The Korteweg-de Vires equation for the bidirectional pedestrian flow model considering the next-nearest-neighbor effect*

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This paper focuses on a two-dimensional bidirectional pedestrian flow model which involves the next-nearest-neighbor effect. The stability condition and the Korteweg-de Vries (KdV) equation are derived to describe the density wave of pedestrian congestion by linear stability and nonlinear analysis. Through theoretical analysis, the soliton solution is obtained.

Keywords: bidirectional pedestrian flow, lattice hydrodynamic model, KdV equation

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1. Introduction

For decades of years, vehicular traffic problems have been studied and many typical models have been proposed to describe the traffic conditions, such as the hydrodynamic model, micro-simulation model, lattice gas model, macroscopic continuum model, car-following model, and artificial intelligence model,[1–9] to solve problems as serious traffic congestion and effectively in order to avoid risks of accidents, and predicate the quantity of traffic flow. Based on the studies, scholars have found that the pedestrian traffic is similar to traffic flow. Those traffic flow models have been used to describe the pedestrian flow and many interesting phenomena have been found out, such as self-organization, noise-induced ordering, and collective phenomena in panic situations.[10,11] Pedestrian movement is an important factor in the analysis and design of channels, traffic intersections, bridges, markets, and other public buildings.

It is necessary to know the characteristics, especially the pedestrian flow rate for rush hour and panic escape in order to avoid the unexpected accident. Therefore, pedestrian dynamics have attracted considerable attention of scientists and engineers. Many pedestrian traffic models have been proposed. In 1999, Nagatani[12] has brought out the one-dimensional lattice hydrodynamic model, which incorporates the idea of the microscopic optimal velocity. Then, the author extended it to two-dimensional model such as BML model[13] in two-lane traffic flow,[14] to describe two-lane traffic flow. Considering the nearest-neighbor sites effect, Xue Yu proposed a new one-dimensional lattice model[15] in traffic flow. Based on it, Tian got a bidirectional model[16] and Xue got a path-changing bidirectional model[17] in pedestrian flow. Then Wen applied the idea of the next-nearest-neighbor effect and got a new two-dimensional bidirectional lattice model in pedestrian flow.[18] Besides, the author got a modified Korteweg-de Vries (mKdV) equation to describe traffic jamming in the form of kink–antikink density wave from the lattice hydrodynamic model. However, the KdV equation is seldom derived.

In this paper, the next-nearest-neighbor effect with two-dimensional bidirectional lattice hydrodynamic models is introduced in Section 2. Then, the linear stability theory is used to obtain the neutral stability condition in Section 3, the KdV equation around the neutral stability line is conducted in Section 4. Some conclusions are given in the last section.

2. Model and method

In two-dimensional bidirectional pedestrian flow, four types of pedestrians are considered, i.e., the eastbound pedestrian (that moves only to the positive x direction), the westbound pedestrian (that moves only to the negative x direction), the northbound pedestrian (that moves only to the positive y direction), and the southbound pedestrian (that moves only to the negative y direction).

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The fraction of the eastbound and westbound pedestrians in all of the pedestrians is c, the fraction of the eastbound pedestrian in the eastbound and westbound pedestrians is c_1, and the fraction of the northbound pedestrian in the northbound and southbound pedestrians is c_2.

The density of pedestrians at site (j,m) at time t are denoted as $\rho_{x+}(j,m,t)$, $\rho_{x-}(j,m,t)$, $\rho_{y+}(j,m,t)$, and $\rho_{y-}(j,m,t)$, respectively. The flux of positive and negative directional pedestrians at site (j,m) at time t are denoted as $Q_{x+}(j,m,t)$, $Q_{x-}(j,m,t)$, $Q_{y+}(j,m,t)$, and $Q_{y-}(j,m,t)$, respectively. The total average density is ρ_0. The continuity equations of the two-dimensional bidirectional pedestrian flow are given by the following equations

$$\partial_t\rho_{x+}(j,m,t)+cc_1\rho_0[Q_{x+}(j,m,t) - Q_{x+}(j-1,m,t)]=0, \tag{1}$$

$$\partial_t\rho_{x-}(j,m,t)+c(1-c_1)\rho_0[Q_{x-}(j,m,t) - Q_{x-}(j+1,m,t)]=0, \tag{2}$$

$$\partial_t\rho_{y+}(j,m,t)+(1-c)c_2\rho_0[Q_{y+}(j,m,t) - Q_{y+}(j,m-1,t)]=0, \tag{3}$$

$$\partial_t\rho_{y-}(j,m,t)+(1-c)(1-c_2)\rho_0[Q_{y-}(j,m,t) - Q_{y-}(j,m+1,t)]=0. \tag{4}$$

According to Ref. [14] the flux along each direction is determined by the total optimal current with delay time τ. The flux at site (j,m) is determined by the total optimal current flux at site $(j+1,m)$ and site $(j+2,m)$ to the northbound pedestrian. Thus, the evolution equations are extended as follows:[18]

$$Q_{x+}(j,m,t+\tau)=cc_1\rho_0V(\rho(j+1,m,t)), \tag{5}$$

$$Q_{x-}(j,m,t+\tau)=c(1-c_1)\rho_0V(\rho(j-1,m,t)), \tag{6}$$

$$Q_{y+}(j,m,t+\tau)=(1-c)c_2\rho_0V(\rho(j,m+1,t)), \tag{7}$$

$$Q_{y-}(j,m,t+\tau)=(1-c)(1-c_2)\rho_0V(\rho(j,m-1,t)), \tag{8}$$

where τ denotes the delay time, which is the inverse of the sensitivity a. The optimal velocity function $V(\rho(j,m,t))$ is defined as follows:

$$V(\rho(j,m,t)) = \tanh\left(\frac{2}{\rho_0}-\frac{\rho(j,m,t)}{\rho_0^2}-\frac{1}{\rho_c}\right) + \tanh\left(\frac{1}{\rho_c}\right), \tag{9}$$

where ρ_c is the inverse of the safety distance.

According to Ref. [18] when we consider the influences of the next-nearest-neighbor site, the evolution equations (5)–(8) should be rewritten. The flux at site (j,m) is determined not only by its neighbor sites at time t, but also the neighbor sites and the next-nearest-neighbor sites at the time $t-\tau$. Thus, we can rewrite the new equations as follows:

$$Q_{x+}(j,m,t+\tau) = cc_1[(1-\gamma)\rho_0V(\rho(j+1,m,t)) + \gamma\rho_0V(\rho(j+2,m,t))], \tag{10}$$

$$Q_{x-}(j,m,t+\tau) = c(1-c_1)[(1-\gamma)\rho_0V(\rho(j-1,m,t)) + \gamma\rho_0V(\rho(j-2,m,t))], \tag{11}$$

$$Q_{y+}(j,m,t+\tau) = (1-c)c_2[(1-\gamma)\rho_0V(\rho(j,m+1,t)) + \gamma\rho_0V(\rho(j,m+2,t))], \tag{12}$$

$$Q_{y-}(j,m,t+\tau) = (1-c)(1-c_2)[(1-\gamma)\rho_0V(\rho(j,m-1,t)) + \gamma\rho_0V(\rho(j,m-2,t))], \tag{13}$$

where the coefficient γ represents the magnitude of the effect between the site (j,m) and both of the nearest site and the next-nearest site. And $0<\gamma\le 0.5$, which means if γ is bigger than 0.5, we focus more concern on the next-nearest site than the nearest one, which is not true for the real pedestrian flow. By inserting Eqs. (1)–(4) into Eqs. (10)–(13) respectively and applying $\rho(j,m,t)=\rho_{x+}(j,m,t)+\rho_{x-}(j,m,t)+\rho_{y+}(j,m,t)+\rho_{y-}(j,m,t)$, we get the total density equation:

$$\begin{aligned}
&\rho(j,m,t+2\tau)-\rho(j,m,t+\tau)+\tau\rho_0^2(cc_1)^2\\
&\quad\times[V(\rho(j+1,m,t))-V(\rho(j,m,t))]\\
&\quad+\gamma\tau\rho_0^2(cc_1)^2[V(\rho(j+2,m,t))-2V(\rho(j+1,m,t))\\
&\quad+V(\rho(j,m,t))]+\tau\rho_0^2[c(1-c_1)]^2[V(\rho(j-1,m,t))\\
&\quad-V(\rho(j,m,t))]+\gamma\tau\rho_0^2[c(1-c_1)]^2[V(\rho(j-2,m,t))\\
&\quad-2V(\rho(j-1,m,t))+V(\rho(j,m,t))]\\
&\quad+\tau\rho_0^2[(1-c)c_2]^2[V(\rho(j,m+1,t))-V(\rho(j,m,t))]\\
&\quad+\gamma\tau\rho_0^2[(1-c)c_2]^2[V(\rho(j,m+2,t))\\
&\quad-2V(\rho(j,m+1,t))+V(\rho(j,m,t))]\\
&\quad+\tau\rho_0^2[(1-c)(1-c_2)]^2[V(\rho(j,m-1,t))\\
&\quad-V(\rho(j,m,t))]+\gamma\tau\rho_0^2[(1-c)(1-c_2)]^2\\
&\quad\times[V(\rho(j,m-2,t))-2V(\rho(j,m-1,t))\\
&\quad+V(\rho(j,m,t))]=0.
\end{aligned} \tag{14}$$

As $c=0$, $c_2=1$, $\gamma=0$ or $c=1$, $c_1=1$, $\gamma=0$, equation (14) reduces to the density equation of the one-dimensional lattice hydrodynamic equation for traffic flow.[12] In the limit $c_1=1$, $c_2=1$, $\gamma=0$, equation (14) reduces to the density equation of the two-dimensional lattice hydrodynamic equation.[13] When $\gamma=0$, equation (14) is the density equation of the two-dimensional bidirectional lattice hydrodynamic equation.[17]

3. Linear stability analysis

We use the linear stability method to the model described by Eq. (14). It is supposed that the steady state is a uniform pedestrian flow with a constant density ρ_0 and constant $V(\rho_0)$

along any directions. Obviously the steady-state solution for Eq. (14) is given by

$$\begin{aligned}&\rho(j,m,t)=\rho_0,\\&v_{x+}(j,m,t)=v_{x-}(j,m,t)=v_{y+}(j,m,t)\\&\qquad=v_{y-}(j,m,t)=V(\rho_0).\end{aligned}\tag{15}$$

Suppose $y(j,m,t)$ to be a small deviation from the steady-state solution:

$$\rho(j,m,t)=\rho_0+y(j,m,t).\tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (14), we get the following linearized equation

$$\begin{aligned}&y(j,m,t+2\tau)-y(j,m,t+\tau)+\tau\rho_0^2(cc_1)^2V'(\rho_0)\\&\quad\times[y(j+1,m,t)-y(j,m,t)]+\gamma\tau\rho_0^2(cc_1)^2V'(\rho_0)\\&\quad\times[y(j+2,m,t)-2y(j+1,m,t)+y(j,m,t)]\\&\quad+\tau\rho_0^2[c(1-c_1)]^2V'(\rho_0)[y(j-1,m,t)-y(j,m,t)]\\&\quad+\gamma\tau\rho_0^2[c(1-c_1)]^2V'(\rho_0)[y(j-2,m,t)-2y(j-1,m,t)\\&\quad+y(j,m,t)]+\tau\rho_0^2[(1-c)c_2]^2V'(\rho_0)[y(j,m+1,t)\\&\quad-y(j,m,t)]+\gamma\tau\rho_0^2[(1-c)c_2]^2V'(\rho_0)[y(j,m+2,t)\\&\quad-2y(j,m+1,t))+y(j,m,t)]+\tau\rho_0^2[(1-c)(1-c_2)]^2\\&\quad\times V'(\rho_0)[y(j,m-1,t)-y(j,m,t)]\\&\quad+\gamma\tau\rho_0^2[(1-c)(1-c_2)]^2V'(\rho_0)[y(j,m-2,t)\\&\quad-2y(j,m-1,t)+y(j,m,t)]=0,\end{aligned}\tag{17}$$

where

$$V'(\rho_0)=\left.\frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho}\right|_{\rho=\rho_0}$$

is the derivative of the optimal velocity $V(\rho)$ at $\rho=\rho_0$.

By expanding $y(j,m,t)=\exp[\mathrm{i}k(j+m)+zt]$, $z=z_1(\mathrm{i}k)+z_2(\mathrm{i}k)^2+\cdots$, inserting them into Eq. (17), and neglecting the higher order terms, the first-order and second-order terms of the coefficient in the expression of $\mathrm{i}k$ are derived as follows:

$$\begin{aligned}z_1=&-\rho_0^2V'\{(cc_1)^2-[c(1-c_1)]^2\\&+[(1-c)c_2]^2-[(1-c)(1-c_2)]^2\},\end{aligned}\tag{18}$$

$$\begin{aligned}z_2=&-\frac{3}{2}\tau(\rho_0^2V')^2\{(cc_1)^2-[c(1-c_1)]^2+[(1-c)c_2]^2\\&-[(1-c)(1-c_2)]^2\}^2-\left(\frac{1}{2}+\gamma\right)\rho_0^2V'\{(cc_1)^2\\&+[c(1-c_1)]^2+[(1-c)c_2]^2\\&+[(1-c)(1-c_2)]^2\}.\end{aligned}\tag{19}$$

If $z_2>0$, the uniform flow is always stable. If $z_2<0$, the uniform steady flow is unstable for long wavelength modes. Therefore we have the following unstable condition

$$\begin{aligned}\tau>&-\{(1+2\gamma)\{(cc_1)^2+[c(1-c_1)]^2+[(1-c)c_2]^2\\&+[(1-c)(1-c_2)]^2\}\}/\{3\rho_0^2V'\{(cc_1)^2-[c(1-c_1)]^2\\&+[(1-c)c_2]^2-[(1-c)(1-c_2)]^2\}^2\}.\end{aligned}\tag{20}$$

The neutral stability condition for this steady state is given by

$$\begin{aligned}\tau=&-\{(1+2\gamma)\{(cc_1)^2+[c(1-c_1)]^2+[(1-c)c_2]^2\\&+[(1-c)(1-c_2)]^2\}\}/\{3\rho_0^2V'\{(cc_1)^2-[c(1-c_1)]^2\\&+[(1-c)c_2]^2-[(1-c)(1-c_2)]^2\}^2\}.\end{aligned}\tag{21}$$

The phase diagram in (ρ,a)-space is given in Fig. 1. In Fig. 1, there are two regions: one is the stable region above the neutral stability curve, and the other is the unstable region below the neutral stability line. The critical point $\rho_0=0.3$ and $\rho_c=0.25$.

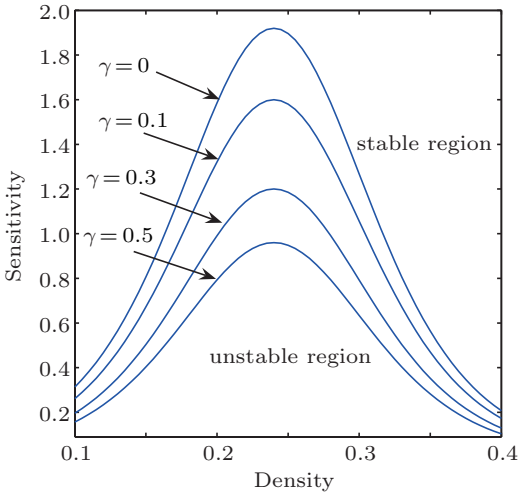


Fig. 1. (color online) The phase diagram in (ρ,a)-space.

4. Nonlinear analysis

We use the reductive perturbation method to Eq. (14) and focus on the behavior around the neutral line to derive the soliton solution. Firstly, we give slow scales for space variables j, m, and time variable t, and define the slow variables X and T as

$$X=\varepsilon(j+m+bt),\quad T=\varepsilon^3t,\quad 0<\varepsilon\ll 1,\tag{22}$$

where b is a constant to be determined later. Then we have

$$\rho(j,m,t)=\rho_0+\varepsilon^2R(X,T).\tag{23}$$

Next, substituting Eqs. (22) and (23) into Eq. (14) and making the Taylor expansion to the sixth-order of ε. We obtain the following nonlinear partial differential equation

$$\varepsilon^3\{b+g\rho_0^2V'\}\partial_XR+\varepsilon^4\left\{\frac{3}{2}b^2\tau+\frac{1+2\gamma}{2}f\rho_0^2V'\right\}\partial_X^2R$$

$$+\varepsilon^5\left\{\partial_T R+\left[\frac{7b^3\tau^2}{6}+\frac{1+6\gamma}{6}g\rho_0^2V'\right]\partial_X^3 R+g\rho_0^2V''R\partial_X R\right\}+\varepsilon^6\left\{\left[\frac{5b^4\tau^3}{8}+\frac{1+14\gamma}{24}f\rho_0^2V'\right]\partial_X^4 R+\frac{1+2\gamma}{4}f\rho_0^2V''\partial_X^2R^2+3b\tau\partial_X\partial_T R\right\}=0. \quad (24)$$

Here,

$$V'=\left.\frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho}\right|_{\rho=\rho_0},\quad V''=\left.\frac{\mathrm{d}^2V(\rho)}{\mathrm{d}\rho^2}\right|_{\rho=\rho_0},$$
$$g=(cc_1)^2-[c(1-c_1)]^2+[(1-c)c_2]^2-[(1-c)(1-c_2)]^2,$$
$$f=(cc_1)^2+[c(1-c_1)]^2+[(1-c)c_2]^2+[(1-c)(1-c_2)]^2.$$

Now, we study the traffic flow near the neutral stability line. Let us put $b=-g\rho_0^2V'$ in Eq. (24) then the third-order term equals zero. Here,

$$\frac{\tau}{\tau_s}=1+\varepsilon^2, \quad (25)$$

where

$$\tau_s=-\frac{f(1+2\gamma)}{3g^2\rho_0^2V'}=\frac{f(1+2\gamma)}{3gb},$$

equation (24) is rewritten as

$$\varepsilon^5(\partial_T R+g_1\partial_X^3R+g_2R\partial_X R)+\varepsilon^6(g_3\partial_X^2R+g_4\partial_X^4R+g_5\partial_X^2R^2)=0. \quad (26)$$

Here the coefficients g_i are given in Table 1.

Making the following transformations,

$$T=\sqrt{g_1}T',\quad X=\sqrt{g}_1X',\quad R(X,T)=\frac{1}{g_2}R'(X',T'), \quad (27)$$

we get the standard KdV equation from Eq. (26)

$$\partial_{T'}R'+\partial_{X'}^3R'+R'\partial_{X'}R'+\varepsilon\left(\frac{g_3}{\sqrt{g_1}}\partial_{X'}^2R'+\frac{g_4}{(g_1)^{3/2}}\partial_{X'}^4R'+\frac{g_5}{g_2\sqrt{g_1}}\partial_{X'}^2R'^2\right)=0. \quad (28)$$

Then, by ignoring the $O(\varepsilon)$ terms in Eq. (28), we obtain the KdV equation with the soliton solution

$$R_0'(X',T')=A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{12}}\left(X'-\frac{AT'}{3}\right)\right]. \quad (29)$$

Assuming

$$R'(X,T')=R_0'(X,T')+\varepsilon R_1'(X',T'),$$

we then consider the $O(\varepsilon)$ correction. In order to determine the selected value A, it is necessary to coincide with the solvability condition

$$(R_0',M[R_0'])\equiv\int_{-\infty}^{+\infty}\mathrm{d}XR_0'M[R_0']=0, \quad (30)$$

where

$$M[R_0']=\frac{g_3}{\sqrt{g_1}}\partial_{X'}^2R'+\frac{g_4}{(g_1)^{3/2}}\partial_{X'}^4R'+\frac{g_5}{g_2\sqrt{g_1}}\partial_{X'}^2R'^2.$$

Then the selected amplitude A is determined as[19]

$$A=\frac{21g_1g_2g_3}{-5g_2g_4+24g_1g_5}. \quad (31)$$

We thus derived the KdV soliton solution as follows:

$$\rho(j,m,t)=\rho_0+\varepsilon^2\frac{A}{g_2}\,\mathrm{sech}^2\left(\varepsilon\sqrt{\frac{A}{12g_1}}\right)\times\left[j+m+\left(V'+\frac{\varepsilon^2A}{3}\right)t\right], \quad (32)$$

where $\varepsilon^2=\tau/\tau_s-1$. The soliton solution represents the coexisting phase, which consists of the freely moving phase with low density and the jammed (or congested) phase with high density.

In the limit of $c=0,c_2=1,\gamma=0$ or $c=1,c_1=1,\gamma=0$, equation (32) reduces to the solution of the density equation of the one-dimensional lattice hydrodynamic equation for pedestrian flow. In the limit $c_1=1$, $c_2=1$, $\gamma=0$, equation (32) reduces to the solution of the density equation of the two-dimensional lattice hydrodynamic equation. When $\gamma=0$, equation (32) is the solution of the density equation for the two-dimensional bidirectional pedestrian lattice hydrodynamic equation.

Table 1. The coefficients g_i of the model.

g_1	g_2	g_3
$\frac{7b^3\tau^2}{6}+\frac{1+6\gamma}{6}g\rho_0^2V'$	$g\rho_0^2V''$	$-\frac{1+2\gamma}{2}f\rho_0^2V'$
g_4	g_5	
$-\frac{23b^4\tau^3}{8}-\frac{1+6\gamma}{2}b\tau\rho_0^2V'g+\frac{1+14\gamma}{24}\rho_0^2V'f$	$\left[\frac{1+2\gamma}{4}-\frac{3\rho_0^2gb\tau}{2}\right]V''$	

5. Summary

Based on the two-dimensional bidirectional lattice hydrodynamic pedestrian flow model, the stability condition is obtained by using the linear stability analysis. We have derived the KdV equation near the neutral stability line, which could describe the soliton waves in pedestrian flow. Additionally the phase diagram reveals that the critical point a_c decreases with the increase of the fraction of γ. That is to say, the next-nearest-neighbor effect should be considered, which is helpful to the stability of the pedestrian flow.

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