Stochastic bounded consensus tracking of leader-follower multi-agent systems with measurement noises based on sampled data with general sampling delay*

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In this paper we provide a unified framework for consensus tracking of leader-follower multi-agent systems with measurement noises based on sampled data with a general sampling delay. First, a stochastic bounded consensus tracking protocol based on sampled data with a general sampling delay is presented by employing the delay decomposition technique. Then, necessary and sufficient conditions are derived for guaranteeing leader-follower multi-agent systems with measurement noises and a time-varying reference state to achieve mean square bounded consensus tracking. The obtained results cover no sampling delay, a small sampling delay and a large sampling delay as three special cases. Last, simulations are provided to demonstrate the effectiveness of the theoretical results.

Keywords: leader-follower multi-agent systems, stochastic bounded consensus tracking, measurement noises, general sampling delay

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1. Introduction

During the past decade, distributed coordinated control of multi-agent systems, due to its broad applications, has received considerable attention in various fields such as physics, biology, computer science, and control engineering. As a fundamental issue in distributed coordinated control, the consensus is to design a proper control law called a consensus protocol to ensure that all agents eventually reach an agreement on certain quantities of interest. For most of the existing consensus protocols,[1–7] the final common value to be achieved is a function of initial states of all agents and is inherently a prior unknown constant. This is the so-called χ-consensus.[8]

However, in many practical applications, it is required that all agents communicating with their neighbors eventually converge to a desired time-varying reference state. This is the so-called leader-follower consensus or consensus tracking. Due to the ineffectiveness of χ-consensus protocols to directly deal with consensus tracking, many consensus tracking protocols of leader-follower multi-agent systems are proposed, and up to now a lot of attention has focused on consensus tracking of leader-follower multi-agent systems,[9–23] where it is assumed that each agent accurately measures the states of its neighbors and the leader if the agent has access to the leader. Considering the fact that real networks are often in uncertain communication environments with various source noises, channel noises, and sink noises, in Refs. [24]–[28] the consensus tracking of leader-follower multi-agent systems with measurement noises was further investigated. Huang and Manton[24] proposed the stochastic approximation type protocols with the decreasing consensus gains, and obtained the sufficient conditions guaranteeing discrete-time leader-follower multi-agent systems with measurement noises to reach the mean square consensus tracking by using the stochastic Lyapunov analysis. Employing the stochastic analysis and algebra graph theory, Ma *et al.*[25] derived sufficient conditions for guaranteeing continuous-time leader-follower multi-agent systems with measurement noises to achieve the mean square consensus tracking. Hu and Feng[26] developed a distributed tracking control scheme with distributed estimators based on a novel velocity decomposition technology for a continuous-time leader-follower multi-agent system with measurement noises, demonstrating that the closed loop tracking control system is stochastically stable in the mean square and the estimation errors converge to zero in the mean square as well. Sun and Ruan[27] and Sun *et al.*[28] investigated the consensus tracking of a continuous-time leader-follower multi-agent system with noise perturbation and delays, and obtained the satisfactory condition for almost surely achieving consensus

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tracking.

For the above work about consensus tracking of leader-follower multi-agent systems with measurement noises,[24–28] in the continuous-time multi-agent systems the continuous-time consensus tracking protocols are used, and in the discrete-time multi-agent systems the discrete-time consensus tracking protocols are adopted. However, due to the application of digital sensors and controllers, in many cases, though the system itself is a continuous process, only sampled data at discrete sampling instants are available for the synthesis of control laws. Thus, it is necessary to study the consensus tracking problems of leader-follower multi-agent systems based on sampled data. Recently, Wu *et al.*[29] considered the stochastic bounded consensus tracking problems of leader-follower multi-agent systems with measurement noises based on sampled data without sampling delay, and obtained the necessary and sufficient conditions for mean square bounded consensus tracking. Further, Wu *et al.*[30] considered the stochastic bounded consensus tracking problems of leader-follower multi-agent systems with measurement noises based on sampled data with a small sampling delay, and the obtained results cover those of Ref. [29]. However, just as was pointed out in Refs. [31] and [32], in real applications, the sampling delay induced by the signal sampling process might be large, that is to say, the sampling delay is more than a sampling period. Therefore, it is necessary to study the consensus tracking problems of leader-follower multi-agent systems with measurement noises based on sampled data with a large sampling delay.

Based on the above consideration, in this paper we investigate the stochastic bounded consensus tracking problems of leader-follower multi-agent systems with measurement noises based on sampled data with a general sampling delay, which covers no sampling delay, a small sampling delay, and a large sampling delay as its three special cases. First, the stochastic bounded consensus tracking protocol based on sampled data with a general sampling delay is presented by using the delay decomposition technique. Then, the augmented matrix method, the probability limit theory, the algebra graph theory, and some other techniques are employed to derive the necessary and sufficient conditions guaranteeing the mean square bounded consensus tracking.

2. Preliminaries and problem statement

2.1. Algebra graph theory

Let $G=(V,E,A)$ be a weighted undirected graph with a set of nodes $V=\{v_1,v_2,\ldots,v_N\}$, a set of edges $E\subseteq V\times V$, and the weighted adjacency matrix $A=[a_{ij}]$ with nonnegative adjacency elements a_{ij}. The node indexes of G belong to a finite index set $I=\{1,2,\ldots,N\}$. An edge of G is denoted by $e_{ij}=(v_i,v_j)$. The adjacency elements associated with the edges are positive, i.e., $e_{ij}\in E\Leftrightarrow a_{ij}>0$. Moreover, we assume $a_{ii}=0$ for all $i\in I$. For the undirected graph G, the adjacency matrix A is symmetric, i.e., $a_{ij}=a_{ji}$. The set of neighbors of node v_i is denoted by $N_i=\{v_j\in V:e_{ij}\in E\}$. The degree of node v_i is defined as $d_i=\sum\limits_{j\in N_i}a_{ij}$. The Laplacian matrix of G is defined as $L=D-A$, where $D=\mathrm{Diag}(d_1,d_2,\ldots,d_N)$ is the degree matrix of G with diagonal elements d_i and zero off-diagonal elements. An important fact of L is that all the row sums are zero and thus L has a right eigenvector 1_N associated with the zero eigenvalue, where 1_N denotes the N-dimensional column vector with all elements being equal to 1. A path between two distinct nodes v_i and v_j means a sequence of distinct edges of the form $(v_i,v_{k_1}),(v_{k_1},v_{k_2}),\ldots,(v_{k_l},v_j)$. A graph is called being connected if there is a path between any two distinct nodes of the graph. An important property of the Laplacian matrix, which is instrumental in the convergence analysis of consensus protocols, is that the graph G is connected if and only if $\mathrm{rank}(L)=N-1$. Thus, for a connected graph G, L has one and only one zero eigenvalue and all the other eigenvalues of L are positive.

The following notations will be used throughout this paper. 0_N denotes the N-dimensional column vectors with all zeros; I_N and $0_{N\times N}$ denote the N-dimensional identity matrix and the N-dimensional matrix with all zeros, respectively; for a given vector or matrix M, M^{T} and $\|M\|_2$ denote its transpose and 2-norm M, respectively; R denotes a set of real numbers; for a given matrix $M\in R^{N\times N}$, $\det(M)$ and $\lambda_i(M),i\in I$ denote its determinant and N eigenvalues, respectively; for a given random variable X, $E(X)$ denotes its mathematical expectation.

2.2. Problem statement

In a multi-agent system with N agents, an agent and an available information flow between two agents are considered as a node and an edge in an undirected graph, respectively. Consider the system of dynamic agents, described by

$$\dot{x}_i(t)=u_i(t),i\in I, \tag{1}$$

where $x_i(t)\in R$ and $u_i(t)\in R$ represent, respectively, the state of the i-th agent and the associated control input called the consensus protocol.

In Refs. [29] and [30], Wu *et al.* proposed the following continuous-time consensus tracking protocol

$$u_i(t)=-\alpha_i(x_i(t)-x_i^*(t))+\sum_{j\in N_i}a_{ij}(x_{ij}(t)-x_i(t)),i\in I, \tag{2}$$

where $x_i^*(t)=x^*(t)+\rho_i\eta_i(t)$ and $x_{ij}(t)=x_j(t)+\sigma_{ij}\omega_{ij}(t),j\in N_i$ denote, respectively, the state measurements that the i-th agent receives from the virtual leader and the j-th agent, $x^*(t)$ is the state of the virtual leader, $\{\eta_i(t):i\in I\}$ and $\{\omega_{ij}(t):j\in N,i\in I\}$ are two mutually independent standard

Gaussian white noises, $\rho_i > 0$ and $\sigma_{ij} > 0$ represent, respectively, the intensities of $\eta_i(t)$ and $\omega_{ij}(t)$, and α_i is positive if the i-th agent has access to the virtual leader, and 0 otherwise.

Denote the sampling period and the sampling delay by h and τ, respectively. Decomposing the sampling delay τ with respect to the sampling period h yields

$$\tau = mh + \varepsilon, \tag{3}$$

where m is a nonnegative integer and $\varepsilon \in [0, h)$. It follows from Eq. (3) that

$$t - \tau \in \begin{cases} [kh - mh - \varepsilon, kh - mh) \subset [kh - mh - h, kh - mh), & \text{if } t \in [kh, kh + \varepsilon), \\ [kh - mh, kh - mh + h - \varepsilon) \subset [kh - mh, kh - mh + h), & \text{if } t \in [kh + \varepsilon, kh + h). \end{cases} \tag{4}$$

Therefore, using the period sampling technology and zero-order hold circuit, the following consensus tracking protocol based on sampled data with the general sampling delay is induced from the continuous-time consensus tracking protocol (2):

$$u_i(t) = \begin{cases} u_i(kh - mh - h) = -\alpha_i(x_i(kh - mh - h) - x_i^*(kh - mh - h)) \\ \quad + \sum\limits_{j \in N_i} a_{ij}(x_{ij}(kh - mh - h) - x_i(kh - mh - h)), & \text{if } t \in [kh, kh + \varepsilon), \\ u_i(kh - mh) = -\alpha_i(x_i(kh - mh) - x_i^*(kh - mh)) \\ \quad + \sum\limits_{j \in N_i} a_{ij}(x_{ij}(kh - mh) - x_i(kh - mh)), & \text{if } t \in [kh + \varepsilon, kh + h). \end{cases} \tag{5}$$

Definition 1 System (1) under protocol (5) is said to achieve the mean square bounded consensus tracking, if system (1) applying protocol (5) has the following property: $\lim_{t\to\infty} E\left((x_i(t) - x^*(t))^2\right) \le C < \infty, \forall i \in I$, where C is a bounded positive constant that is independent of t.

3. Convergence analysis

In this section, employing the augmented matrix method, the probability limit theory, the algebra graph theory, and some other techniques, we derive the necessary and sufficient conditions that guarantee that system (1) applying protocol (5) reaches the mean square bounded consensus tracking. Before moving on, we need the following assumptions and lemmas.

Assumption 1 The communication network topology G composed of N agents is fixed, undirected and connected, and at least one agent has access to the virtual leader.

Assumption 2 The reference state of the virtual leader satisfies the Lipschitz continuity condition, i.e., there is a positive constant ξ^* such that $|x^*(a) - x^*(b)| \le \xi^* |a - b|, \forall a, b \in [0 + \infty)$.

Remark 1 Assumption 2 that the dynamics of the virtual leader satisfies the Lipschitz continuity condition is general, which has been used in many references. In Ref. [30], for first-order multi-agent systems, it was assumed that the position state of the virtual leader satisfies the Lipschitz continuity condition. In Ref. [18], for second-order multi-agent systems, it was assumed that the acceleration state of the virtual leader satisfies the Lipschitz continuity condition.

Lemma 1 Let M_1, M_2, M_3, $M_4 \in R^{l\times l}$, where l denotes the dimension of a matrix. If M_1, M_2, M_3, and M_4 commute pairwise, then

$$\det\left(\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}\right) = \det(M_1 M_4 - M_2 M_3).$$ [33]

Lemma 2 All eigenvalues of a matrix $M \in R^{l\times l}$ are within the unit circle, if and only if there is a matrix norm $\|\cdot\|_*$ such that $\|M\|_* < 1$.[34]

Lemma 3 All roots of the equation $s^3 + a_2 s^2 + a_1 s + a_0 = 0$, where a_0, a_1, $a_2 \in R$, are within the unit circle, if and only if the inequalities $1 + a_2 + a_1 + a_0 > 0, -1 + a_2 - a_1 + a_0 < 0$, $|a_0| < 1$ and $|a_0^2 - 1| > |a_0 a_2 - a_1|$ hold simultaneously.[35]

Lemma 4 Assume that the communication network topology G composed of N agents is fixed, undirected and connected, and that at least one agent has access to the virtual leader, then the matrix $(\Delta + L)$ must be positive-definite, where $\Delta = \mathrm{Diag}(\alpha_1, \ldots \alpha_N)$, L is the Laplacian matrix of G, and $\alpha_i, i \in I$ are defined as above.[9]

Now, we give the main results.

Theorem 1 Under Assumption 1 and Assumption 2, the multi-agent system (1), applying the consensus tracking protocol (5), achieves the mean square bounded consensus tracking, if and only if all roots of the equation

$$s^{m+2} - s^{m+1} + (h - \varepsilon)\lambda_i(\Delta + L)s + \varepsilon\lambda_i(\Delta + L) = 0, i \in I \tag{6}$$

are within the unit circle.

Proof When $t \in [kh, kh + \varepsilon)$, from Eqs. (1) and (5) we obtain

$$x_i(t) = x_i(kh) + (t - kh)u_i(kh - mh - h), \tag{7}$$

which further leads to

$$x_i(kh + \varepsilon) = x_i(kh) + \varepsilon u_i(kh - mh - h),$$

$$k=0,1,2,\ldots;i\in I. \tag{8}$$

Similarly, when $t\in[kh+\varepsilon,kh+h)$, from Eqs. (1) and (5) we obtain

$$x_i(t)=x_i(kh+\varepsilon)+(t-kh-\varepsilon)u_i(kh-mh), \tag{9}$$

which further results in

$$\begin{aligned} x_i(kh+h)&=x_i(kh+\varepsilon)+(h-\varepsilon)u_i(kh-mh),\\ &k=0,1,2,\ldots;i\in I. \end{aligned} \tag{10}$$

Combining Eqs. (8) and (10) yields the dynamics at the sampling instants

$$\begin{aligned} x_i(kh+h)&=x_i(kh)+\varepsilon u_i(kh-mh-h)+(h-\varepsilon)u_i(kh-mh),\\ &k=0,1,2,\ldots;i\in I. \end{aligned} \tag{11}$$

Notice $L1_N=0_N$, then we can rewrite the dynamics (11) in a compact form

$$\begin{aligned} x(kh+h)&=x(kh)-\varepsilon(\Delta+L)(x(kh-mh-h)\\ &\quad -x^*(kh-mh-h)1_N)+\varepsilon n(kh-mh-h)\\ &\quad -(h-\varepsilon)(\Delta+L)(x(kh-mh)-x^*(kh-mh)1_N)\\ &\quad +(h-\varepsilon)n(kh-mh),\quad k=0,1,2,\ldots, \end{aligned} \tag{12}$$

where

$$\begin{aligned} x(kh)&=(x_1(kh),x_2(kh),\ldots,x_N(kh))^{\mathrm{T}},\\ n(kh)&=(n_1(kh),n_2(kh),\ldots,n_N(kh))^{\mathrm{T}} \end{aligned}$$

and

$$n_i(kh)=\alpha_i\rho_i\eta_i(kh)+\sum_{j\in N_i}a_{ij}\sigma_{ij}\omega_{ij}(kh),i\in I.$$

Denote $\tilde{x}(kh)=x(kh)-x^*(kh)1_N$. Combining Eq. (12) and the auxiliary equations

$$\begin{cases} \tilde{x}(kh)=\tilde{x}(kh),\\ \tilde{x}(kh-h)=\tilde{x}(kh-h),\\ \ldots,\\ \tilde{x}(kh-mh)=\tilde{x}(kh-mh) \end{cases}$$

yields the dynamics about the tracking error vector $\delta(kh)\triangleq\left(\tilde{x}^{\mathrm{T}}(kh),\tilde{x}^{\mathrm{T}}(kh-h),\ldots,\tilde{x}^{\mathrm{T}}(kh-mh-h)\right)^{\mathrm{T}}$:

$$\delta(kh+h)=P\delta(kh)+\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N+\varepsilon n(kh-mh-h)+(h-\varepsilon)n(kh-mh)\\ 0_{(m+1)N}\end{pmatrix}, \tag{13}$$

where

$$P=\begin{bmatrix} I_N & 0_{N\times N} & \ldots & 0_{N\times N} & -(h-\varepsilon)(\Delta+L) & -\varepsilon(\Delta+L)\\ I_N & 0_{N\times N} & \ldots & 0_{N\times N} & 0_{N\times N} & 0_{N\times N}\\ 0_{N\times N} & I_N & 0_{N\times N} & \ldots & 0_{N\times N} & 0_{N\times N}\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0_{N\times N} & \ldots & 0_{N\times N} & I_N & 0_{N\times N} & 0_{N\times N}\\ 0_{N\times N} & 0_{N\times N} & \ldots & 0_{N\times N} & I_N & 0_{N\times N}\end{bmatrix}_{(m+2)N\times(m+2)N}.$$

From Eq. (13), we have

$$\delta(nh)=P^n\delta(0)+\sum_{k=0}^{n-1}P^{n-k-1}\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N+\varepsilon n(kh-mh-h)+(h-\varepsilon)n(kh-mh)\\ 0_{(m+1)N}\end{pmatrix}. \tag{14}$$

Denote the characteristic polynomial of P by $F(s)$. According to Lemma 1 we have

$$\begin{aligned} F(s)&=\det\left(sI_{(m+2)N}-P\right)\\ &=\det\left(\begin{bmatrix} (s-1)I_N & 0_{N\times N} & \ldots & 0_{N\times N} & (h-\varepsilon)(\Delta+L) & \varepsilon(\Delta+L)\\ -I_N & sI_N & \ldots & 0_{N\times N} & 0_{N\times N} & 0_{N\times N}\\ 0_{N\times N} & -I_N & sI_N & \ldots & 0_{N\times N} & 0_{N\times N}\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0_{N\times N} & \ldots & 0_{N\times N} & -I_N & sI_N & 0_{N\times N}\\ 0_{N\times N} & 0_{N\times N} & \ldots & 0_{N\times N} & -I_N & sI_N\end{bmatrix}_{(m+2)N\times(m+2)N}\right)\\ &=\det\left(\begin{bmatrix} (s-1)s^mI_N+(h-\varepsilon)(\Delta+L) & \varepsilon(\Delta+L)\\ -I_N & sI_N\end{bmatrix}_{2N\times 2N}\right)\\ &=\det\left((s^{m+2}-s^{m+1})I_N+((h-\varepsilon)s+\varepsilon)(\Delta+L)\right)\\ &=\prod_{i=1}^{N}\left\{s^{m+2}-s^{m+1}+((h-\varepsilon)s+\varepsilon)\lambda_i(\Delta+L)\right\} \end{aligned}$$

$$= \prod_{i=1}^{N} \left\{ s^{m+2} - s^{m+1} + (h-\varepsilon)\lambda_i(\Delta+L)s + \varepsilon\lambda_i(\Delta+L) \right\}.$$

Therefore, $\rho(P) < 1$ if and only if all roots of Eq. (6) are within the unit circle, where $\rho(P)$ represents the spectral radius of P. From Lemma 2, when all roots of Eq. (6) are within the unit circle, there is a matrix norm $\|\cdot\|_*$ such that

$$\|P\|_* < 1. \tag{15}$$

From Eq. (14), we have

$$\begin{aligned} E\left(\|\delta(nh)\|_2^2\right) \le\ & 2\|P^n\|_2^2\|\delta(0)\|_2^2 \\ & + 2E\left(\left\|\sum_{k=0}^{n-1} P^{n-k-1}\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N + \varepsilon n(kh-mh-h) + (h-\varepsilon)n(kh-mh) \\ 0_{(m+1)N} \end{pmatrix}\right\|_2^2\right) \\ \le\ & 2\|P^n\|_2^2\|\delta(0)\|_2^2 + 4\left\|\sum_{k=0}^{n-1} P^{n-k-1}\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N \\ 0_{(m+1)N} \end{pmatrix}\right\|_2^2 \\ & + 4E\left(\left\|\sum_{k=0}^{n-1} P^{n-k-1}\begin{pmatrix} \varepsilon n(kh-mh-h) + (h-\varepsilon)n(kh-mh) \\ 0_{(m+1)N} \end{pmatrix}\right\|_2^2\right). \end{aligned} \tag{16}$$

By the equivalence of matrix norms, there are constants $C_* > 0$ and $C_2 > 0$ such that

$$\frac{1}{C_2}\|\cdot\|_* \le \|\cdot\|_2 \le C_*\|\cdot\|_*.$$

Thus, we have the following estimations:

$$\|P^n\|_2^2 \le C_*^2\|P^n\|_*^2 \le C_*^2\|P\|_*^{2n}, \tag{17}$$

$$\begin{aligned} & \left\|\sum_{k=0}^{n-1} P^{n-k-1}\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N \\ 0_{(m+1)N} \end{pmatrix}\right\|_2^2 \le C_*^2\left\|\sum_{k=0}^{n-1} P^{n-k-1}\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N \\ 0_{(m+1)N} \end{pmatrix}\right\|_*^2 \\ \le\ & C_*^2\left(\sum_{k=0}^{n-1}\left\|P^{n-k-1}\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N \\ 0_{(m+1)N} \end{pmatrix}\right\|_*\right)^2 \le C_*^2\left(\sum_{k=0}^{n-1}\left\|P^{n-k-1}\right\|_*\left\|\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N \\ 0_{(m+1)N} \end{pmatrix}\right\|_*\right)^2 \\ \le\ & C_*^2C_2^2\left(\sum_{k=0}^{n-1}\left\|P^{n-k-1}\right\|_*\left\|\begin{pmatrix} -(x^*(kh+h)-x^*(kh))1_N \\ 0_{(m+1)N} \end{pmatrix}\right\|_2\right)^2 \\ \le\ & C_*^2C_2^2\xi^{*2}h^2N\left(\sum_{k=0}^{n-1}\left\|P^{n-k-1}\right\|_*\right)^2 = C_*^2C_2^2\xi^{*2}h^2N\left(\frac{1-\|P\|_*^{n-1}}{1-\|P\|_*}\right)^2 \end{aligned} \tag{18}$$

and

$$\begin{aligned} & E\left(\left\|\sum_{k=0}^{n-1} P^{n-k-1}\begin{pmatrix} \varepsilon n(kh-mh-h) + (h-\varepsilon)n(kh-mh) \\ 0_{(m+1)N} \end{pmatrix}\right\|_2^2\right) \\ \le\ & \sum_{k=0}^{n-1}\left\|P^{n-k-1}\right\|_2^2 E\left(\left\|\begin{pmatrix} \varepsilon n(kh-mh-h) + (h-\varepsilon)n(kh-mh) \\ 0_{(m+1)N} \end{pmatrix}\right\|_2^2\right) \\ \le\ & C_*^2\sum_{k=0}^{n-1}\|P\|_*^{2(n-k-1)} E\left(\left\|\begin{pmatrix} \varepsilon n(kh-mh-h) + (h-\varepsilon)n(kh-mh) \\ 0_{(m+1)N} \end{pmatrix}\right\|_2^2\right) \\ \le\ & C_*^2(\varepsilon^2+(h-\varepsilon)^2)\sum_{i\in I}(\alpha_i^2\rho_i^2 + \sum_{j\in N_i} a_{ij}^2\sigma_{ij}^2)\sum_{k=0}^{n-1}\|P\|_*^{2(n-k-1)} \\ =\ & C_*^2(\varepsilon^2+(h-\varepsilon)^2)\sum_{i\in I}(\alpha_i^2\rho_i^2 + \sum_{j\in N_i} a_{ij}^2\sigma_{ij}^2)\frac{1-\|P\|_*^{2(n-1)}}{1-\|P\|_*^2}. \end{aligned} \tag{19}$$

Combining Eqs. (15)–(19) yields

$$\begin{aligned}
&\limsup_{n\to\infty} E\left(\|\delta(nh)\|_2^2\right)\\
&\le 2C_*^2\|\delta(0)\|_2^2 \lim_{n\to\infty}\|P\|_*^{2n}\\
&\quad +4C_*^2C_2^2\xi^{*^2}h^2N\lim_{n\to\infty}\left(\frac{1-\|P\|_*^{n-1}}{1-\|P\|_*}\right)^2\\
&\quad +4C_*^2(\varepsilon^2+(h-\varepsilon)^2)\sum_{i\in I}(\alpha_i^2\rho_i^2\\
&\quad +\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2)\lim_{n\to\infty}\frac{1-\|P\|_*^{2(n-1)}}{1-\|P\|_*^2}\\
&= 4C_*^2C_2^2\xi^{*^2}h^2N\frac{1}{(1-\|P\|_*)^2}+4C_*^2(\varepsilon^2+(h-\varepsilon)^2)\sum_{i\in I}(\alpha_i^2\rho_i^2\\
&\quad +\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2)\frac{1}{1-\|P\|_*^2}, \qquad (20)
\end{aligned}$$

which further leads to

$$\begin{aligned}
&\limsup_{n\to\infty} E\left(\|x(nh)-x^*(nh)1_N\|_2^2\right)\\
&\le \frac{4C_*^2}{m+2}\left[\frac{C_2^2\xi^{*^2}h^2N}{(1-\|P\|_*)^2}\right.\\
&\quad \left.+\frac{(\varepsilon^2+(h-\varepsilon)^2)\sum_{i\in I}(\alpha_i^2\rho_i^2+\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2)}{1-\|P\|_*^2}\right]\triangleq\tilde{C}<\infty. \qquad (21)
\end{aligned}$$

It follows from inequality (21) that

$$\begin{aligned}
&\limsup_{n\to\infty} E\left(\|u(nh)\|_2^2\right)\le 2\|\Delta+L\|_2^2\\
&\quad\times\limsup_{n\to\infty}\|x(nh)-x^*(nh)1_N\|_2^2+2\sum_{i\in I}\left(\alpha_i^2\rho_i^2+\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2\right)\\
&\le 2\tilde{C}\|\Delta+L\|_2^2+2\sum_{i\in I}\left(\alpha_i^2\rho_i^2+\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2\right). \qquad (22)
\end{aligned}$$

From Eq. (7) we have that when $t\in[kh,kh+\varepsilon)$, $x(t)-x(kh)=(t-kh)u(kh-mh-h)$, which together with inequality (22) further induces

$$\begin{aligned}
&\lim_{t\to\infty}\sup_{kh\le t<kh+\varepsilon} E\left(\|x(t)-x(kh)\|_2^2\right)\\
&= 2\varepsilon^2\left[\tilde{C}\|\Delta+L\|_2^2+\sum_{i\in I}(\alpha_i^2\rho_i^2+\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2)\right]. \qquad (23)
\end{aligned}$$

Similarly, from Eqs. (8) and (10) we have that when $t\in[kh+\varepsilon,kh+h)$, $x(t)-x(kh)=\varepsilon u(kh-mh-h)+(t-kh-\varepsilon)u(kh-mh)$, which together with inequality (22) further yields

$$\begin{aligned}
&\lim_{t\to\infty}\sup_{kh+\varepsilon\le t<kh+h} E\left(\|x(t)-x(kh)\|_2^2\right)\\
&= 4[\varepsilon^2+(h-\varepsilon)^2]\\
&\quad\times\left[\tilde{C}\|\Delta+L\|_2^2+\sum_{i\in I}\left(\alpha_i^2\rho_i^2+\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2\right)\right]. \qquad (24)
\end{aligned}$$

Combining Eqs. (23) and (24) gives

$$\begin{aligned}
&\lim_{t\to\infty}\sup_{kh\le t<kh+h} E\left(\|x(t)-x(kh)\|_2^2\right)\\
&\le 4[\varepsilon^2+(h-\varepsilon)^2]\\
&\quad\times\left[\tilde{C}\|\Delta+L\|_2^2+\sum_{i\in I}(\alpha_i^2\rho_i^2+\sum_{j\in N_i}a_{ij}^2\sigma_{ij}^2)\right]\triangleq\tilde{\tilde{C}}. \qquad (25)
\end{aligned}$$

This together with inequality (21) implies that

$$\begin{aligned}
&\limsup_{t\to\infty} E(\|x(t)-x^*(t)1_N\|_2^2)\\
&= \limsup_{t\to\infty} E(\|(x(t)-x(kh))+(x(kh)-x^*(kh)1_N)\\
&\quad +(x^*(kh)-x^*(t))1_N\|_2^2)\\
&\le \limsup_{t\to\infty} E([\|(x(t)-x(kh))+(x(kh)-x^*(kh)1_N)\|_2\\
&\quad +\|(x^*(kh)-x^*(t))1_N\|_2]^2)\\
&\le \limsup_{t\to\infty} 2E(\|(x(t)-x(kh))+(x(kh)-x^*(kh)1_N)\|_2^2)\\
&\quad +2\limsup_{t\to\infty}\|(x^*(kh)-x^*(t))1_N\|_2^2\\
&\le 4\limsup_{t\to\infty} E(\|x(t)-x(kh)\|_2^2)\\
&\quad +4\limsup_{t\to\infty} E(\|x(kh)-x^*(kh)1_N\|_2^2)\\
&\quad +2\limsup_{t\to\infty}\|(x^*(kh)-x^*(t))1_N\|_2^2\\
&\le 4\tilde{\tilde{C}}+4\tilde{C}+2\xi^{*^2}h^2N\triangleq C. \qquad (26)
\end{aligned}$$

According to Definition 1, multi-agent system (1), applying the consensus tracking protocol (5), achieves the mean square bounded consensus tracking. The proof is completed.

Remark 2 Protocol (5) provides a unified form for the case of the general sampling delay, and the presentation that all roots of Eq. (6) are within the unit circle also provides a unified form for necessary and sufficient conditions guaranteeing that system (1) applying protocol (5) achieves the mean square bounded consensus tracking. However, it is hard to derive from this presentation the explicit allowable scope of the sampling period that guarantees that system (1) applying the protocol (5) achieves the mean square bounded consensus tracking. Therefore, the results of Theorem 1 are mainly used for testing the effectiveness of the sampling period guaranteeing the mean square bounded consensus tracking.

Remark 3 If $m=0$ and $\varepsilon=0$, then $\tau=0$, i.e., there is no sampling delay. In this particular case, protocol (5) with the general sampling delay degenerates into the protocol without the sampling delay given in Ref. [29]. If $m=0$ and $\varepsilon\neq 0$, then $0<\varepsilon=\tau<h$, i.e., the sampling delay is less than the sampling period. In this particular case, protocol (5) with the general sampling delay degenerates into the protocol with the small sampling delay given in Ref. [30]. If $m=1$ and $\varepsilon\neq 0$, then $h<h+\varepsilon=\tau<2h$, i.e., the sampling delay is more than the sampled period and less than twice the sampling period. In this particular case, protocol (5) with the general sampling delay becomes the consensus tracking protocol with the large sampling delay

$$u_i(t)=\begin{cases}-\alpha_i(x_i(kh-2h)-x_i^*(kh-2h))\\ +\sum\limits_{j\in N_i}a_{ij}(x_{ij}(kh-2h)\\ -x_i(kh-2h)),t\in[kh,kh+\tau-h),\\ -\alpha_i(x_i(kh-h)-x_i^*(kh-h))\\ +\sum\limits_{j\in N_i}a_{ij}(x_{ij}(kh-h)\\ -x_i(kh-h)),t\in[kh+\tau-h,kh+h),\\ k=0,1,2,\ldots;\ i\in I,\end{cases}\tag{27}$$

and we have the following corollary.

Corollary 1 Under Assumption 1 and Assumption 2, multi-agent system (1), applying the consensus tracking protocol (27), achieves the mean square bounded consensus tracking, if and only if

$$h<\frac{1}{\max_{i\in I}\{\lambda_i(\Delta+L)\}}\tag{28}$$

and

$$h<\tau<h+\min\left\{h,\frac{\sqrt{1-h\max_{i\in I}\{\lambda_i(\Delta+L)\}}}{\max_{i\in I}\{\lambda_i(\Delta+L)\}}\right\}\tag{29}$$

hold simultaneously.

Proof When $h<h+\varepsilon=\tau<2h$, Eq. (6) degenerates into

$$s^3-s^2+(2h-\tau)\lambda_i(\Delta+L)s+(\tau-h)\lambda_i(\Delta+L)=0,\quad i\in I.\tag{30}$$

From Lemma 3, Eq. (30) has all roots within the unit circle if and only if

$$\begin{cases}h\lambda_i(\Delta+L)>0,\\ 2(\tau-h)\lambda_i(\Delta+L)<2+h\lambda_i(\Delta+L),\\ (\tau-h)\lambda_i(\Delta+L)<1,\\ \left|(\tau-h)^2\lambda_i^2(\Delta+L)-1\right|>h\lambda_i(\Delta+L),\quad i\in I.\end{cases}\tag{31}$$

Under Assumption 1, further simplifying Eq. (30) shows that inequality (31) holds if and only if inequalities (28) and (29) hold simultaneously, where we have used Lemma 4. The proof is completed.

4. Simulations

In this section, numerical simulations are provided to illustrate the effectiveness of the above theoretical results. Consider a multi-agent system composed of four agents and one virtual leader labeled as agent 0 with the network topology shown in Fig. 1.

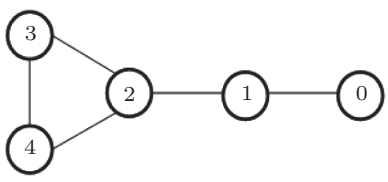


Fig. 1. Network topology composed of four agents and a virtual leader.

Without loss of generality, all weights of edges are assumed to be 1, the intensities of measurement noises are equal to 1, the initial states of four agents are randomly chosen, and the state of the virtual leader is $x^*(t)=\sqrt{1+t}$. We illustrate the effectiveness of Theorem 1 by verifying the effectiveness of Corollary 1. It follows from inequality (28) that the maximal allowable upper bound of the sampling period guaranteeing that system (1) applying the protocol (27) achieves the mean square bounded consensus tracking is 0.243. From inequality (29) we have that for $h=0.2$, the maximal allowable scope of the sampling delay guaranteeing that system (1) applying the protocol (27) achieves the mean square bounded consensus tracking is (0.2, 0.3022). The numerical results are shown in Figs. 2 and 3, respectively. It can be seen from Fig. 2 that when $(h,\tau)=(0.2,0.25)$, which simultaneously satisfies inequalities (28) and (29), multi-agent system (1), applying the consensus tracking protocol (27), achieves the mean square bounded consensus tracking. From Fig. 3 we find that when $(h,\tau)=(0.3,0.25)$, which does not simultaneously satisfy inequalities (28) and (29), multi-agent system (1), applying the consensus tracking protocol (27), cannot achieve the mean square bounded consensus tracking. Thus, the results of Corollary 1 are numerically verified, which further verifies the results of Theorem 1.

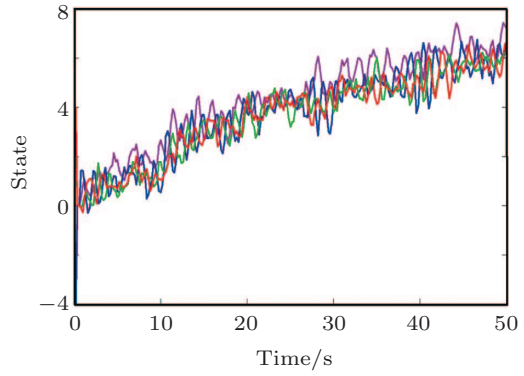


Fig. 2. (color online) States of system (1) using protocol (27) with $h=0.2$ and $\tau=0.25$.

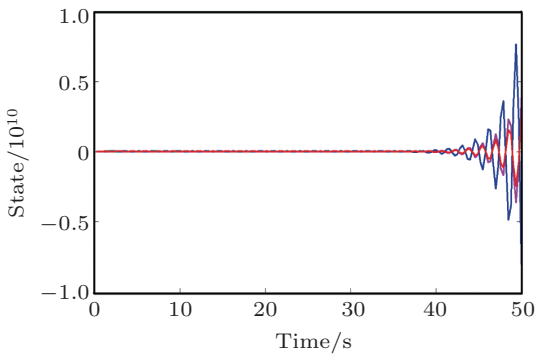


Fig. 3. (color online) States of system (1) using protocol (27) with $h=0.3$ and $\tau=0.25$.

5. Conclusions

In this paper, we investigate the stochastic bounded consensus tracking problems of leader-follower multi-agent systems with measurement noises and the virtual leader with the time-varying reference state based on sampled data with the general sampling delay. Using the delay decomposition technique, we derive the consensus tracking protocol based on sampled data with the general sampling delay. Applying the augmented matrix method, the probability limit theory, the algebra graph theory, and some other techniques, we obtain the necessary and sufficient conditions guaranteeing the mean square bounded consensus tracking. One of the future research directions is to consider the case of time-varying sampling delay.

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