Cubature Kalman filters: Derivation and extension

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This paper focuses on the cubature Kalman filters (CKFs) for the nonlinear dynamic systems with additive process and measurement noise. As is well known, the heart of the CKF is the third-degree spherical–radial cubature rule which makes it possible to compute the integrals encountered in nonlinear filtering problems. However, the rule not only requires computing the integration over an n-dimensional spherical region, but also combines the spherical cubature rule with the radial rule, thereby making it difficult to construct higher-degree CKFs. Moreover, the cubature formula used to construct the CKF has some drawbacks in computation. To address these issues, we present a more general class of the CKFs, which completely abandons the spherical–radial cubature rule. It can be shown that the conventional CKF is a special case of the proposed algorithm. The paper also includes a fifth-degree extension of the CKF. Two target tracking problems are used to verify the proposed algorithm. The results of both experiments demonstrate that the higher-degree CKF outperforms the conventional nonlinear filters in terms of accuracy.

Keywords: nonlinear filtering, cubature Kalman filters, cubature rules, state estimation, fully symmetric points

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1. Introduction

The Kalman filter (KF)[1] is an efficient recursive filter that estimates the state of a linear dynamic system from a series of noisy measurements. It has been used in a wide range of engineering and econometric applications from target tracking and fault diagnosis to the estimation of structural macroeconomic models[2,3] and it is an important topic in modern control theory and control system engineering. However, most physical systems are inherently nonlinear in nature, the nonlinearity can seriously degrade the KF performance, and even provoke the filter divergence. As a result, some attention has been shifted from the linear filtering as a target for new areas, with concerns about the nonlinear filters.

During the past few decades, huge efforts have been made to develop efficient filters for nonlinear and non-Gaussian systems. The Bayesian probabilistic inference provides an optimal solution framework for dynamic state estimation problems, but the solution requires propagating the full density function (PDF), which makes researchers give up obtaining the optimal filters analytically.[4–6] As a consequence, some approximations, in the Bayesian framework, have been adopted to develop the suboptimal numerical filtering techniques.

When confronting nonlinear filtering problems, the first commonly used approach is to linearize. The extended Kalman filter (EKF),[7,8] which has been used as the state-of-the-art filter in many engineering areas, typically applies the KF to nonlinear dynamic systems by simply linearizing all the nonlinear models, thus trying to avoid the nonlinear aspects of such problems. The EKF can give particularly poor performance if the dynamic systems are highly nonlinear.

As better alternatives to the EKF, a large number of nonlinear filters based on the idea of Bayesian sampling provide a wide range of choices. Such filters can be roughly classified into two categories by different sampling strategies–deterministic sampling and random sampling. The former includes the unscented Kalman filter (UKF),[9,10] the Gauss–Hermite filter (GHF), the central difference filter (CDF),[4] etc. This type of filter utilizes the Gaussian assumption to approximate the PDF. A well-known filter using random sampling is the particle filter (PF),[11,12] which also includes the Gaussian particle filter (GPF),[13] the quasi-Gaussian particle filter (QGPF),[6] etc. These filters approach PDF by using a certain number of particles. The heavily computational load and the curse of the dimensionality problem put constraints to these filtering algorithms in practical applications. Some other algorithms, such as the H_2/H_∞ filtering[14] and the robust filtering techniques[15] and their variations[16] also draw some attention.

Based on the third-degree spherical–radial cubature rule, Arasaratnam and Haykin, *et al.* formulated and elaborated the cubature Kalman filters (CKF) in 2009.[17,18] The CKF employs deterministic sampling to evaluate the intractable integrals encountered in filtering problems. It achieves great success in solving nonlinear state estimation problems,[19–21] since its inception. The CKF receives increasing attention be-

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cause it is numerically more stable and more accurate than the UKF,[17] but some drawbacks of the spherical–radial cubature rule in computation cannot be ignored.

Additionally, the third-degree-based CKF may be not sufficiently accurate for many filtering problems with large uncertainties. The improvement in error performance of the higher-degree cubature rules over that of the third-degree rule and is well documented,[22–26] which provides solutions for improving the CKF performance. Unfortunately, the third-degree spherical–radial cubature rule is inconvenient to deduce higher-degree filters due to its complicated structure. Therefore, an easier derivation of a general class of the CKFs is presented in this paper.

The rest of the paper is organized as follows. In Section 2, the derivation of the CKFs, which completely abandons the spherical–radial cubature rule, is presented. The fifth-degree extension of the filter is outlined in Section 3. Two experiments are carried out in Section 4 to test and validate the proposed algorithms. Conclusions of the paper are drawn in Section 5.

2. Derivation of the CKF

2.1. Nonlinear filtering problems

Consider the nonlinear filtering problems with additive process and measurement noise, whose state-space model can be expressed by the pair of difference equations in discrete-time[17]

$$\boldsymbol{x}_k = \boldsymbol{f}(\boldsymbol{x}_{k-1}, \boldsymbol{u}_{k-1}) + \boldsymbol{v}_{k-1}, \tag{1}$$

$$\boldsymbol{z}_k = \boldsymbol{h}(\boldsymbol{x}_k, \boldsymbol{u}_k) + \boldsymbol{w}_k. \tag{2}$$

Equations (1) and (2) are the process equation and the measurement equation respectively, in which $\boldsymbol{x}_k \in \mathbb{R}^{n_{\boldsymbol{x}}}$ is the state at time k, $\boldsymbol{u}_k \in \mathbb{R}^{n_{\boldsymbol{u}}}$ is the control input, $\boldsymbol{f}$ and $\boldsymbol{h}$ are some nonlinear functions; $\boldsymbol{z}_k \in \mathbb{R}^{n_{\boldsymbol{z}}}$ is the measurement; $\boldsymbol{v}_{k-1}$ and $\boldsymbol{w}_k$ are white noise with zero mean and covariance $\boldsymbol{Q}_{k-1}$ and $\boldsymbol{R}_k$, respectively.

Under the Gaussian assumption in the Bayesian filtering framework, the core issue of nonlinear filtering problems is to compute the multi-dimensional weighted integrals whose integrands are all in the form of nonlinear function $\times$ Gaussian density, viz.

$$I(\boldsymbol{f}) = \int_{\boldsymbol{\Omega}} \boldsymbol{f}(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{3}$$

where $w(\boldsymbol{x}) = \exp(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})$ is the weighting function and $\boldsymbol{\Omega} \subseteq \mathbb{R}^n$ is the region of integration. Due to the difficulty in obtaining the analytical solution, research is focused on developing some numerical integration formulae.

2.2. The third-degree spherical–radial cubature rule

In the CKF, the following integral is considered:

$$\begin{aligned} I(\boldsymbol{f}) &= \int_{\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}) \exp\left(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} \\ &= \int_0^{\infty} \int_{U_n} \boldsymbol{f}(r\boldsymbol{y}) r^{n-1} \exp\left(-r^2\right) \mathrm{d}\sigma(\boldsymbol{y}) \mathrm{d}r, \end{aligned} \tag{4}$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$, $U_n = \{\boldsymbol{y} \in \mathbb{R}^n | y_1^2 + \cdots + y_n^2 = 1\}$ and $\sigma(\cdot)$ is the spherical surface measure or the area element on U_n.

In some sense, the major contribution of Ref. [17] is the third-degree spherical–radial cubature rule, which can be summarized as follows.

i) For the radial rule, it can be approximated by

$$\int_0^{\infty} f(x) x^{n-1} \exp\left(-x^2\right) \mathrm{d}x \approx w_1 f(x_1), \tag{5}$$

where w_1 is the weight, and $x_1 \in \mathbb{R}^+$.

ii) The spherical integral can be approximated by $2n$ points,

$$\int_{U_n} \boldsymbol{f}(\boldsymbol{y}) \mathrm{d}\sigma(\boldsymbol{y}) \approx w \sum_{i=1}^{2n} \boldsymbol{f}[u]_i, \tag{6}$$

where $w > 0$ is the weight corresponding to the generator $[u]$. Then,

$$\begin{aligned} I(\boldsymbol{f}) &= \int_{\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}) \exp(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\ &= \frac{\sqrt{\pi^n}}{2n} \sum_{i=1}^{2n} \boldsymbol{f}\left(\sqrt{\frac{n}{2}}[1]_i\right). \end{aligned} \tag{7}$$

As can be seen from Eq. (6), the spherical cubature rule requires computing the integration over an n-dimensional spherical region U_n. Moreover, once extending the spherical–radial cubature rule to the higher-degree ones, the orders of the spherical cubature formula (6) and the radial formula (5) should be enhanced simultaneously, resulting in complicated forms. Additionally, the integration formula (6) also has some known shortcomings, i.e., the evaluation points (cubature points) are eventually outside $\mathbb{R}^n$ for sufficiently large n, and adherence to positive weight formulae can lead to other inconvenient properties, such as evaluation points outside the domain, or complex evaluation points.[27]

2.3. A general class of CKFs

Regarding integral (3), a typical third-degree cubature formula integrating over the field $\mathbb{R}^n$ is given by[27]

$$\begin{aligned} I(\boldsymbol{f}) &= \int_{\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}) \exp\left(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} \\ &\approx W_0 \boldsymbol{f}[0] + W_1 \sum_{i=1}^{2n} \boldsymbol{f}[u], \end{aligned} \tag{8}$$

where $I(\boldsymbol{f})$ is given by Eq. (3), W_0 and W_1 are the weights corresponding to the generators $[0]$ and $[u]$. Equation (8) is a fully symmetric numerical integration formula, and it possesses the following properties:

Property 1 If $\boldsymbol{f}$ is a monomial containing any odd power of coordinate variables then $I(\boldsymbol{f}) = 0$.

Property 2 If $\boldsymbol{f}$ is a monomial comprised of only even powers of coordinate variables then $I(\boldsymbol{f})$ depends only on the exponents and not on the ordering of the variables.

Owing to the symmetry of the integration formula, the above properties are reasonably clear. So that we can easily conclude that if a fully symmetric integration formula is exact for all monomials up to any even degree $2m$, then it is also exact for all monomials of degree $2m+1$.

Remark 1 A fully symmetric integration formula is said to be of degree $2m+1$ when it is exact for all monomials of the form $x_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$, with $\sum_{i=1}^{n} d_i \leq 2m+1$ being an odd integer.

Hence, if we demand that equation (8) be exact of degree 3, it suffices to consider the values of all $\boldsymbol{f}$ in $\{1, x_1^2\}$, namely, consider the monomials $\boldsymbol{f}(\boldsymbol{x}) = 1$ and $\boldsymbol{f}(\boldsymbol{x}) = x_1^2$, then the equations to be satisfied are

$$\begin{pmatrix} 1 & 2n \\ 0 & 2u^2 \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \end{pmatrix} = \begin{pmatrix} I_0 \\ I_2 \end{pmatrix}, \tag{9}$$

where

$$I_0 = \int_{\mathbb{R}^n} \exp\left(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} = \sqrt{\pi^n}, \tag{10}$$

$$I_2 = \int_{\mathbb{R}^n} x_1^2 \exp\left(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} = \sqrt{\pi^n}/2, \tag{11}$$

here equation (11) uses the following property of the gamma function.

Property 3 In general, for non-negative values of n we have

$$\Gamma(1/2+n) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \tag{12}$$

where $n!!$ denotes the double factorial, and $n!! = 1$ when $n = 0$.

Since u, W_0, and W_1 are unknown, equation (9) is underdetermined. There are an infinite number of solutions for this equation. In particular, W_0 vanishes if and only if $u = \sqrt{nI_2/I_0} = \sqrt{n/2}$, which eventually yields $W_1 = I_0/(2n) = \sqrt{\pi^n}/(2n)$. Furthermore, recall that the noise is assumed to be zero mean, convert the integrals into the standard normal distribution, then we will have

$$\begin{aligned} I_N(\boldsymbol{f}) &= \int_{\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}) N(\boldsymbol{0}; \boldsymbol{I}) \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{\sqrt{\pi^n}} \int_{\mathbb{R}^n} \boldsymbol{f}\left(\sqrt{2}\boldsymbol{x}\right) \exp\left(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} \\ &= \sum_{i=1}^{m} \omega_i \boldsymbol{f}(\xi_i), \end{aligned} \tag{13}$$

where $\omega_i = 1/(2n)$, and $\xi_i = \sqrt{n}[1]_i$, $i = 1, \ldots, 2n$. The result conforms to the conventional CKF.

With the different generators and the corresponding weights, we can construct a general class of the third-degree CKFs with $2n$+1 evaluations, but only one with $2n$ points. In this line of thinking the conventional CKF is a special case of the proposed method.

3. A fifth-degree extension of the CKF

The importance of the third-degree cubature rules over higher-degree rules was emphasized in Refs. [17] and [28]. And the authors stated that their "experience with higher-degree rules has indicated that they yield no improvement or make the performance worse", which can be a highly subjective point of view.

Up to now, many records have indicated that higher accuracy can be obtained by using higher-degree cubature rules.[22–26] It is an apparent fact that if an integration formula is exact of degree $2m+1$, where $m > 1$, then it is also exact of degree 3, which implies that higher-degree rules can deliver more accurate outputs, or at least yield comparably average performance. Hence, research on constructing higher-degree CKFs will make sense.

The drawbacks of the spherical–radial cubature rule are underlined in the previous section. In this section, we intend to give the derivation of the fifth-degree CKF in our theoretical framework that completely abandons the spherical–radial cubature rule.

To consider integral (3) again, the following typical fifth-degree cubature formula is used:[27]

$$\begin{aligned} I(\boldsymbol{f}) \approx\ & \hat{W}_0 \boldsymbol{f}[0] + \hat{W}_1 \sum_{i=1}^{2n} \boldsymbol{f}[v] \\ & + \hat{W}_{1,1} \sum_{i=1}^{2n(n-1)} \boldsymbol{f}[v, v], \quad n > 1, \end{aligned} \tag{14}$$

where $\hat{W}_0$, $\hat{W}_1$, and $\hat{W}_{1,1}$ are weights corresponding to the generators $[0]$, $[v]$, and $[v, v]$, respectively.

According to Property 1 and Property 2, if we demand that equation (14) be exact of degree 5 then it is sufficient to consider $\boldsymbol{f}$ that consists of only even powers of variables. As for the fifth-degree rule, the even order monomials can be $\{1, x_i^2, x_i^4, x_i^2 x_j^2\}$, $i \neq j$. Knowing that the integration $I(\boldsymbol{f})$ depends only on the exponents, we consider $\boldsymbol{f}$ in the monomial set $\{1, x_1^2, x_1^4, x_1^2 x_2^2\}$, so that the equations to be satisfied are

$$\begin{pmatrix} 1 & 2n & 2n(n-1) \\ 0 & 2v^2 & 4(n-1)v^2 \\ 0 & 2v^4 & 4(n-1)v^4 \\ 0 & 0 & 4v^4 \end{pmatrix} \begin{pmatrix} \hat{W}_0 \\ \hat{W}_1 \\ \hat{W}_{1,1} \end{pmatrix} = \begin{pmatrix} I_0 \\ I_2 \\ I_4 \\ I_{2,2} \end{pmatrix}, \tag{15}$$

where

$$I_4 = \int_{\mathbb{R}^n} x_1^4 \exp\left(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} = 3\sqrt{\pi^n}/4, \tag{16}$$

$$I_{2,2} = \int_{\mathbb{R}^n} x_1^2 x_2^2 \exp\left(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} = \sqrt{\pi^n}/4. \tag{17}$$

The unique solution of Eq. (15) is

$$v = \sqrt{3/2}, \quad \hat{W}_0 = [1 - (7-n)n/18]/\sqrt{\pi^n},$$

$$\hat{W}_1 = (4-n)\sqrt{\pi^n}/18, \quad \hat{W}_{1,1} = \sqrt{\pi^n}/36. \tag{18}$$

In a way similar to that in Eq. (13), we obtain the cubature point sets and the corresponding weights as follows:

$$\xi_i = \begin{cases} [0]_i, & i = 1, \\ \left[\sqrt{3}\right]_i, & i = 2,\ldots,2n+1, \\ \left[\sqrt{3},\sqrt{3}\right]_i, & i = 2(n+1),\ldots,2n^2+1, \end{cases}$$
$$\omega_i = \begin{cases} 1-(7-n)n/18, & i = 1, \\ (4-n)/18, & i = 2,\ldots,2n+1, \\ 1/36, & i = 2(n+1),\ldots,2n^2+1. \end{cases} \tag{19}$$

Substituting Eq. (14) into Eq. (8), we can construct the fifth-degree CKF. The recursive procedure of the fifth-degree CKF is kept unchanged, it still consists of two updates, i.e., the time update and the measurement update. So the computational load depends on the number of evaluation points. Similarly, we can construct the ninth-degree CKF, the eleventh-degree CKF, etc., but they are not discussed here for the following reasons:

i) The higher the degree, the greater the amount of computation. As mentioned in Ref. [27], a general construction for fully symmetric formulas of degree $2m+1$ requires at least $(2n)^m[1+O(1/n)]/m!$ evaluation points, where $n \to \infty$. It is clear that the minimum number of evaluation points will increase rapidly as n increases for any fully symmetric integration formula with a degree higher than five, which implies the expensive computational cost. For instance, consider the case where $n = 20$, the seventh-degree CKF adopts a lower bound of evaluation points of about 1.07×10^4 and the ninth-degree CKF needs at least 1.07×10^5 evaluations, by contrast, the fifth-degree CKF requires only 801 points. So the cubature rules higher than the fifth-degree are not commonly used in engineering;

ii) Moreover, most of the fully symmetric cubature formulae higher than the fifth degree are underdetermined, it is difficult to determine the optimal solution.

To sum up, we claim that the fifth-degree CKF is the balance between filtering accuracy and computational cost. It is uniquely determined and more accurate than the conventional CKF. As a compromise, the fifth-degree CKF adopts $2n^2+1$ evaluation points, whereas the third-degree CKF takes no more than $2n+1$ evaluations.

4. Simulations

In this section, the algorithms are validated via two target tracking problems, including the bearing-only tracking (BOT) that has been extensively investigated in Refs. [5], [6], [29], and [30], and the target tracking problem discussed in Refs. [17] and [31].

4.1. Bearing-only tracking

The bearings-only model is well motivated and arises in defense and engineering applications. The bearings only tracking problem considers tracking a moving target within a two-dimensional plane, of which the standard second-order model is formulated as follows:

$$\boldsymbol{x}_k = \boldsymbol{\phi}\boldsymbol{x}_{k-1} + \boldsymbol{\Gamma}v_{k-1}, \quad k = 1,\ldots,N, \tag{20}$$

where $\boldsymbol{x}_k = (x,\dot{x},y,\dot{y})_k^{\mathrm{T}}$, $\boldsymbol{v}_{k-1} = (v_x,v_y)_{k-1}^{\mathrm{T}}$, and

$$\boldsymbol{\phi} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{\Gamma} = \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \\ 0 & 0.5 \\ 0 & 1 \end{bmatrix}.$$

Here, x and y denote Cartesian coordinates of the moving target, $\dot{x}$ and $\dot{y}$ represent the velocities in the x and y directions, respectively. The system noise is assumed to be of Gaussian distribution with $\boldsymbol{v}_{k-1} \sim N(0, Q\boldsymbol{I}_2)$, where $\boldsymbol{I}_2$ is the 2×2 identity matrix. A fixed observer at the origin of the plane takes noise measurements of the target bearing

$$z_k = \arctan(y_k/x_k) + w_k, \tag{21}$$

where the measurement noise is $w_k \sim N(0,R)$. The reference data are generated using $Q = 0.0005^2$, $R = 0.001^2$, and $N = 35$. The initial state vector of the dynamic system is $\boldsymbol{x}_0 = (-0.05, 0.001, 0.7, -0.055)^{\mathrm{T}}$ and the initial estimate is $\hat{\boldsymbol{x}}_{0|0} = \boldsymbol{x}_0$ with covariance $\boldsymbol{P}_{0|0} = \mathrm{diag}\{0.1^2, 0.005^2, 0.1^2, 0.01^2\}$. For a fair comparison, we carry out 200 independent Monte Carlo runs. The tracking trajectories of all the filters, including the EKF, the UKF, the conventional CKF, and the fifth-degree CKF are shown in Fig. 1.

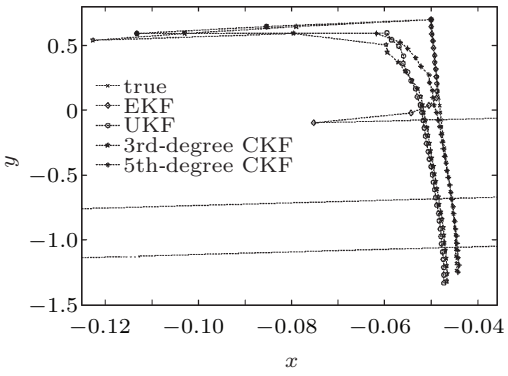


Fig. 1. Tracking trajectories of the moving target.

Figure 1 indicates that the fifth-degree CKF has better performance than its counterpart. And its trajectory is the closest to the actual one. In this simulation, even if large errors are introduced into the initial covariance $\boldsymbol{P}_{0|0}$, the filters perform satisfactorily after $k \geq 5$ except the EKF. Furthermore, we define the root-mean square error (RMSE) averaged across all time instances as

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{k=0}^{N}\left(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k}\right)^2}, \tag{22}$$

where $\hat{\boldsymbol{x}}_{k|k}$ is the estimate of $\boldsymbol{x}_k$ at the n-th Monte Carlo run. The RMSE is used to quantitatively evaluate the filtering performances of the four nonlinear filtering algorithms. RMSEs of the four algorithms are presented in Fig. 2.

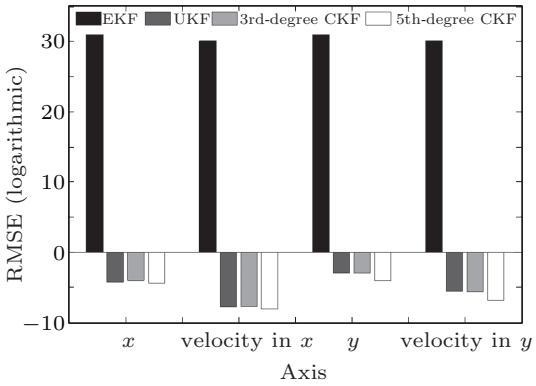


Fig. 2. Averaged RMSEs of all four coordinates.

Figure 2 further points out that the fifth-degree CKF has the smallest errors in all four coordinates. In contrast, the EKF has terrible errors, which make it impossible to track any target. And the CKF has comparable performance to the UKF.

Define the RMSE in position as

$$\begin{aligned}&\mathrm{RMSE}_{\mathrm{pos}}(k)\\&=\sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\left(\boldsymbol{x}_n-\hat{\boldsymbol{x}}_{n|n}\right)^2+\left(\boldsymbol{y}_n-\hat{\boldsymbol{y}}_{n|n}\right)^2\right)},\end{aligned}\tag{23}$$

where $(\boldsymbol{x}_n,\boldsymbol{y}_n)$ and $\left(\hat{\boldsymbol{x}}_{n|n},\hat{\boldsymbol{y}}_{n|n}\right)$ are the actual and the estimated positions at the n-th Monte Carlo run, respectively. In a way similar to that for the RMSE in position, we can easily write the RMSE in velocity.

The RMSEs in position and velocity are shown in Fig. 3, which clearly exhibits the best performance of the fifth-degree CKF in terms of filtering accuracy. The figure also reports that the CKF performs slightly better than the UKF. Moreover, the curves of the $\mathrm{RMSE}_{\mathrm{pos}}$ and the $\mathrm{RMSE}_{\mathrm{vel}}$ of the fifth-degree are smoother than those of the other three algorithms, especially in the time interval $k\in[0,10]$.

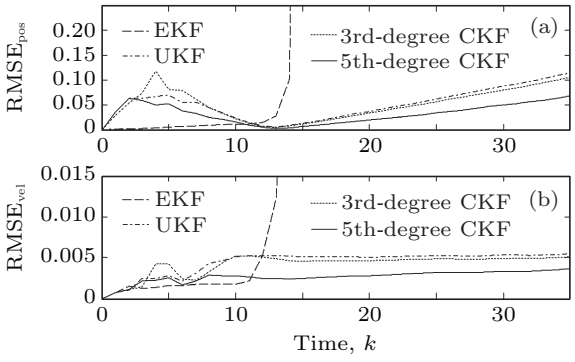


Fig. 3. (a) RMSEs in position; (b) RMSEs in velocity.

4.2. Target tracking

This target tracking scenario includes measurements of the range and bearing. The kinematics of an aircraft turning motion can be modeled by[15,25]

$$\boldsymbol{x}_k=\boldsymbol{\phi}\boldsymbol{x}_{k-1}+\boldsymbol{v}_{k-1},\quad k=1,\ldots,N,\tag{24}$$

where $\boldsymbol{x}_k=(x,\dot{x},y,\dot{y},\Omega)_k^{\mathrm{T}}$, $\boldsymbol{v}_{k-1}\sim N(\boldsymbol{0},\boldsymbol{Q})$ is the white noise with zero mean and a nonsingular covariance $\boldsymbol{Q}=\mathrm{diag}\{q_1M,q_1M,q_2T\}$, where

$$\boldsymbol{\phi}=\begin{bmatrix}1&\dfrac{\sin\Omega T}{\Omega}&0&-\dfrac{1-\cos\Omega T}{\Omega}&0\\0&\cos\Omega T&0&-\sin\Omega T&0\\0&\dfrac{1-\cos\Omega T}{\Omega}&1&\dfrac{\sin\Omega T}{\Omega}&0\\0&\sin\Omega T&0&\cos\Omega T&0\\0&0&0&0&1\end{bmatrix},$$

$$M=\begin{bmatrix}\dfrac{T^3}{3}&\dfrac{T^2}{2}\\\dfrac{T^2}{2}&T\end{bmatrix}.$$

Here Ω is the turn rate, T is the time-interval between two consecutive measurements. A radar is fixed at the origin of the plane to measure the range r and bearing θ

$$\begin{pmatrix}r_k\\\theta_k\end{pmatrix}=\begin{pmatrix}\sqrt{x_k^2+y_k^2}\\\arctan(y_k/x_k)\end{pmatrix}+\boldsymbol{w}_k,\tag{25}$$

where the measurement noise $\boldsymbol{w}_k\sim N(\boldsymbol{0},\boldsymbol{R})$ with zero mean and covariance $\boldsymbol{R}=\mathrm{diag}\{\sigma_r^2,\sigma_\theta^2\}$. The reference data are $T=1$ s, $\Omega=-3^\circ$ s^{-1}, $q_1=0.1$ m^2/s^3, $\sigma_r=10$ m, $q_2=1.75\times10^{-4}$/s^3, $N=100$ s, and $\sigma_\theta=\sqrt{10}$ mrad. The initial state is $\boldsymbol{x}_0=(1000\text{ m},300\text{ m/s},1000\text{ m},0\text{ m/s},-3^\circ/\text{s})^{\mathrm{T}}$ and the initial estimate is $\hat{\boldsymbol{x}}_{0|0}=\boldsymbol{x}_0$ with covariance $\boldsymbol{P}_{0|0}=\mathrm{diag}\{100\text{ m}^2,10\text{ m}^2/\text{s}^2,100\text{ m}^2,10\text{ m}^2/\text{s}^2,100\text{ mrad}^2/\text{s}^2\}$.

For a fair comparison, we make 200 independent Monte Carlo runs. Corresponding to Eq. (23), we may easily write formulae for RMSE of the turn rate. The RMSEs in position, velocity, and turn rate are shown in Figs. 4(a), 4(b), and 4(c), respectively. There is not much difference between filters in the interval 0 s–50 s. Hence, only the RMSEs in the time interval of 50 s–100 s are presented. Owing to the fact that the EKF is sensitive to information available in the previous time step and the UKF was often found to halt its operation, we do not present the results of the EKF and the UKF here.

Figure 4(a) indicates that the fifth-degree CKF outputs have more accurate position information than the third-degree CKF. And figure 4(b) illustrates that the difference in velocity between filters is smaller than that in position. The fifth-degree CKF also marginally outperforms the third-degree CKF in turn rate as shown in Fig. 4(c).

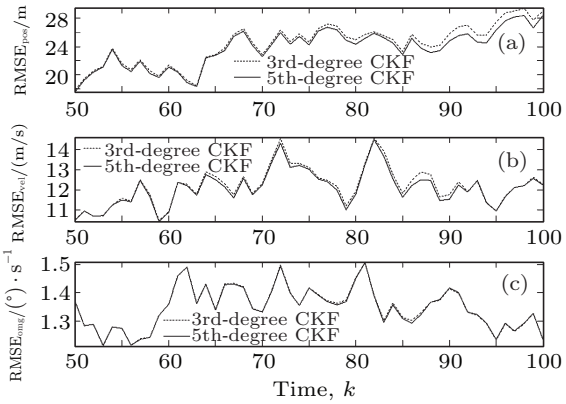


Fig. 4. RMSEs in time interval of 50 s–100 s: (a) RMSEs in position; (b) RMSEs in velocity; (c) RMSEs in turn rate.

Remark 2 Both of the simulations demonstrate that the fifth-degree CKF outperforms the conventional CKF, not "yield no improvement or make the performance worse".

5. Conclusions

The paper focuses on the derivation and extension of the CKF. Since the conventional third-degree spherical–radial cubature rule is restricted within a complicated integration region, an easier derivation of the CKF, which completely abandons the third-degree spherical–radial cubature rule, is presented. It clearly shows that the conventional CKF is a special case of the proposed method.

Owing to the fact that the CKF can benefit from higher-degree cubature rules, one may easily think of constructing higher-degree CKFs. However, when the extending of the spherical–radial cubature rule to higher-degree ones is considered, the orders of the spherical cubature formula and the radial formula should be enhanced simultaneously, resulting in complicated forms, so the fifth-degree extension of the CKF which abandons the spherical–radial cubature rule is also included.

Two target tracking problems are used to validate the proposed method. The results of both experiments demonstrate the improved performance of the fifth-degree CKF over the conventional nonlinear filters.

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