^\dagger-a^*a}\vdots$, so from

$$\langle m|D(\alpha)|n\rangle=\sqrt{\frac{n!}{m!}}\alpha^{m-n}\mathrm{e}^{-|\alpha|^2/2}\mathrm{L}_n^{m-n}(|\alpha|^2),\quad m>n, \quad (15)$$

where L_n^{m-n} is the Laguerre polynomial, and from Eq. (10) we have

$$D(\alpha)=\vdots\ \mathrm{e}^{\alpha a^\dagger-a^*a}\vdots$$
$$=\sum_{m,n}\frac{2}{m!}\alpha^{m-n}\mathrm{e}^{-|\alpha|^2/2}\mathrm{L}_n^{m-n}(|\alpha|^2)\vdots \mathrm{H}_{m,n}\left(2a^\dagger,2a\right)$$
$$\times\exp\left[-2a^\dagger a\right]\vdots , \quad (16)$$

this indicates the following new relation

$$\sum_{m,n}\frac{2}{m!}\alpha^{m-n}\mathrm{e}^{-|\alpha|^2/2}\mathrm{L}_n^{m-n}(|\alpha|^2)\mathrm{H}_{m,n}\left(2\lambda^*,2\lambda\right)$$
$$\times\exp\left[-2|\lambda|^2\right]=\mathrm{e}^{\alpha\lambda^*-a^*\lambda}. \quad (17)$$

Due to the relationship between the two-variable Hermite polynomials and the associated Laguerre polynomials

$$\mathrm{H}_{m,n}(\xi,\kappa)=\begin{cases}n!\,(-1)^n\,\xi^{m-n}\mathrm{L}_n^{m-n}(\xi\kappa), & m>n,\\ m!\,(-1)^m\,\kappa^{n-m}\mathrm{L}_m^{n-m}(\xi\kappa), & m<n,\end{cases} \quad (18)$$

where

$$\mathrm{L}_n^{\alpha}(x)=\sum_k\binom{\alpha+n}{n-k}\frac{(-x)^k}{k!}, \quad (19)$$

we can further reform Eq. (17) as

$$\mathrm{e}^{-2|\lambda|^2-|\alpha|^2/2}\sum_{m,n}2^{m-n+1}(-1)^n\frac{n!}{m!}(\alpha\lambda^*)^{m-n}\mathrm{L}_n^{m-n}(|\alpha|^2)$$
$$\times\mathrm{L}_n^{m-n}\left(4|\lambda|^2\right)=\mathrm{e}^{\alpha\lambda^*-a^*\lambda}, \quad (20)$$

which is a new generating function relation about the Laguerre polynomial.

When $A=\Delta(\alpha,\alpha^*)$, the Wigner operator, due to $\Delta(\alpha,\alpha^*)=\frac{1}{2}\vdots\ \delta\left(a^\dagger-\alpha^*\right)\delta(a-\alpha)\vdots$,[11] so from

$$\langle m|\Delta(\alpha,\alpha^*)|n\rangle=\frac{1}{\pi\sqrt{n!m!}}\mathrm{e}^{-2|\alpha|^2}\mathrm{H}_{m,n}(2\alpha,2\alpha^*) \quad (21)$$

and Eq. (10) we have

$$\Delta(\alpha,\alpha^*)=\sum_{m,n}\frac{2}{\pi n!m!}\mathrm{e}^{-2|\alpha|^2}\mathrm{H}_{m,n}(2\alpha,2\alpha^*)\vdots \mathrm{H}_{m,n}\left(2a^\dagger,2a\right)\exp\left[-2a^\dagger a\right]\vdots$$
$$=\vdots\sum_{m,n}\frac{2^{2m-2n+1}}{\pi}\frac{n!}{m!}\left(\alpha a^\dagger\right)^{m-n}\mathrm{e}^{-2|\alpha|^2}\mathrm{L}_n^{m-n}\left(4|\alpha|^2\right)\mathrm{L}_n^{m-n}\left(4a^\dagger a\right)\exp\left[-2a^\dagger a\right]\vdots$$
$$=\frac{1}{2}\vdots\ \delta\left(a^\dagger-\alpha^*\right)\delta(a-\alpha)\vdots , \quad (22)$$

which exhibits a new expansion of Dirac delta function in terms of the Laguerre polynomials

$$\delta(\lambda^*-\alpha^*)\,\delta(\lambda-\alpha)=\mathrm{e}^{-2|\alpha|^2-2|\lambda|^2}\sum_{m,n}\frac{2^{2(m-n+1)}}{\pi}\frac{n!}{m!}(\alpha\lambda^*)^{m-n}\mathrm{L}_n^{m-n}\left(4|\alpha|^2\right)\mathrm{L}_n^{m-n}\left(4|\lambda|^2\right). \quad (23)$$

When $A = S = \exp[\frac{\lambda}{2}(a^2 - a^{\dagger 2})]$, the squeezing operator, due to its normal ordering form

$$S =: \exp\left[\frac{a^2 - a^{\dagger 2}}{2}\tanh\lambda + (\operatorname{sech}\lambda - 1)a^\dagger a\right]: \tag{24}$$

using Eq. (3) we have

$$\begin{aligned} S &= 2\operatorname{sech}^{1/2}\lambda \,\vdots \int \frac{\mathrm{d}^2\beta}{\pi}\langle -\beta| : \exp\left[\frac{a^2 - a^{\dagger 2}}{2}\tanh\lambda + (\operatorname{sech}\lambda - 1)a^\dagger a\right] : |\beta\rangle \\ &\quad \times \exp\left[2\beta^* a - 2\beta a^\dagger + 2a^\dagger a\right]\mathrm{d}^2\beta \,\vdots \\ &= \operatorname{sech}\frac{\lambda}{2} \,\vdots\, \exp\left(2\mathrm{i}QP\tanh\frac{\lambda}{2}\right) \,\vdots\, = \operatorname{sech}\frac{\lambda}{2} \,\vdots\, \exp\left[(a + a^\dagger)(a - a^\dagger)\tanh\frac{\lambda}{2}\right] \,\vdots\, . \end{aligned} \tag{25}$$

Then from

$$\langle m|S|n\rangle = (\operatorname{sech}\lambda)^{m+1/2}\left(\frac{1}{2}\tanh\lambda\right)^{(n-m)/2}\sum_{l=0}^{[m/2]}\frac{\sqrt{n!m!}(-\sinh^2\lambda)^l}{4^l l!(l + \frac{n-m}{2})!(m-2l)!}, \quad n \geqslant m \tag{26}$$

and Eq. (10) we have the new expansion formula

$$\begin{aligned} &\operatorname{sech}\frac{\lambda}{2}\exp\left[(\alpha^2 - \alpha^{*2})\tanh\frac{\lambda}{2}\right] \\ &= 2\sum_{m,n}(\operatorname{sech}\lambda)^{m+1/2}\left(\frac{\tanh\lambda}{2}\right)^{(n-m)/2}\sum_{l=0}^{[m/2]}\frac{(-\sinh^2\lambda)^l}{4^l l!(l + \frac{n-m}{2})!(m-2l)!}\mathrm{H}_{m,n}(2\alpha^*, 2\alpha)\exp(-2|\alpha|^2). \end{aligned} \tag{27}$$

When $A = D(\alpha)|k\rangle\langle k|D^\dagger(\alpha)$, the displaced Fock operator, by observing

$$\begin{aligned} D(\alpha)|n\rangle &= D(\alpha)\frac{a^{\dagger n}}{\sqrt{n!}}D^{-1}(\alpha)D(\alpha)|0\rangle \\ &= \frac{(a^\dagger - \alpha^*)^n}{\sqrt{n!}}|\alpha\rangle. \end{aligned} \tag{28}$$

Its completeness relation is obvious when we employ the IWOP technique

$$\begin{aligned} &\int \frac{\mathrm{d}^2\alpha}{\pi}D(\alpha)|k\rangle\langle k|D^{-1}(\alpha) \\ &= \int \frac{\mathrm{d}^2\alpha}{\pi}\frac{(a^\dagger - \alpha^*)^k}{\sqrt{k!}}|\alpha\rangle\langle\alpha|\frac{(a-\alpha)^k}{\sqrt{k!}} \\ &= \int \frac{\mathrm{d}^2\alpha}{\pi} : \frac{(a^\dagger - \alpha^*)^k(a-\alpha)^k}{k!}\mathrm{e}^{-(a^\dagger - \alpha^*)(a-\alpha)} : \\ &= \int \frac{\mathrm{d}^2\alpha}{\pi}\frac{|\alpha|^{2k}}{k!}\mathrm{e}^{-|\alpha|^2} = 1. \end{aligned} \tag{29}$$

From the fact that the Weyl ordering is invariant under unitary transformations, we know

$$\begin{aligned} &D(\alpha)|k\rangle\langle k|D^\dagger(\alpha) \\ &= 2(-)^k D(\alpha) \,\vdots\, \mathrm{L}_k\left(4a^\dagger a\right)\mathrm{e}^{-2a^\dagger a} \,\vdots\, D^{-1}(\alpha) \\ &= 2(-1)^k \,\vdots\, \mathrm{L}_k\left[4\left(a^\dagger - \alpha^*\right)(a-\alpha)\right]\mathrm{e}^{-2\left(a^\dagger - \alpha^*\right)(a-\alpha)} \,\vdots\, , \end{aligned} \tag{30}$$

so comparing Eqs. (29) and (30) we obtain

$$\begin{aligned} &2\int \frac{\mathrm{d}^2\alpha}{\pi} \,\vdots\, \mathrm{L}_k\left[4\left(a^\dagger - \alpha^*\right)(a-\alpha)\right]\mathrm{e}^{-2\left(a^\dagger - \alpha^*\right)(a-\alpha)} \,\vdots \\ &= (-1)^k \end{aligned} \tag{31}$$

or a new integration

$$2\int \frac{\mathrm{d}^2\alpha}{\pi}\mathrm{L}_k\left[4|\alpha|^2\right]\mathrm{e}^{-2|\alpha|^2} = (-1)^k. \tag{32}$$

Then using Eq. (15)

$$\begin{aligned} &\langle m|D(\alpha)|k\rangle\langle k|D^\dagger(\alpha)|n\rangle \\ &= k!\sqrt{\frac{1}{m!n!}}\alpha^m\alpha^{*n}|\alpha|^{-2k}\mathrm{e}^{-|\alpha|^2}\mathrm{L}_k^{m-k}(|\alpha|^2)\mathrm{L}_k^{n-k}(|\alpha|^2) \end{aligned} \tag{33}$$

and Eq. (10) we have

$$\begin{aligned} &2(-1)^k \,\vdots\, \mathrm{L}_k\left[4\left(a^\dagger - \alpha^*\right)(a-\alpha)\right]\mathrm{e}^{-2\left(a^\dagger - \alpha^*\right)(a-\alpha)} \,\vdots \\ &= \sum_{m,n}k!\sqrt{\frac{1}{m!n!}}\alpha^m\alpha^{*n}|\alpha|^{-2k}\mathrm{e}^{-|\alpha|^2}\mathrm{L}_k^{m-k}(|\alpha|^2)\mathrm{L}_k^{n-k}(|\alpha|^2)\frac{2}{\sqrt{n!m!}} \,\vdots\, \mathrm{H}_{m,n}\left(2a^\dagger, 2a\right)\exp\left[-2a^\dagger a\right] \,\vdots \\ &= \,\vdots\, \sum_{m,n}\frac{2k!}{m!}\alpha^m(-\alpha^*)^n|\alpha|^{-2k}\mathrm{e}^{-|\alpha|^2}\mathrm{L}_k^{m-k}(|\alpha|^2)\mathrm{L}_k^{n-k}(|\alpha|^2)(2a^\dagger)^{m-n}\mathrm{L}_n^{m-n}\left(4a^\dagger a\right)\exp\left[-2a^\dagger a\right] \,\vdots\, , \end{aligned} \tag{34}$$

from which we have the new expansion formula

$$(-1)^k \mathrm{L}_k\left[4\left(\lambda^*-\alpha^*\right)(\lambda-\alpha)\right] \mathrm{e}^{-2(\lambda^*-\alpha^*)(\lambda-\alpha)}$$
$$= \exp\left[-2|\lambda|^2-|\alpha|^2\right] \sum_{m,n} 2^{m-n} \frac{k!}{m!} \alpha^m (-\alpha^*)^n \lambda^{*(m-n)} |\alpha|^{-2k} \mathrm{L}_k^{m-k}(|\alpha|^2) \mathrm{L}_k^{n-k}(|\alpha|^2) \mathrm{L}_n^{m-n}\left(4|\lambda|^2\right). \quad (35)$$

4. Weyl ordered form of a^{-1}

Dirac[12] first introduced the Bose annihilator a^{-1} and noticed that it obeys $aa^{-1}=1$, $a^{-1}a\neq 1$, since $a|0\rangle=0$, where $|0\rangle$ is the vacuum state. In Ref. [13] the inverse operators in Fock space are studied by Fan via a coherent state approach. Let $|z\rangle$ be the un-normalized coherent state $|z\rangle=\mathrm{e}^{za^\dagger}|0\rangle$, with the aid of the contour integration expression of the Fock state $|n\rangle$

$$|n\rangle = \frac{\sqrt{n!}}{2\pi i}\oint_c \mathrm{d}z \frac{|z\rangle}{z^{n+1}}, \quad (36)$$

and noticing that $a^{-1}|z\rangle = z^{-1}|z\rangle$ holds in the sense of the contour integration with the contour encircling $z=0$, Fan[14] obtained

$$a^{-1}|n\rangle = \frac{\sqrt{n!}}{2\pi i}\oint_c dz \frac{1}{z^{n+2}} \mathrm{e}^{za^\dagger}|0\rangle = \frac{1}{\sqrt{n+1}}|n+1\rangle, \quad (37)$$

which means

$$a^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n+1\rangle\langle n| \quad (38)$$

satisfying

$$aa^{-1}=1,\ a^{-1}a = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n+1\rangle\langle n|a = \sum_{n=0}^{\infty} |n+1\rangle\langle n+1| = 1-|0\rangle\langle 0|. \quad (39)$$

When $A=a^{-1}$, from

$$\langle m|a^{-1}|n\rangle = \langle m| \sum_{n'=0}^{\infty} \frac{1}{\sqrt{n'+1}} |n'+1\rangle\langle n'|\, n\rangle = \frac{1}{\sqrt{n+1}}\delta_{m,n+1} \quad (40)$$

and Eq. (10) we derive

$$a^{-1} = \sum_{n=0}^{\infty} \frac{2}{(n+1)!} {:}\, \mathrm{H}_{n+1,n}\left(2a^\dagger, 2a\right) \exp\left[-2a^\dagger a\right] {:}\,. \quad (41)$$

On the other hand, using Eq. (3) we have

$$a^{-1} = 2\int \frac{\mathrm{d}^2\beta}{\pi} {:}\, \langle -\beta|a^{-1}|\beta\rangle \times \exp\left[2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)\right] {:}$$
$$= 2\int \frac{\mathrm{d}^2\beta}{\pi} {:}\, \frac{1}{\beta} \times \exp\left[-2|\beta|^2 + 2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)\right] {:}\,. \quad (42)$$

Comparing Eq. (41) with Eq. (42) we obtain

$$2\int \frac{\mathrm{d}^2\beta}{\pi} {:}\, \frac{1}{\beta} \exp\left[-2|\beta|^2 + 2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)\right] {:} = \sum_{n=0}^{\infty} \frac{2}{(n+1)!} {:}\, \mathrm{H}_{n+1,n}\left(2a^\dagger, 2a\right) \exp\left[-2a^\dagger a\right] {:}\,, \quad (43)$$

which implies a new integration formula

$$\int \frac{\mathrm{d}^2\beta}{\pi} \frac{1}{\beta} \exp\left[-2|\beta|^2 + 2\left(\beta^*\lambda - \lambda^*\beta\right)\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \mathrm{H}_{n+1,n}\left(2\lambda^*, 2\lambda\right) \exp\left[-4|\lambda|^2\right]. \quad (44)$$

In summary, by virtue of the Weyl ordering method we find a new formalism of optical field operator expansion in number state representation. New expansion of miscellaneous optical fields are therefore exhibited. Some new generating functions of special polynomials are derived.

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