Codimension-two bifurcation of axial loaded beam bridge subjected to an infinite series of moving loads*

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A novel model is proposed which comprises of a beam bridge subjected to an axial load and an infinite series of moving loads. The moving loads, whose distance between the neighbouring ones is the length of the beam bridge, coupled with the axial force can lead the vibration of the beam bridge to codimension-two bifurcation. Of particular concern is a parameter regime where non-persistence set regions undergo a transition to persistence regions. The boundary of each stripe represents a bifurcation which can drive the system off a kind of dynamics and jump to another one, causing damage due to the resulting amplitude jumps. The Galerkin method, averaging method, invertible linear transformation, and near identity nonlinear transformations are used to obtain the universal unfolding for the codimension-two bifurcation of the mid-span deflection. The efficiency of the theoretical analysis obtained in this paper is verified via numerical simulations.

Keywords: mid-span deflection, beam bridge, infinite series of moving loads, codimension-two bifurcation

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1. Introduction

The study on the mid-span of a beam bridge under moving loads has attracted a lot of interest and the theory of moving load has been extensively developed.[1–8] Theoretical analytical method and finite element are the main methods for studying the beam bridge vibration.[1,2] Fryba[3,4] used the theory of a Euler–Bernoulli beam to obtain the analytical solution of constant force, moving mass, and moving load which act on the beam. Yang[5] adopted a modal superposition approach to derive the solutions for a vehicle–bridge system and concluded that sufficiently accurate solutions confirmed that accurate solutions can be obtained by considering only the first mode. Rao[6] analyzed the dynamic maximum deflection of a beam and pointed out that the dynamics deflection on the combination resonance was four or five times the one on single-mode resonance. Ouyang[7] presented some typical moving-load dynamic problems through a simply supported beam excited by a moving load. Azizi[8] employed the spectral element method in frequency domain to analyze the continuous beams and bridges subjected to a moving load. The procedure is simplified for a moving load with a constant amplitude and velocity.

It is well known that a number of studies have been performed to predict the dynamic response of the mid-span of a beam bridge subjected to moving loads by employing beams and the vibration of a Bernoulli–Euler beam exhibits distinctly nonlinear behavior. Nayfeh[9] developed a multi-scale method to solve the equation of the beam with no moving loads. Yanmeni Wayou[10] described a Euler–Bernoulli beam under moving loads with a duffing-like equation. The method of multiple scales is adopted to highlight the parametric resonance. The influence of the mass parameter, the damping, and the load velocity on the beam deflections were examined. Ansari[11] investigated frequency responses under the condition of internal or external resonance with the multi-scale method and studied the influences of nonlinear stiffness damping as well as magnitude of moving load on the frequency response. Hu[12] investigated the convergence of the Galerkin method for an elastic beam subjected to a moving load.

Furthermore, codimension bifurcation is a kind of classical nonlinear phenomena and has been extensively studied in pure and applied sciences and engineering (see Refs. [13] and [14]). Codimension-two generalized-Hopf bifurcation and fold-Hopf bifurcation were investigated in Ref. [15]. Zhang[16] considered a three-codimensional unfolding with Z_2 symmetry on the basis of singularity theory. Zhang[17] applied the simplest normal form of an electromechanical-coupled nonlinear dynamic system to analyze the codimension-two bifurcation. The codimension-two bifurcation for the quad bundle conductor's galloping was analyzed in Ref. [18]. Tian[19] detected the codimension-two bifurcation for a recently proposed SD oscillator. Recently, the codimension-two bifurcations of equilibria of a two-dimensional Hindmarsh–Rose model was discussed in Ref. [20]. Codimension-two homoclinic bifurcation of the Hindmarsh–Rose model of neural action potential was studied in Ref. [21]. In this year, the dynam-

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ical complexity of a discrete SIS epidemic model with standard incidence by the qualitative analysis and numerical simulations was investigated in Ref. [22]. These results demonstrated the usefulness of the techniques for analyzing such a complicated behavior in applications. However, although a number of analytical investigations have been carried out to study the dynamic behavior of bridge decks traversed by moving vehicles, there is limited research on nonlinear vibrations of the beam bridge.[23] A key point to explain the beam bridge collapse is that a nonlinear response can lead to various complex dynamical phenomena, such as different modes of vibration, bifurcation, and chaotic motion.

The motivation of this paper is to explore the vibration and stability of the axial loaded beam bridge subjected to an infinite series of travelling loads with constant velocity, which repeat at regular time intervals. We do not restrict ourselves to investigate the effects of various parameters, such as the load velocity, load frequency, damping, and so on, but we consider the more complex behavior expected in a nonlinear system. We explore persistence sets, non-persistence sets, and a codimension-two bifurcation set.

This paper is organized as follows. In the next section, a mathematical model describing the dynamic interactions between the axial loaded beam bridge and an infinite series of moving loads is proposed. A static axial force enhances the nonlinear effect of the beam bridge and an infinite series of travelling loads make the beam bridge extend to infinity. In Section 3, the universal unfolding of the normal form with codimension-two bifurcation is obtained. Furthermore, the codimension-two bifurcation phenomena for the mid-span deflection are presented by using the relationship of the parameters between the mid-span deflection and the universal unfolding. In the last section, numerical results are presented to show the efficiency of our theoretical analysis.

2. Preliminary dynamic analysis

Consider the mid-span deflection of a beam bridge subjected to infinite equidistant successive moving loads (as shown in Fig. 1). Assuming that moving loads pass with a constant velocity v, the equation of motion can be written in the following form

$$EI\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}-S(x)\frac{\partial^2 u}{\partial x^2}+c_1\frac{\partial u}{\partial t} = \left(Mg-M\frac{\partial^2 u}{\partial t^2}\right)\delta(x-vt), \quad \left(0\le t\le\frac{L}{v}\right), \tag{1}$$

where EI, $u(x,t)$, $\rho A, c_1$, M are the flexural rigidity, the vertical displacement, mass of unit length, damper of the beam, and the equivalent mass of moving load, respectively.

$$S(x,t)=\pm\frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2$$

is axial force.[24] The use of δ function is to accommodate a pointwise concentrated load in place of a distributed load.

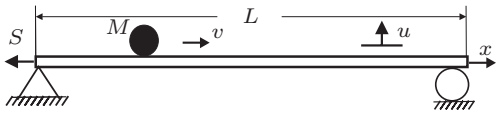


Fig. 1. Bridge-moving loads model.

Assuming that only the first (symmetric) vibrational mode is considered, which leads that u can take the following form

$$u=q(t)\sin\frac{\pi x}{L}, \tag{2}$$

where $q(t)$ is the amplitude of the mode. Substituting Eq. (2) into Eq. (1) and integrating over $[0,L]$ satisfies

$$\left(1+\frac{2M}{\rho AL}\sin^2\frac{\pi v}{L}t\right)\ddot{q}+\frac{EI}{\rho A}\left(\frac{\pi}{L}\right)^4 q\pm\frac{\pi^3 E}{8\rho L^3}q^3+\frac{c_1}{\rho A}\dot{q} = \frac{2Mg}{\rho AL}\sin\frac{\pi v}{L}t, \quad \left(0\le t\le\frac{L}{v}\right). \tag{3}$$

We can use $|\sin(\pi v/L)t|$ $(t\ge 0)$ to substitute $\sin(\pi v/L)t$ $(0\le t\le L/v)$ in order to discuss the vibration of the beam bridge subjected to an infinite series of moving loads with constant velocity, which repeats at time L/v intervals.[23] And let

$$\mu=\frac{M}{L\rho A},\ f=\frac{2Mg}{L\rho A\omega_1^2},\ \omega_1^2=\frac{EI\pi^4}{L^4\rho A},\ \gamma_1=\frac{c_1}{\rho A\omega_1},$$
$$\beta=\mp\frac{E\pi^3}{8\rho L^3\omega_1^2},\ \omega=\frac{\pi v}{L\omega_1},\ t\to\frac{t}{\omega_1}, \tag{4}$$

equation (3) can be rewritten as

$$(1+2\mu\sin^2\omega t)\ddot{q}+q-\beta q^3+\gamma_1\dot{q}=f|\sin\omega t|. \tag{5}$$

Let

$$q=a\cos(2\omega t+\phi),\ \ \dot{q}=-2a\omega\sin(2\omega t+\phi),$$

and apply the average method, system (5) yields

$$\begin{cases}\dot{a}=-\dfrac{a\gamma_1}{2+\mu}+\dfrac{4f\sin\phi}{3(2+\mu)\pi\omega},\\ a\dot{\phi}=\dfrac{a(4-3a^2\beta-16\omega^2(1+\mu))}{8(2+\mu)\omega}+\dfrac{4f\cos\phi}{3(2+\mu)\pi\omega}.\end{cases} \tag{6}$$

3. Codimension-two bifurcation

Let

$$\begin{cases}x=a\cos\phi,\\ y=a\sin\phi.\end{cases} \tag{7}$$

Substituting Eq. (7) into Eq. (6) yields

$$\begin{cases}\dot{x}=-a_1x-a_2y+a_3(x^2+y^2)y,\\ \dot{y}=a_2x-a_1y-a_3(x^2+y^2)x+\dfrac{4f}{3(2+\mu)\pi\omega},\end{cases} \tag{8}$$

where

$$a_1=\frac{\gamma_1}{2+\mu},\ \ a_2=\frac{1-4w^2(1+\mu)}{2(2+\mu)\omega},\ \ a_3=\frac{3\beta}{8(2+\mu)\omega}.$$

Let

$$x=x_0+m,\ \ y=y_0+n, \tag{9}$$

where (x_0,y_0) is an equilibrium point of Eq. (8). Then, equation (8) can be written as

$$\begin{cases}\dot m=A_1m+A_2n+A_5m^2+2A_6mn\\ \quad +3A_5n^2+A_7(m^2n+n^3),\\ \dot n=A_3m+A_4n-3A_6m^2-2A_5mn\\ \quad -A_6n^2-A_7(m^3+mn^2),\end{cases} \tag{10}$$

where

$$\begin{aligned}&A_1=-a_1+2a_3x_0y_0,\ \ A_2=-a_2+a_3(x_0^2+3y_0^2),\\ &A_3=a_2-a_3(3x_0^2+y_0^2),\ \ A_4=-a_1-2a_3x_0y_0,\\ &A_5=a_3y_0,\ \ A_6=a_3x_0,\ \ A_7=a_3.\end{aligned} \tag{11}$$

Obviously, $(0,0)$ is an equilibrium point of Eq. (10). Clearly, when

$$A_1+A_4=0,\ \ A_1A_4-A_2A_3=0, \tag{12}$$

the Jacobi matrix of Eq. (10) has double zero roots.

Substituting

$$A_3=-\frac{A_1^2}{A_2}+\varepsilon_1,\ \ A_4=-A_1+\varepsilon_2, \tag{13}$$

into Eq. (10) and introducing the linear transformation

$$\begin{pmatrix}m\\ n\end{pmatrix}=\begin{pmatrix}-\dfrac{A_2}{A_1} & \dfrac{A_2}{A_1^2}\\ 1 & 0\end{pmatrix}\begin{pmatrix}m_1\\ n_1\end{pmatrix}$$

satisfies

$$\begin{cases}\dot m_1=\left(-\dfrac{A_2}{A_1}\varepsilon_1+\varepsilon_2\right)m_1-\left(1-\dfrac{A_2}{A_1^2}\varepsilon_1\right)n_1+a_{20}m_1^2\\ \quad +a_{11}m_1n_1+a_{02}n_1^2+a_{30}m_1^3+a_{21}m_1^2n_1\\ \quad +a_{12}m_1n_1^2+a_{03}n_1^2,\\ \dot n_1=(A_1\varepsilon_2-A_2\varepsilon_1)m_1+\dfrac{A_2}{A_1}\varepsilon_1n_1+b_{20}m_1^2+b_{11}m_1n_1\\ \quad +b_{02}n_1^2+b_{30}m_1^3+b_{21}m_1^2n_1+b_{12}m_1n_1^2+b_{03}n_1^3,\end{cases} \tag{14}$$

where

$$\begin{aligned}a_{20}&=\frac{2A_2A_5}{A_1}-A_6-\frac{6A_2^2A_6}{A_1^2},\\ a_{11}&=\frac{-2A_1A_2A_5+6A_2^2A_6}{A_1^3},\\ a_{02}&=-\frac{3A_2^2A_6}{A_1^4},\\ a_{30}&=\frac{A_1^2A_2A_7+A_2^3A_7}{A_1^3},\\ a_{21}&=\frac{-A_1^2A_2A_7-3A_2^3A_7}{A_1^4},\\ a_{12}&=\frac{3A_2^3A_7}{A_1^5},\\ a_{03}&=\frac{-A_2^3A_7}{A_1^6},\\ b_{20}&=\frac{3(A_1^3A_5+A_1A_2^2A_5-A_1^2A_2A_6-A_2^3A_6)}{A_1A_2},\\ b_{11}&=\frac{2(-2A_1A_2A_5+A_1^2A_6+3A_2^2A_6)}{A_1^2},\\ b_{02}&=\frac{A_1A_2A_5-3A_2^2A_6}{A_1^3},\\ b_{30}&=\frac{A_1^4A_7+2A_1^2A_2^2A_7+A_2^4A_7}{A_1^2A_2},\\ b_{21}&=\frac{-3(A_1^2A_2A_7+A_2^3A_7)}{A_1^3},\\ b_{12}&=\frac{A_1^2A_2A_7+3A_2^3A_7}{A_1^4},\\ b_{03}&=-\frac{A_2^3A_7}{A_1^5}.\end{aligned}$$

Hence, with the aid of the Mathematica program in Ref. [17], the normal form of Eq. (14) can be obtained as follows:

$$\begin{cases}\dot m_1=n_1,\\ \dot n_1=(A_2\varepsilon_1-A_1\varepsilon_2)m_1+\varepsilon_2n_1+am_1^2+bm_1n_1,\end{cases} \tag{15}$$

where

$$\begin{aligned}a&=[(2a_{20}+b_{11})(3A_2^2\varepsilon_1^2+6A_1^2\varepsilon_2-2A_1A_2\varepsilon_1\varepsilon_2\\ &\quad -A_1^2(6A_2\varepsilon_1+\varepsilon_2^2))]/3A_1^2,\\ b&=\frac{(2a_{20}+b_{11})(9A_2\varepsilon_1+\varepsilon_2(-9A_1+2\varepsilon_2))}{2\varepsilon_2}.\end{aligned}$$

Let

$$\begin{cases}u=\dfrac{A_2\varepsilon_1-A_1\varepsilon_2}{2a}+m_1,\\ v=n_1,\end{cases}$$

and normal form (15) can be transformed into the following form

$$\begin{cases}\dot u=v,\\ \dot v=-\dfrac{(A_2\varepsilon_1-A_1\varepsilon_2)^2}{4a}\\ \quad +\left(-\dfrac{b(A_2\varepsilon_1-A_1\varepsilon_2)}{2a}+\varepsilon_2\right)v+au^2+buv.\end{cases} \tag{16}$$

The scale transformations may be introduced as follows:

$$u\to\frac{a}{b^2}u,\ \ v\to\frac{a^2}{b^3}v,\ \ t\to\frac{b}{a}t,$$

which leads that system (16) can be rewritten in the form

$$\begin{cases}\dot u=v,\\ \dot v=\mu_1+\mu_2v+u^2+uv,\end{cases} \tag{17}$$

where

$$\mu_1 = -\frac{b^4(A_2\varepsilon_1 - A_1\varepsilon_2)^2}{4a^4},$$
$$\mu_2 = \frac{b}{a}\left(-\frac{b(A_2\varepsilon_1 - A_1\varepsilon_2)}{2a} + \varepsilon_2\right).$$

System (16) is the universal unfolding for the norm form

$$\begin{cases} \dot{u} = v, \\ \dot{v} = u^2 + uv \end{cases} \tag{18}$$

being the folded dynamics near the origin of codimension-two bifurcation.[16]

The following analysis is carried out in order to obtain the unfolded dynamics with a bifurcation diagram for the codimension-two bifurcation.

Firstly, it is easy to find that, for $\mu_1 \le 0$, the equilibria for Eq. (17) can be obtained and written as

$$(x_\pm, 0) = (\pm\sqrt{-\mu_1}, 0),$$

while only equilibrium point $(0,0)$ is saddle-mode as $\mu_1 = 0$, $\mu_2 \neq 0$. Hence, the saddle-node bifurcation occurs at $\mu_1 = 0$, $\mu_2 \neq 0$.

Secondly, when $\mu_1 < 0$, the Jacobian at the unique equilibrium $(x_\pm, 0)$ is given by

$$Df(x_\pm, 0) = \begin{bmatrix} 0 & 1 \\ \pm 2\sqrt{-\mu_1} & \mu_2 \pm \sqrt{-\mu_1} \end{bmatrix}.$$

From Ref. [19], we know that the Hopf bifurcation occurs at $\mu_1 = -\mu_2^2$.

Furthermore, let

$$\mu_1 = -\varepsilon^4\delta, \quad (\delta \neq 0), \quad \mu_2 = \varepsilon^2\kappa,$$
$$u \to \varepsilon^2 u, \quad v \to \varepsilon^3 v, \quad t \to \frac{t}{\varepsilon},$$

equation (17) can be rewritten as

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\delta + \varepsilon\kappa v + u^2 + \varepsilon uv. \end{cases} \tag{19}$$

For $\varepsilon = 0$, system (19) is a Hamiltonian system

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\delta + u^2, \end{cases} \tag{20}$$

and the Hamiltonian function is of the form

$$H(u,v) = \frac{v^2}{2} + \delta u - \frac{u^3}{3}. \tag{21}$$

And for convenience, let $\delta = 1$. These lead to the equilibria $(\pm 1, 0)$, where $(1,0)$ is the saddle point, while $(-1,0)$ is center as shown in Fig. 2. The homoclinic orbit can be written in the parametric form as

$$\begin{cases} u = 3\tanh^2\left(\frac{\sqrt{2}}{2}t\right) - 2, \\ v = 3\sqrt{2}\,\mathrm{sech}^2\left(\frac{\sqrt{2}}{2}t\right)\tanh\left(\frac{\sqrt{2}}{2}t\right). \end{cases} \tag{22}$$

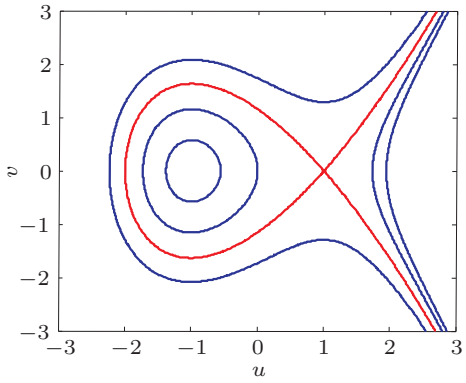


Fig. 2. (color online) Hamilton phase portrait: $\delta = 1$.

For $\varepsilon \neq 0$, system (19) is dissipative, the homoclinic orbit is preserved, iff

$$M(k) = \int_{-\infty}^{\infty} v(kv + uv)\,\mathrm{d}t = 0. \tag{23}$$

It is known that, the homoclinic orbit occurs on $k = 5/7$, i.e. $\mu_1 = -(49/25)\mu_2$. Thus, the bifurcations of Eq. (19) in μ_1–μ_2 plane can be listed as follows.

(i) $SN^+ \bigcup SN^- = \{\mu_1 = 0, \mu_2 \neq 0\}$, the saddle-node bifurcation;

(ii) $H = \{\mu_1 = -\mu_2^2\}$, Hopf bifurcation;

(iii) $HL = \{\mu_1 = -(49/25)\mu_2^2\}$, the homoclinic bifurcation.

The bifurcation diagram described above divides the μ_1–μ_2 plane into four persistence regions, in each of which the system is structurally stable. The bifurcation diagram, codimension-two bifurcation, and phase portraits are well understood in Fig. 3.

Noticing that

$$\begin{aligned} \mu_1 &= -\frac{b^4(A_2\varepsilon_1 - A_1\varepsilon_2)^2}{4a^4}, \\ \mu_2 &= \frac{b}{a}\left(-\frac{b(A_2\varepsilon_1 - A_1\varepsilon_2)}{2a} + \varepsilon_2\right), \\ a &= [(2a_{20} + b_{11})(3A_2^2\varepsilon_1^2 + 6A_1^2\varepsilon_2 - 2A_1A_2\varepsilon_1\varepsilon_2 \\ &\quad - A_1^2(6A_2\varepsilon_1 + \varepsilon_2^2))]/3A_1^2, \\ b &= \frac{(2a_{20} + b_{11})(9A_2\varepsilon_1 + \varepsilon_2(-9A_1 + 2\varepsilon_2))}{2\varepsilon_2}. \end{aligned}$$

One can obtain the bifurcation diagram for system (15) in original parameter space plane ε_1–ε_2 for $A_1 = 1$, $A_2 = -1$ in the following (see Fig. 4).

i) The system behaves as the saddle-node bifurcation when $(\varepsilon_1, \varepsilon_2)$ on the branch SN with the unique equilibrium. If $\varepsilon_1 = 0$, $(0,0)$ is a saddle-node for $\varepsilon_2 \neq 0$ (see region SN).

ii) When $(\varepsilon_1, \varepsilon_2)$ touches H_1 and H_2, there appears the Hopf bifurcation which bifurcates into an unstable limit cycle encircling the stable focus (see regions VIII and IV). And there is a saddle and a source in regions VII and V.

iii) The unstable limit cycles may extend into a homoclinic orbit when $(\varepsilon_1, \varepsilon_2)$ tends to HL_1, HL_2, HL_3, and HL_4 from II, IV, and VII, respectively.

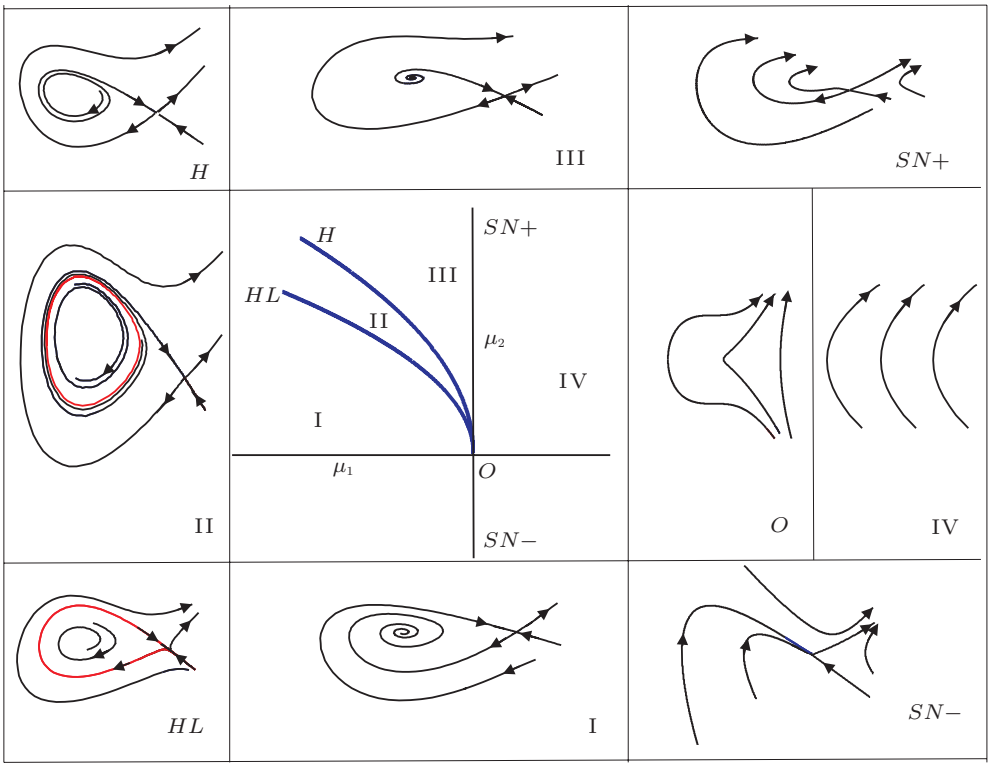


Fig. 3. (color online) Bifurcation set of system (17).

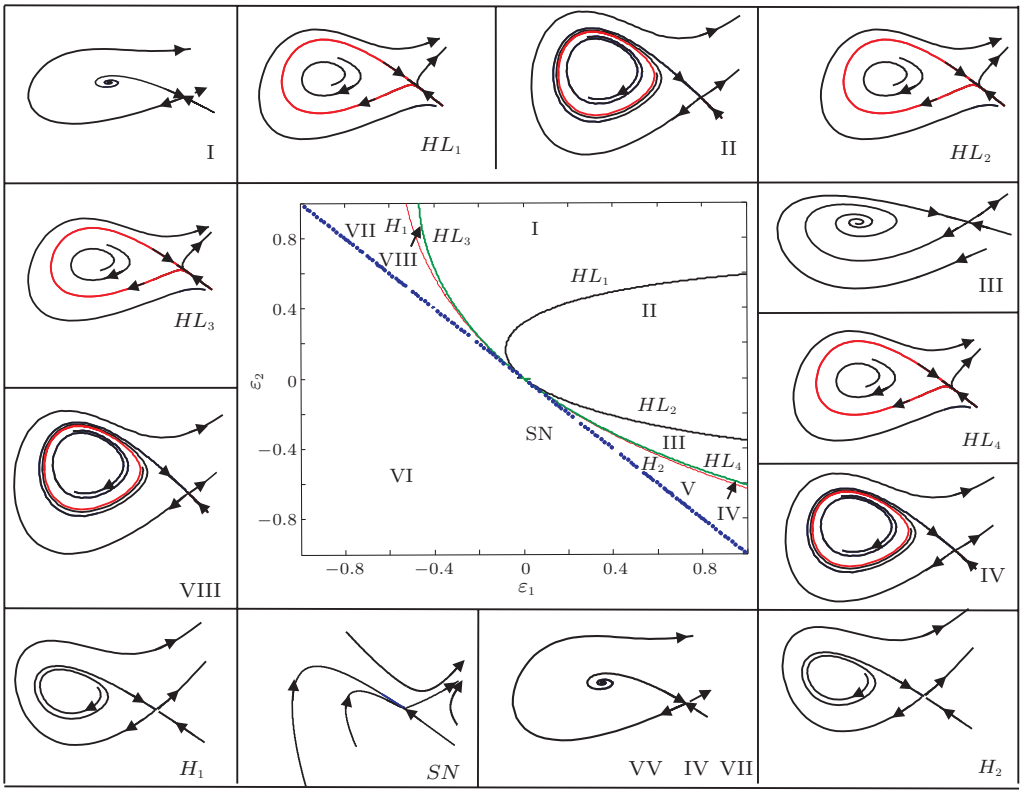


Fig. 4. (color online) The ε_1–ε_2 bifurcation set of system (14).

The numerical method is used to verify codimension-two bifurcation (see Fig. 5). Figure 5 is the phase portraits of system (8) for $E = 2.6911656521719704 \times 10^{10}$ N·m^2, $I = 0.00140625$ m^4, $\rho = 7.2 \times 10^3$ kg/m^3, $A = 0.2$ m^2, $L = 40$ m, $g = 10$ N/kg, $v = 26.0378$ m/s, $c_1 = 0.3024$ N·s/m. The homoclinic orbit are presented in Fig. 5(a) for $M = 5.76 \times 10^3$ kg. And the limit circle is presented in Fig. 5(b) for $M = 3.76 \times 10^3$ kg.

In summary, from Figs. 4 and 5, we know that regions III, V, VI, and VII are ideal states in which the amplitude of the mid-span decreases gradually. The limit cycle in regions II and IV are not acceptable cases because the small periodical vibration of mid span could be unstable to destroy the stability of the structure. Other situations, such as sources I and VIII, are terrible cases, which make the amplitude increase consistently and make the beam bridge collapse. Furthermore, from the sense to the physics of the beam bridge, ε_2 yields $\varepsilon_2 < 0$. Therefore, the parameters, which satisfy $\varepsilon_2 < 0$ and lie in regions III, V, and VI, can be chosen in the design of the beam bridge to keep vibration stability in real engineering.

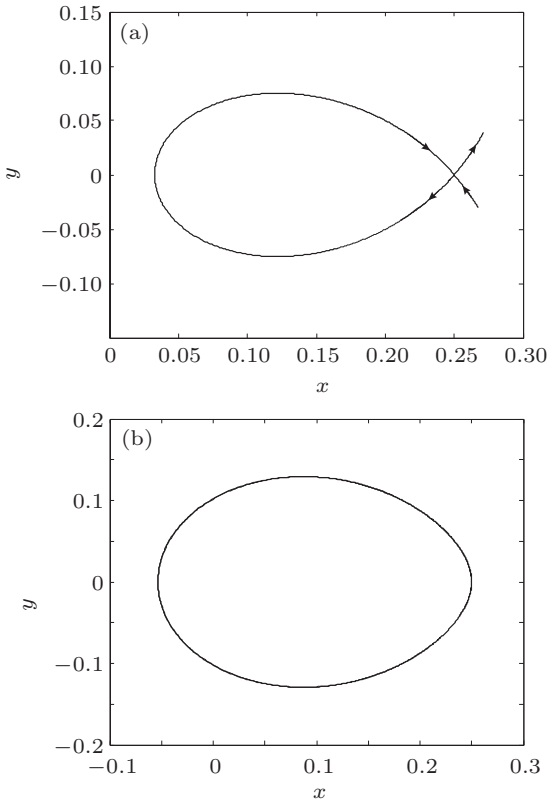


Fig. 5. (a) Homoclinic orbit; (b) Limit circle.

4. Conclusions

In this paper, the complicated nonlinear dynamics of a beam subjected to an axial load and an infinite series of moving loads have been investigated. The moving loads and the nonlinear characteristic of the axial force can lead the vibration of the system to codimension-two bifurcation. Analyses were performed to investigate a parameter regime where non-persistence set regions undergo a transition to persistence regions. Taking into account the large difference of amplitudes corresponding to these two possible states, when the conditions of the parameters are close enough to the bifurcation boundaries, a trajectory may hop from one persistence state to another, causing potentially dangerous large amplitude jumps which can make the beam bridge collapse. Hence, the future study on the complicated nonlinear dynamics of this beam bridge is being carried out in three aspects. The first is the chaotic behaviors. The second pursued direction is to construct a multi-degree-of-freedom model for the coupling vibration of the vehicle–bridge for detecting the potential hazard produced by an infinite series of moving loads. The third direction is to study the vibration energy absorptions that suppress the vibrations of the beam bridge under an infinite series of moving loads.

References

[1] Song Y F 2000 *Dynamics of Highway Bridge* (Beijing: China Communication Press) (in Chinese)
[2] Li G H 1996 *Steady and Vibraton of Bridge Structure* (Beijing: China Railroad Press) (in Chinese)
[3] Fryba L 1996 *Dynamics of Railway Bridge* (London: Thomas Telford)
[4] Fryba L 1999 *Vibration of Solids and Structures under Moving Loads* (London: Thomas Telford)
[5] Yang Y B and Lin C W 2005 *J. Sound Vib.* **284** 205
[6] Rao R G 2000 *J. Vib. Acoust.* **122** 281
[7] Ouyang H J 2011 *Mechanical Systems and Signal Processing* **25** 2039
[8] Azizi N 2011 *Appl. Math. Model* **36** 3580
[9] Nayfeh A H and Mook D T 1979 *Nonlinear Oscillations* (New York: Wiley-Interscience)
[10] Yanmeni Wayou A N, Tchoukuegno R and Woafo P 2004 *J. Sound Vib.* **273** 1101
[11] Ansari M, Esmailzadeh E and Younesian D 2010 *Nonlinear Dyn.* **61** 163
[12] Ding H, Chen L Q and Yang S P 2012 *J. Sound Vib.* **331** 2426
[13] Moon F C 1992 *Chaotic and Fractal Dynamics* (New York: Wiley)
[14] Yagasaki K 1999 *Nonlinear Mech.* **34** 983
[15] Duan L X and Lu Q S 2005 *Chin. Phys. Lett.* **22** 1325
[16] Zhang Q C, Wang W and Liu F H 2008 *Chin. Phys. B* **17** 4123
[17] Zhang Q C and Tian R L 2009 *Eng. Mech.* **26** 216 (in Chinese)
[18] Liu F H, Zhang Q C and Tan Y 2010 *Chin. Phys. Lett.* **27** 044702
[19] Tian R L, Cao Q J and Yang S P 2010 *Nonlinear Dyn.* **59** 19
[20] Liu X L and Liu S Q 2012 *Nonlinear Dyn.* **67** 847
[21] Linaro D, Alan C and Desroches M, *et al*. 2012 *J. Appl. Dyn. Sys.* **11** 939
[22] Chen Q L, Teng Z D and Wang L, *et al*. 2013 *Nonlinear Dyn.* **71** 55
[23] Tian R L, Yang X W, Cao Q J and Han Y W 2012 *Int. J. Bifur. Chaos* **22** 12501081
[24] Kim S M 2004 *Eng. Struct.* **26** 95