Spin and pseudospin symmetries of the Dirac equation with shifted Hulthén potential using supersymmetric quantum mechanics

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In this paper, we obtain approximate analytical solutions of the Dirac equation for the shifted Hulthén potential within the framework of spin and pseudospin symmetry limits for arbitrary spin–orbit quantum number κ using the supersymmetry quantum mechanics. The energy eigenvalues and the corresponding Dirac wave functions are obtained in closed forms.

Keywords: Dirac equation, supersymmetry method, spin and pseudospin symmetries, shifted Hulthén potential

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1. Introduction

As is well known, the Dirac equation describes spin-1/2 particles in relativistic quantum mechanics. In order to investigate the nuclear shell model, the studies of spin and pseudospin symmetries of the Dirac equation provide us with an invaluable source of knowledge.[1] The concepts of spin and pseudospin symmetries, which were introduced many years ago in nuclear theory,[2,3] have been successfully used to explain the features of deformed nuclei[4] and superdeformation[5] and establish an effective shell-model coupling scheme.[6] Within the framework of the Dirac theory, the spin symmetry occurs when the difference in potential between the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant, i.e., $\Delta(r) = V(r) - S(r) = \text{const}$. On the other hand, the pseudospin symmetry arises when the sum of the terms is a constant, i.e., $\Sigma(r) = V(r) + S(r) = \text{const}$. Useful contributions by other authors can be found in Refs. [7]–[20]. By considering relativistic theories with scalar and vector potentials, Ginocchio[21] showed that these symmetries could explain degeneracies in meson spectra or in single-particle energy levels. The spin symmetry in nuclear theory usually refers to a quasi-degeneracy of the single nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n, l, j = l - 1/2)$, where n, l, and j are the single nucleon radial, orbital, and total angular momentum quantum numbers for a single particle respectively.[22] Also, the pseudospin symmetry implies that $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$ states are degenerate.[22] These symmetries have been already analyzed with a number of potentials including Woods–Saxon,[23] Manning–Rosen,[24] Eckart,[25] harmonic oscillator,[26] Hyllerass,[27] and some others.[28] Such studies use a variety of analytical techniques such as the asymptotic iteration method (AIM), the Nikiforov–Uvarov (NU) method,[29] and supersymmetric quantum mechanics (SUSYQM).[30,31] Here, we will focus on the Hulthén potential which plays a significant role in atom and molecule physics.[32] It has also been used to explain the electronic properties of some alkali halides.[33] Moreover, as it resembles the Coulomb interaction in structure, it has also been investigated within the framework of Dirac.[34] In the present paper, we will study the Dirac equation with shifted Hulthén potential with the spin–orbit coupling term quantum number κ for spin and pseudospin symmetries limit using the SUSYQM. The rest of this paper is organized as follows. In Section 2, the Dirac theory within the framework of spin and pseudospin symmetry limits are presented. Bound state solutions for the energy eigenvalues are presented in Section 3. Discussion about the numerical results is given in Section 4. Finally, some conclusions are drawn from the present study in Section 5.

2. Theory of Dirac equation for spin and pseudospin symmetry

In the relativistic units ($\hbar = c = 1$), the Dirac equation with both scalar $S(r)$ and vector $V(r)$ potentials reads

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta(M + S(r))]\psi(r) = [E - V(r)]\psi(r), \tag{1}$$

where E, M, and $\boldsymbol{p} = -\mathrm{i}\boldsymbol{\nabla}$, denote the energy, particle mass, and momentum operator, respectively. As usual, α and β are the 4×4 Pauli matrices

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2}$$

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where I is the unit matrix and σ_i represents the Pauli matrix. The total angular momentum J and $\hat{K} = -\beta(\alpha \cdot L + 1)$ commute with the Dirac Hamiltonian in a central field, with L being the orbital angular momentum. The eigenvalues of the $\hat{K}$ are $\kappa = -(j+1/2)$ for aligned spin and $\kappa = +(j+1/2)$ for unaligned spin. The wave function can then be classified according to the angular momentum j and the spin–orbit quantum number κ as

$$\psi_{nk}(r,\theta,\phi) = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r) & Y^{l}_{jm}(\theta,\phi) \\ iG_{n\kappa}(r) & Y^{\tilde{l}}_{jm}(\theta,\phi) \end{pmatrix}, \tag{3}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ respectively are the upper and the lower components of the wave function, $Y^{l}_{jm}(\theta,\phi)$ and $Y^{\tilde{l}}_{jm}(\theta,\phi)$ represent the spherical harmonics, n is the radial quantum number, and m is the projection of the angular momentum on the z axis. In addition, l and $\tilde{l}$ are respectively called the orbital angular momentum spin quantum number and orbital angular momentum pseudospin quantum number. The substitution of Eq. (3) into Eq. (1) yields two coupling equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)F_{n\kappa}(r) = [M + E_{n\kappa} - \Delta(r)]G_{n\kappa}(r), \tag{4}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)G_{n\kappa}(r) = [M - E_{n\kappa} + \Sigma(r)]F_{n\kappa}(r), \tag{5}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$. By eliminating $G_{n\kappa}(r)$ in favor of $F_{n\kappa}(r)$, we have

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) + \frac{\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)}{M + E_{n\kappa} - \Delta(r)}\right\}F_{n\kappa}(r) = 0, \tag{6}$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) - \frac{\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)}{M - E_{n\kappa} + \Sigma(r)}\right\}G_{n\kappa}(r) = 0. \tag{7}$$

3. Bound state solutions

In this section, we intend to study the bound-state solutions in the spin and pseudospin symmetry limit using the SUSYQM.

3.1. Spin symmetry limit

In the spin symmetry limit, $\mathrm{d}\Delta(r)/\mathrm{d}r = 0$ or $\Delta(r) = C_\mathrm{s} =$ constant, and equation (6) reduces to

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} - (M + E^\mathrm{s}_{n\kappa} - \Delta(r))(M - E^\mathrm{s}_{n\kappa} + \Sigma(r))\right\} \times F^\mathrm{s}_{n\kappa}(r) = 0, \tag{8}$$

where $\kappa = l$ for $\kappa < 0$ and $\kappa = -(l+1)$ for $\kappa > 0$. We take the sum potential $\Sigma(r)$ as the shifted Hulthén potential[34]

$$\Sigma(r) = \frac{V^\mathrm{s}_0 + 1/(2b^2)}{\mathrm{e}^{r/b} - 1} + \frac{V^\mathrm{s}_1}{(\mathrm{e}^{r/b} - 1)^2}, \tag{9}$$

which differs from the Hulthén potential by the second term on the right-hand side. In the potential relation, b is the range of the potential and V_0 represents the depth of the potential well. Bear in mind the outcomes of the Kratzer potential in the ad-hoc inverse square term for small distances. Substitution of the potential into Eq. (8) yields

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} - (M + E^\mathrm{s}_{n\kappa} - C_\mathrm{s})(M - E^\mathrm{s}_{n\kappa}) - (M + E^\mathrm{s}_{n\kappa} - C_\mathrm{s}) \times \left(\frac{V^\mathrm{s}_0 + 1/(2b^2)}{\mathrm{e}^{r/b} - 1} + \frac{V^\mathrm{s}_1}{(\mathrm{e}^{r/b} - 1)^2}\right)\right\}F^\mathrm{s}_{n\kappa}(r) = 0. \tag{10}$$

Equation (10) cannot be exactly solved due to the existence of the exponential and the centrifugal terms together. Therefore, we introduce the approximation (Fig. 1)[35]

$$\frac{1}{r^2} \approx b^2\left[d_0 + \frac{1}{\mathrm{e}^{r/b} - 1} + \frac{1}{(\mathrm{e}^{r/b} - 1)^2}\right]. \tag{11}$$

Inserting the latter into Eq. (10), we obtain

$$-\frac{\mathrm{d}^2 F^\mathrm{s}_{n\kappa}}{\mathrm{d}r^2} + V_\mathrm{eff}(r)F^\mathrm{s}_{n\kappa}(r) = \tilde{E}^\mathrm{s}_{n\kappa}F^\mathrm{s}_{n\kappa}(r), \tag{12}$$

where

$$V_\mathrm{eff}(r) = \frac{A^\mathrm{s}\,\mathrm{e}^{r/b} + B^\mathrm{s}}{(\mathrm{e}^{r/b} - 1)^2}, \tag{13}$$

$$A^\mathrm{s} = b^2\kappa(\kappa+1) + MV^\mathrm{s}_0 + V^\mathrm{s}_0 E^\mathrm{s}_{n\kappa} - V^\mathrm{s}_0 C_\mathrm{s} + \frac{M}{2b^2} + \frac{E^\mathrm{s}_{n\kappa}}{2b^2} - \frac{C_\mathrm{s}}{2b^2}, \tag{14}$$

$$B^\mathrm{s} = -MV^\mathrm{s}_0 - V^\mathrm{s}_0 E^\mathrm{s}_{n\kappa} + V^\mathrm{s}_0 C_\mathrm{s} - \frac{M}{2b^2} - \frac{E^\mathrm{s}_{n\kappa}}{2b^2} + \frac{C_\mathrm{s}}{2b^2} + MV^\mathrm{s}_1 + V^\mathrm{s}_1 E^\mathrm{s}_{n\kappa} - V^\mathrm{s}_1 C_\mathrm{s}, \tag{15}$$

with

$$\tilde{E}^\mathrm{s}_{n\kappa} = -d_0 b^2\kappa(\kappa+1) + E^{\mathrm{s}2}_{n\kappa} - M^2 + C_\mathrm{s}(M - E^\mathrm{s}_{n\kappa}). \tag{16}$$

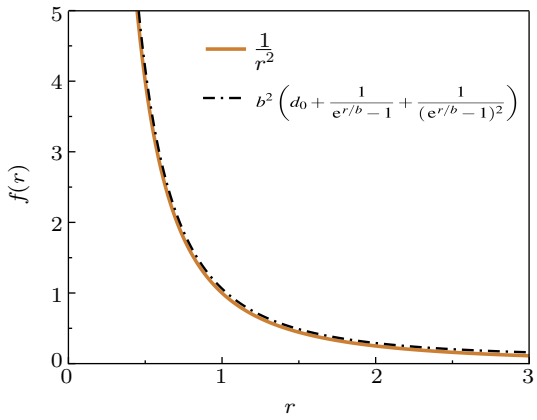


Fig. 1. (color online) The $(1/r^2)$ and its approximation for $b = 1$ and $d_0 = 0.1$.

In the SUSYQM formulation, the ground-state wave function $F^{\rm s}_{0,\kappa}(r)$ is given by

$$F^{\rm s}_{0,\kappa}(r)=\exp\Big(-\int W(r)\mathrm{d}r\Big), \tag{17}$$

in which the integrand is called the superpotential and the Hamiltonian is composed of the raising and lowering operators as follows:

$$H_-=\hat{A}^+\hat{A}=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+V_-(r), \tag{18a}$$

$$H_+=\hat{A}\hat{A}^+=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+V_+(r), \tag{18b}$$

with

$$\hat{A}=\frac{\mathrm{d}}{\mathrm{d}r}-W(r), \tag{19}$$

$$\hat{A}^+=-\frac{\mathrm{d}}{\mathrm{d}r}-W(r), \tag{20}$$

$$V_\pm(r)=W^2(r)\mp W'(r). \tag{21}$$

Thus, we have to first solve the associated Riccati equation

$$W^2(r)\mp W'(r)=V_{\rm eff}(r)-\tilde{E}^{\rm s}_{0,\kappa}, \tag{22}$$

for which we propose a solution in the following form:

$$W(r)=p^{\rm s}-\frac{q^{\rm s}}{(\mathrm{e}^{r/b}-1)}. \tag{23}$$

From Eqs. (13) and (21), we can obtain the exact parameters of our study as

$$\begin{aligned}&(p^{\rm s})^2+\frac{(q^{\rm s})^2}{(\mathrm{e}^{r/b}-1)^2}-\frac{2p^{\rm s}q^{\rm s}}{(\mathrm{e}^{r/b}-1)}-\frac{(q^{\rm s}/b)\mathrm{e}^{r/b}}{(\mathrm{e}^{r/b}-1)^2}\\&=\frac{A^{\rm s}\mathrm{e}^{r/b}+B^{\rm s}}{(\mathrm{e}^{2r/b}-1)^2}-\tilde{E}^{\rm s}_{0,\kappa},\end{aligned} \tag{24}$$

or more explicitly,

$$\tilde{E}^{\rm s}_{0,\kappa}=-(p^{\rm s})^2, \tag{25}$$

$$q^{\rm s}=\frac{1+\sqrt{1+4b^2(A^{\rm s}+B^{\rm s})}}{2b}, \tag{26}$$

$$p^{\rm s}=\frac{(-q^{\rm s}/b-A^{\rm s})}{2q^{\rm s}}. \tag{27}$$

From Eqs. (13)–(21) and (25)–(27), we can obtain the supersymmetric partner potentials as

$$\begin{aligned}V_+(r)=&\frac{q^{\rm s}(q^{\rm s}+1/b)\mathrm{e}^{r/b}}{(\mathrm{e}^{r/b}-1)^2}+\frac{-(q^{\rm s})^2-2p^{\rm s}q^{\rm s}}{(\mathrm{e}^{r/b}-1)}\\&+\Big(\frac{-q^{\rm s}/b-A^{\rm s}}{2q^{\rm s}}\Big)^2,\end{aligned} \tag{28}$$

$$\begin{aligned}V_-(r)=&\frac{q^{\rm s}(q^{\rm s}-1/b)\mathrm{e}^{r/b}}{(\mathrm{e}^{r/b}-1)^2}+\frac{-(q^{\rm s})^2-2p^{\rm s}q^{\rm s}}{(\mathrm{e}^{r/b}-1)}\\&+\Big(\frac{-q^{\rm s}/b-A^{\rm s}}{2q^{\rm s}}\Big)^2.\end{aligned} \tag{29}$$

It is not difficult to see that $V_+(r)$ and $V_-(r)$ satisfy the shape-invariant condition

$$V_+(r,a_0)=V_-(r,a_1)+R(a_1), \tag{30}$$

where $a_0=q^{\rm s}$ and a_i is a function of a_0, i.e., $a_1=f(a_0)=a_0+1/b$. Therefore, $a_n=f(a_0)=a_0+n/b$ and we can see that the shape invariance holds via a mapping of the form $q^{\rm s}\to q^{\rm s}+1/b$. From Eq. (30), we have

$$\begin{aligned}R(a_1)&=\Big(\frac{-a_0/b-A^{\rm s}}{2a_0}\Big)^2-\Big(\frac{-a_1/b-A^{\rm s}}{2a_1}\Big)^2,\\R(a_2)&=\Big(\frac{-a_1/b-A^{\rm s}}{2a_1}\Big)^2-\Big(\frac{-a_2/b-A^{\rm s}}{2a_2}\Big)^2,\\R(a_n)&=\Big(\frac{-a_{n-1}/b-A^{\rm s}}{2a_{n-1}}\Big)^2-\Big(\frac{-a_n/b-A^{\rm s}}{2a_n}\Big)^2.\end{aligned} \tag{31}$$

On the other hand, the energy eigenvalues can be obtained as follows:

$$\tilde{E}^{\rm s}_{n\kappa}=\tilde{E}^{\rm s-}_{n\kappa}+\tilde{E}^{\rm s}_{0,\kappa}, \tag{32}$$

where

$$\tilde{E}^{\rm s-}_{n\kappa}=\sum_{k=1}^{n}R(a_k)=\Big(\frac{-a_0/b-A^{\rm s}}{2a_0}\Big)^2-\Big(\frac{-a_n/b-A^{\rm s}}{2a_n}\Big)^2. \tag{33}$$

Substituting Eqs. (16), (23), and (33) into Eq. (32), we obtain a complicated transcendental energy equation for the shifted Hulthén potential as

$$\begin{aligned}&-d_0b^2\kappa(\kappa+1)+E^{\rm s2}_{n\kappa}-M^2+C_{\rm s}(M-E^{\rm s}_{n\kappa})\\&+\left(\frac{-\Big(\dfrac{1+\sqrt{1+4b^2(A^{\rm s}+B^{\rm s})}}{2b}+\dfrac{n}{b}\Big)-A^{\rm s}b}{1+\sqrt{1+4b^2(A^{\rm s}+B^{\rm s})}+2n}\right)^2=0,\end{aligned} \tag{34}$$

where the corresponding upper component is

$$\begin{aligned}F^{\rm s}_{n\kappa}(r)=&N_{n\kappa}(\mathrm{e}^{-r/b})^{\sqrt{w^{\rm s}_3}}(1-\mathrm{e}^{-r/b})^{1/2+\sqrt{w^{\rm s}_1-w^{\rm s}_2+w^{\rm s}_3+1/4}}\\&\times{}_2\mathrm{F}_1\Big(-n,n+2\sqrt{w^{\rm s}_3}+2\sqrt{w^{\rm s}_1-w^{\rm s}_2+w^{\rm s}_3+1/4}+1,\\&2\sqrt{w^{\rm s}_3}+1,\mathrm{e}^{-r/b}\Big),\end{aligned} \tag{35}$$

with

$$\begin{aligned}w^{\rm s}_1=&b^2\Big\{d_0b^2\kappa(\kappa+1)+(M+E^{\rm s}_{n\kappa}-C_{\rm s})\\&\times\Big[M-E^{\rm s}_{n\kappa}-V^{\rm s}_0-\frac{1}{2b^2}+V^{\rm s}_1\Big]\Big\},\\w^{\rm s}_2=&b^2\Big\{2d_0b^2\kappa(\kappa+1)-b^2\kappa(\kappa+1)+(M+E^{\rm s}_{n\kappa}-C_{\rm s})\\&\times\Big[2(M-E^{\rm s}_{n\kappa})-V^{\rm s}_0-\frac{1}{2b^2}\Big]\Big\},\\w^{\rm s}_3=&b^2\{d_0b^2\kappa(\kappa+1)+(M+E^{\rm s}_{n\kappa}-C_{\rm s})(M-E^{\rm s}_{n\kappa})\},\end{aligned} \tag{36}$$

and $N_{n\kappa}$ is the normalization constant. For the lower component, we can simply use

$$G^{\rm s}_{n\kappa}(r)=\frac{1}{M+E^{\rm s}_{n\kappa}-C_{\rm s}}\Big(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}\Big)F^{\rm s}_{n\kappa}(r). \tag{37}$$

3.2. Pseudospin symmetry limit

In the pseudospin symmetry limit, $\mathrm{d}\Sigma(r)/\mathrm{d}r = 0$ or $\Sigma(r) = C_{\rm ps}$ = constant, and equation (7) reduces to

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - (M + E^{\rm ps}_{n\kappa} - \Delta(r))(M - E^{\rm ps}_{n\kappa} + \Sigma(r))\Big\} \times G^{\rm ps}_{n\kappa}(r) = 0, \tag{38}$$

where $\kappa = -\tilde{\ell}$ and $\kappa = \tilde{\ell}+1$ for $\kappa < 0$ and $\kappa > 0$, respectively. As mentioned in the previous section, we work on

$$\Delta(r) = \frac{V^{\rm ps}_0 + 1/(2b^2)}{\mathrm{e}^{r/b}-1} + \frac{V^{\rm ps}_1}{(\mathrm{e}^{r/b}-1)^2}. \tag{39}$$

Substituting Eq. (39) into Eq. (38) we arrive at

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - \Big(M + E^{\rm ps}_{n\kappa} - \frac{V^{\rm ps}_0 + 1/(2b^2)}{\mathrm{e}^{r/b}-1} - \frac{V^{\rm ps}_1}{(\mathrm{e}^{r/b}-1)^2}\Big)(M - E^{\rm ps}_{n\kappa} + C_{\rm ps})\Big\} G^{\rm ps}_{n\kappa}(r) = 0, \tag{40}$$

which can be more neatly written as

$$-\frac{\mathrm{d}^2 G^{\rm ps}_{n\kappa}}{\mathrm{d}r^2} + V_{\rm eff}(r) G^{\rm ps}_{n\kappa}(r) = \tilde{E}^{\rm ps}_{n\kappa} G^{\rm ps}_{n\kappa}(r), \tag{41}$$

with

$$V_{\rm eff}(r) = \frac{A^{\rm ps}\,\mathrm{e}^{r/b} + B^{\rm ps}}{(\mathrm{e}^{r/b}-1)^2}, \tag{42}$$

$$A^{\rm ps} = b^2\kappa(\kappa-1) - MV^{\rm ps}_0 + V^{\rm ps}_0 E^{\rm ps}_{n\kappa} - V^{\rm ps}_0 C_{\rm ps} - \frac{M}{2b^2} + \frac{E^{\rm ps}_{n\kappa}}{2b^2} - \frac{C_{\rm ps}}{2b^2}, \tag{43}$$

$$B^{\rm ps} = MV^{\rm ps}_0 - V^{\rm ps}_0 E^{\rm ps}_{n\kappa} + V^{\rm ps}_0 C_{\rm ps} + \frac{M}{2b^2} - \frac{E^{\rm ps}_{n\kappa}}{2b^2} + \frac{C_{\rm ps}}{2b^2} - MV^{\rm ps}_1 + V^{\rm ps}_1 E^{\rm ps}_{n\kappa} - V^{\rm ps}_1 C_{\rm ps}. \tag{44}$$

The corresponding effective energy in this case is

$$\tilde{E}^{\rm ps}_{n\kappa} = -d_0 b^2\kappa(\kappa-1) + (E^{\rm ps}_{n\kappa})^2 - M^2 + C_{\rm ps}(-M - E^{\rm ps}_{n\kappa}), \tag{45}$$

and the superpotential possess' the form

$$W^{\rm ps}(r) = p^{\rm ps} - \frac{q^{\rm ps}}{(\mathrm{e}^{r/b}-1)}. \tag{46}$$

In this case, the coefficients are explicitly given as

$$\tilde{E}^{\rm ps}_{0,\kappa} = -(p^{\rm ps})^2, \tag{47}$$

$$q^{\rm ps} = \frac{1 + \sqrt{1 + 4b^2(A^{\rm ps} + B^{\rm ps})}}{2b}, \tag{48}$$

$$p^{\rm ps} = \frac{(-q^{\rm ps}/b - A^{\rm ps})}{2q^{\rm ps}}. \tag{49}$$

From Eqs. (42)–(44) and (47)–(49), we can obtain the supersymmetric partner potentials as

$$V_+(r) = \frac{q^{\rm ps}(q^{\rm ps} + 1/b)\,\mathrm{e}^{r/b}}{(\mathrm{e}^{r/b}-1)^2} + \frac{-(q^{\rm ps})^2 - 2p^{\rm ps}q^{\rm ps}}{(\mathrm{e}^{r/b}-1)} + \Big(\frac{-q^{\rm ps}/b - A^{\rm ps}}{2q^{\rm ps}}\Big)^2, \tag{50}$$

$$V_-(r) = \frac{q^{\rm ps}(q^{\rm ps} - 1/b)\,\mathrm{e}^{r/b}}{(\mathrm{e}^{r/b}-1)^2} + \frac{-(q^{\rm ps})^2 - 2p^{\rm ps}q^{\rm ps}}{(\mathrm{e}^{r/b}-1)} + \Big(\frac{-q^{\rm ps}/b - A^{\rm ps}}{2q^{\rm ps}}\Big)^2 \tag{51}$$

with the corresponding mapping being

$$\rho_n = f(\rho_0) = \rho_0 + \frac{n}{b} = q^{\rm ps} + \frac{n}{b}. \tag{52}$$

In other words,

$$R(\rho_1) = \Big(\frac{-\rho_0/b - A^{\rm ps}}{2\rho_0}\Big)^2 - \Big(\frac{-\rho_1/b - A^{\rm ps}}{2\rho_1}\Big)^2,$$
$$R(\rho_2) = \Big(\frac{-\rho_1/b - A^{\rm ps}}{2\rho_1}\Big)^2 - \Big(\frac{-\rho_2/b - A^{\rm ps}}{2\rho_2}\Big)^2,$$
$$R(\rho_n) = \Big(\frac{-\rho_{n-1}/b - A^{\rm ps}}{2\rho_{n-1}}\Big)^2 - \Big(\frac{-\rho_n/b - A^{\rm ps}}{2\rho_n}\Big)^2, \tag{53}$$

and

$$\tilde{E}^{\rm ps}_{n\kappa} = \tilde{E}^{\rm ps-}_{n\kappa} + \tilde{E}^{\rm ps}_{0,\kappa}, \tag{54}$$

where

$$\tilde{E}^{\rm ps-}_{n\kappa} = \sum_{k=1}^{n} R(\rho_k) = \Big(\frac{-\rho_0/b - A^{\rm ps}}{2\rho_0}\Big)^2 - \Big(\frac{-\rho_n/b - A^{\rm ps}}{2\rho_n}\Big)^2. \tag{55}$$

Finally, the energy relation is written as

$$-d_0 b^2\kappa(\kappa-1) + (E^{\rm ps}_{n\kappa})^2 - M^2 + C_{\rm ps}(-M - E^{\rm ps}_{n\kappa}) + \left(\frac{-\Big(\frac{1 + \sqrt{1 + 4b^2(A^{\rm ps} + B^{\rm ps})}}{2b}\Big) - \frac{n}{b} - A^{\rm ps}b}{1 + \sqrt{1 + 4b^2(A^{\rm ps} + B^{\rm ps})} + 2n}\right)^2 = 0, \tag{56}$$

and the corresponding lower component of the wave function becomes

$$G^{\rm ps}_{n\kappa}(r) = N_{n\kappa}(\mathrm{e}^{-r/b})^{\sqrt{w^{\rm ps}_3}}(1 - \mathrm{e}^{-r/b})^{1/2 + \sqrt{w^{\rm ps}_1 - w^{\rm ps}_2 + w^{\rm ps}_3 + 1/4}} \times {}_2\mathrm{F}_1\Big(-n, n + 2\sqrt{w^{\rm ps}_3} + 2\sqrt{w^{\rm ps}_1 - w^{\rm ps}_2 + w^{\rm ps}_3 + 1/4} + 1, 2\sqrt{w^{\rm ps}_3} + 1, \mathrm{e}^{-r/b}\Big), \tag{57}$$

where

$$w^{\rm ps}_1 = b^2\Big\{d_0 b^2\kappa(\kappa-1) + (M - E^{\rm ps}_{n\kappa} + C_{\rm ps}) \times \Big[M + E^{\rm ps}_{n\kappa} + V^{\rm ps}_0 + \frac{1}{2b^2} - V^{\rm ps}_1\Big]\Big\},$$
$$w^{\rm ps}_2 = b^2\Big\{2d_0 b^2\kappa(\kappa-1) - b^2\kappa(\kappa-1) + (M - E^{\rm ps}_{n\kappa} + C_{\rm ps}) \times \Big[2(M + E^{\rm ps}_{n\kappa}) + V^{\rm ps}_0 + \frac{1}{2b^2}\Big]\Big\},$$
$$w^{\rm ps}_3 = b^2\big\{d_0 b^2\kappa(\kappa-1) + (M - E^{\rm ps}_{n\kappa} + C_{\rm ps})(M + E^{\rm ps}_{n\kappa})\big\}. \tag{58}$$

The upper component can be found by using the following relation:

$$F^{\rm ps}_{n\kappa}(r) = \frac{1}{(E^{\rm ps}_{n\kappa} - M + C_{\rm ps})}\Big(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\Big) G^{\rm ps}_{n\kappa}(r). \tag{59}$$

4. Numerical results

We calculate the energy eigenvalues of the Dirac equation in spin and pseudospin symmetry limits in the presence of

the shifted Hulthén potential for various values of the quantum numbers n and κ. The results are reported in Tables 1 and 2. We can see that the detailed degenerate states in the spin symmetry limit are as follows: $(n\mathrm{p}_{1/2}, n\mathrm{p}_{3/2})$ for $\ell = 1$, $(n\mathrm{d}_{3/2}, n\mathrm{d}_{5/2})$ for $\ell = 2$, $(n\mathrm{f}_{5/2}, n\mathrm{f}_{7/2})$ for $\ell = 3$, $(n\mathrm{g}_{7/2}, n\mathrm{g}_{9/2})$ for $\ell = 4$, etc. For the pseudospin symmetry limit one can clearly see that the degeneracy occurs in $n\mathrm{s}_{1/2}, (n-1)\mathrm{d}_{3/2}$ for $\tilde{\ell} = 1$ $(\ell = 0)$, $n\mathrm{p}_{3/2}, (n-1)\mathrm{f}_{5/2}$ for $\tilde{\ell} = 2$ $(\ell = 1)$, $n\mathrm{d}_{5/2}, (n-1)\mathrm{g}_{7/2}$ for $\tilde{\ell} = 3$ $(\ell = 2)$, $n\mathrm{f}_{7/2}, (n-1)\mathrm{h}_{9/2}$ for $\tilde{\ell} = 4$ $(\ell = 3)$, etc. In Figs. 2 and 3, we show the effects of the potential parameters V_0 and V_1 on a bound state, respectively. To reveal the effect of the shift parameter b, we plot the energy eigenvalues versus b for some quantum numbers in Fig. 4. Figures 5 and 6 show the dependences of the bound-state energy level on parameters C_{ps}, C_{s}, and M, respectively. In Fig. 7, we portray the behaviors of energy versus d_0. In Fig. 8, the wave functions are plotted for both $\kappa < 0$ and $\kappa > 0$. Figure 9 shows the effect of the potential parameter of V_1 on the wave function behavior.

Table 1. Energies in the spin symmetry limit for $M = 5$, $V_0 = -5$, $C_{\mathrm{s}} = 2$, $b = 1.01$. The unit 1 fm $= 10^{-15}$ m.

ℓ	$n, \kappa < 0$	(ℓ, j)	$E^{\mathrm{s}}_{n\kappa}$/fm^{-1} $V_1 = 0$	$E^{\mathrm{s}}_{n\kappa}$/fm^{-1} $V_1 = 2$	$n, \kappa > 0$	(ℓ, j)	$E^{\mathrm{s}}_{n\kappa}$/fm^{-1} $V_1 = 0$	$E^{\mathrm{s}}_{n\kappa}$/fm^{-1} $V_1 = 2$
1	$0,-2$	$0\mathrm{p}_{3/2}$	1.452641383	3.943617642	0,1	$0\mathrm{p}_{1/2}$	1.452641383	3.943617642
2	$0,-3$	$0\mathrm{d}_{5/2}$	3.414598024	4.436751147	0,2	$0\mathrm{d}_{3/2}$	3.414598024	4.436751147
3	$0,-4$	$0\mathrm{f}_{7/2}$	4.521744262	4.867384755	0,3	$0\mathrm{f}_{5/2}$	4.521744262	4.867384755
4	$0,-5$	$0\mathrm{g}_{9/2}$	5.100935159	5.179201689	0,4	$0\mathrm{g}_{7/2}$	5.100935159	5.179201689
1	$1,-2$	$1\mathrm{p}_{3/2}$	2.856964322	4.323219777	1,1	$1\mathrm{p}_{1/2}$	2.856964322	4.323219777
2	$1,-3$	$1\mathrm{d}_{5/2}$	4.047623567	4.648967685	1,2	$1\mathrm{d}_{3/2}$	4.047623567	4.648967685
3	$1,-4$	$1\mathrm{f}_{7/2}$	4.754136607	4.960384067	1,3	$1\mathrm{f}_{5/2}$	4.754136607	4.960384067
4	$1,-5$	$1\mathrm{g}_{9/2}$	5.158499939	5.204640613	1,4	$1\mathrm{g}_{7/2}$	5.158499939	5.204640613
1	$2,-2$	$2\mathrm{p}_{3/2}$	3.663803907	4.542542428	2,1	$2\mathrm{p}_{1/2}$	3.663803907	4.542542428
2	$2,-3$	$2\mathrm{d}_{5/2}$	4.408405099	4.778533795	2,2	$2\mathrm{d}_{3/2}$	4.408405099	4.778533795
3	$2,-4$	$2\mathrm{f}_{7/2}$	4.889727493	5.018979039	2,3	$2\mathrm{f}_{5/2}$	4.889727493	5.018979039
4	$2,-5$	$2\mathrm{g}_{9/2}$	5.192042069	5.220380174	2,4	$2\mathrm{g}_{7/2}$	5.192042069	5.220380174

Table 2. Energies in the pseudospin symmetry limit for $M = 5$, $V_0 = 5$, $C_{\mathrm{ps}} = -2$, $b = 1.01$, and $d_0 = 0.1$.

$E^{\mathrm{ps}}_{n\kappa}$/fm^{-1} $V_1 = -2$	$E^{\mathrm{ps}}_{n\kappa}$/fm^{-1} $V_1 = 0$	$(\ell+2, j+1)$	$n-1, \kappa > 0$	$E^{\mathrm{ps}}_{n\kappa}$/fm^{-1} $V_1 = -2$	$E^{\mathrm{ps}}_{n\kappa}$/fm^{-1} $V^{\mathrm{ps}}_1 = 0$	(ℓ, j)	$n, \kappa < 0$	$\tilde{\ell}$
−3.918599886	−2.130007071	$0\mathrm{d}_{3/2}$	0,2	−3.918599886	−2.130007071	$1\mathrm{s}_{1/2}$	$1,-1$	1
−4.349493078	−3.517691395	$0\mathrm{f}_{5/2}$	0,3	−4.349493078	−3.517691395	$1\mathrm{p}_{3/2}$	$1,-2$	2
−4.764769243	−4.423608183	$0\mathrm{g}_{7/2}$	0,4	−4.764769243	−4.423608183	$1\mathrm{d}_{5/2}$	$1,-3$	3
−5.102818246	−4.991615988	$0\mathrm{h}_{9/2}$	0,5	−5.102818246	−4.991615988	$1\mathrm{f}_{7/2}$	$1,-4$	4
−4.248384478	−3.088611672	$1\mathrm{d}_{3/2}$	1,2	−4.248384478	−3.088611672	$2\mathrm{s}_{1/2}$	$2,-1$	1
−4.555901649	−4.013331131	$1\mathrm{f}_{5/2}$	1,3	−4.555901649	−4.013331131	$2\mathrm{p}_{3/2}$	$2,-2$	2
−4.872099148	−4.647428237	$1\mathrm{g}_{7/2}$	1,4	−4.872099148	−4.647428237	$2\mathrm{d}_{5/2}$	$2,-3$	3
−5.145103440	−5.071848745	$1\mathrm{h}_{9/2}$	1,5	−5.145103440	−5.071848745	$2\mathrm{f}_{7/2}$	$2,-4$	4
−4.458221134	−3.69332198	$2\mathrm{d}_{3/2}$	2,2	−4.458221134	−3.69332198	$3\mathrm{s}_{1/2}$	$3,-1$	1
−4.692259282	−4.327948071	$2\mathrm{f}_{5/2}$	2,3	−4.692259282	−4.327948071	$3\mathrm{p}_{3/2}$	$3,-2$	2
−4.944843292	−4.792038156	$2\mathrm{g}_{7/2}$	2,4	−4.944843292	−4.792038156	$3\mathrm{d}_{5/2}$	$3,-3$	3
−5.173898974	−5.124131414	$2\mathrm{h}_{9/2}$	2,5	−5.173898974	−5.124131414	$3\mathrm{f}_{7/2}$	$3,-4$	4

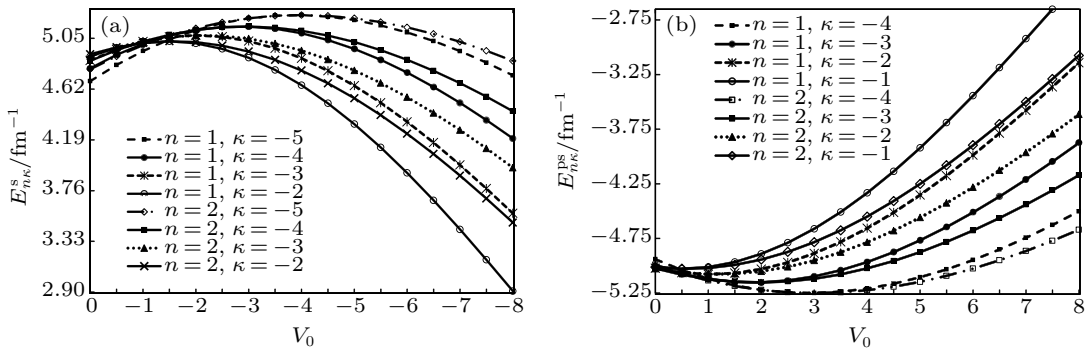


Fig. 2. (a) Curves of energy versus V_0 for the spin symmetry limit for $M = 5$, $V_1 = 2$, $C_{\mathrm{s}} = 2$, $b = 1.01$, and $d_0 = 0.1$. (b) Curves of energy versus V_0 for the pseudospin symmetry limit for $M = 5$, $V_1 = -2$, $C_{\mathrm{ps}} = -2$, $b = 1.01$, and $d_0 = 0.1$.

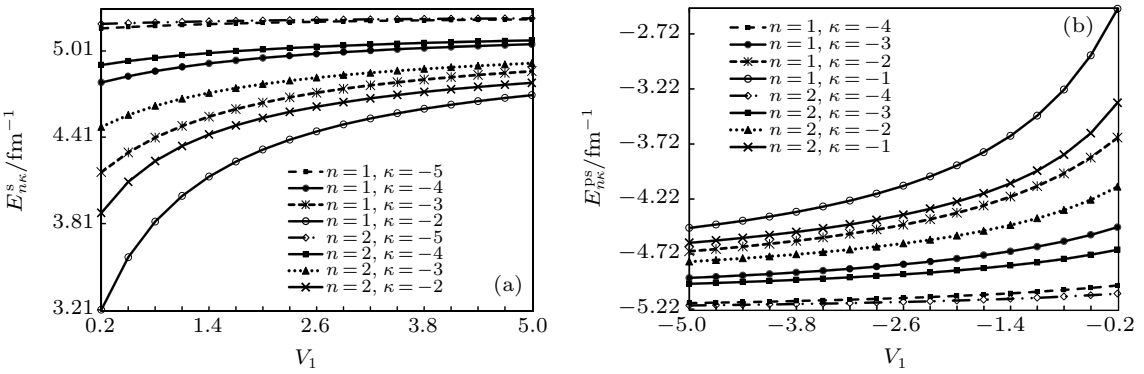


Fig. 3. (a) Curves of energy versus V_1 for the spin symmetry limit for $M = 5$, $V_0 = -5$, $C_s = 2$, $b = 1.01$, and $d_0 = 0.1$. (b) Curves of energy versus V_1 for the pseudospin symmetry limit for $M = 5$, $V_0 = 5$, $C_{ps} = -2$, $b = 1.01$, and $d_0 = 0.1$.

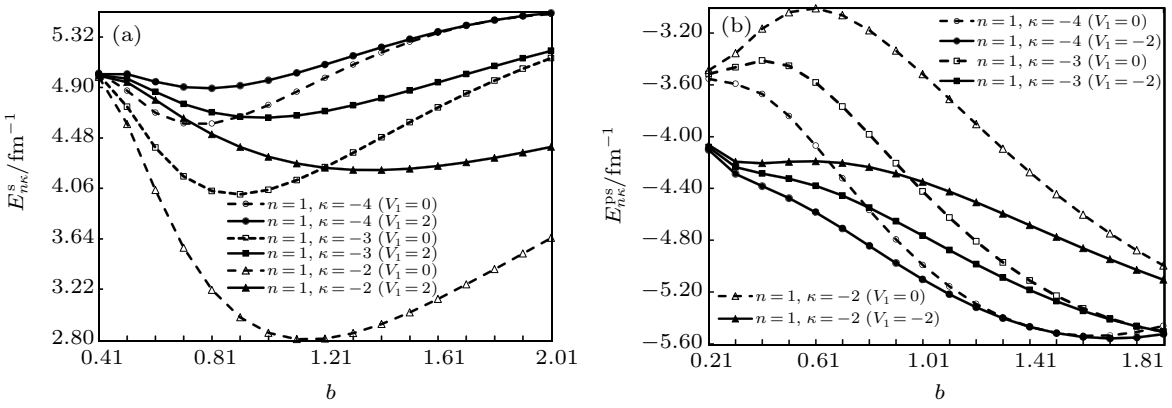


Fig. 4. (a) Curves of energy versus b for the spin symmetry limit for $M = 5$, $V_0 = -5$, $C_s = 2$, and $d_0 = 0.1$. (b) Curves of energy versus b for the pseudospin symmetry limit for $M = 5$, $V_0 = 5$, $C_{ps} = -2$, and $d_0 = 0.1$.

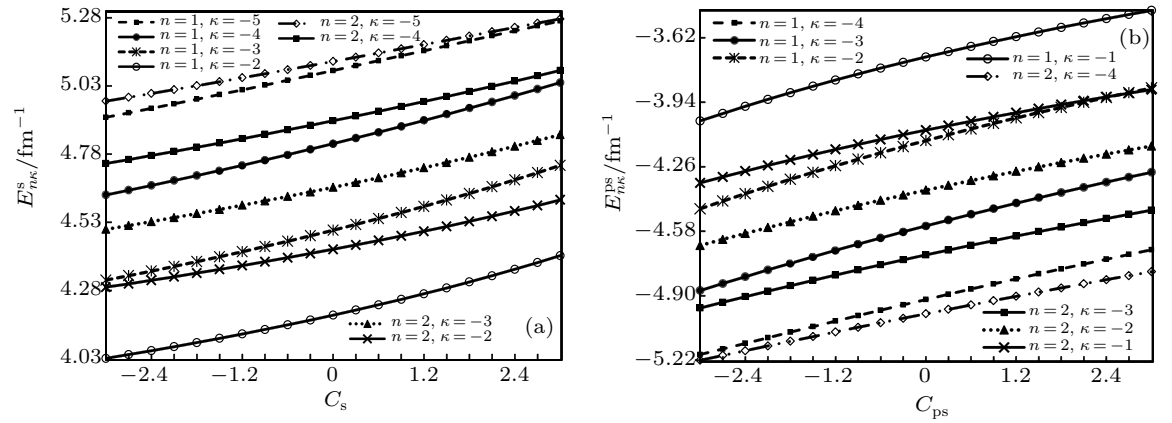


Fig. 5. (a) Curves of energy versus C_s for the spin symmetry limit for $M = 5$, $V_0 = -5$, $V_1 = 2$, $b = 1.01$, $d_0 = 0.1$. (b) Curves of energy versus C_{ps} for the pseudospin symmetry limit for $M = 5$, $V_0 = 5$, $V_1 = -2$, $b = 1.01$, $d_0 = 0.1$.

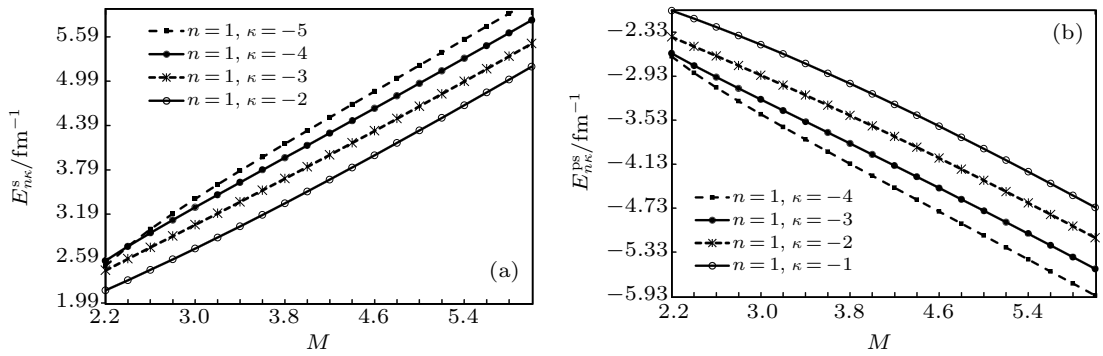


Fig. 6. (a) Curves of energy versus M for the spin symmetry limit for $V_0 = -5$, $V_1 = 2$, $C_s = 2$, $b = 1.01$, and $d_0 = 0.1$. (b) Curves of energy versus M for the pseudospin symmetry limit for $V_0 = 5$, $V_1 = -2$, $C_{ps} = -2$, $b = 1.01$, and $d_0 = 0.1$.

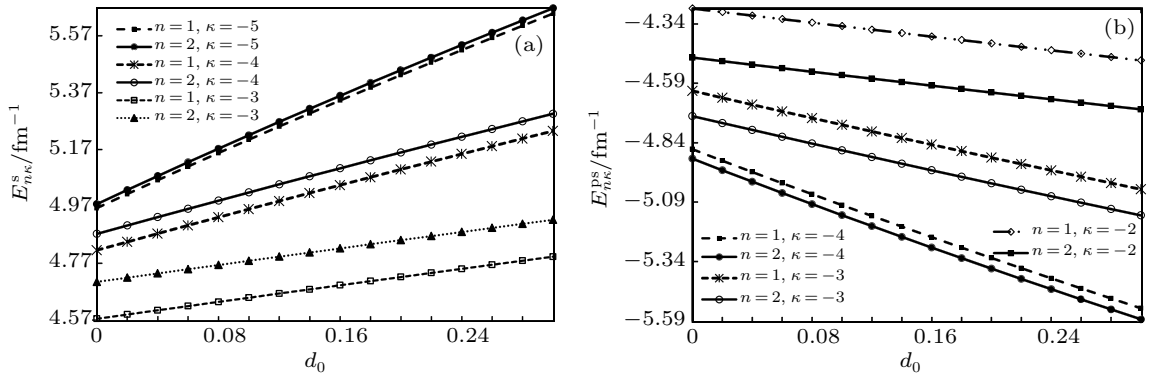


Fig. 7. (a) Curves of energy versus d_0 for the spin symmetry limit for $M=5$, $V_0=-5$, $V_1=2$, $C_{\rm s}=2$, and $b=1.01$. (b) Curves of energy versus d_0 for the pseudospin symmetry limit for $M=5$, $V_0=5$, $V_1=-2$, $C_{\rm ps}=-2$, and $b=1.01$.

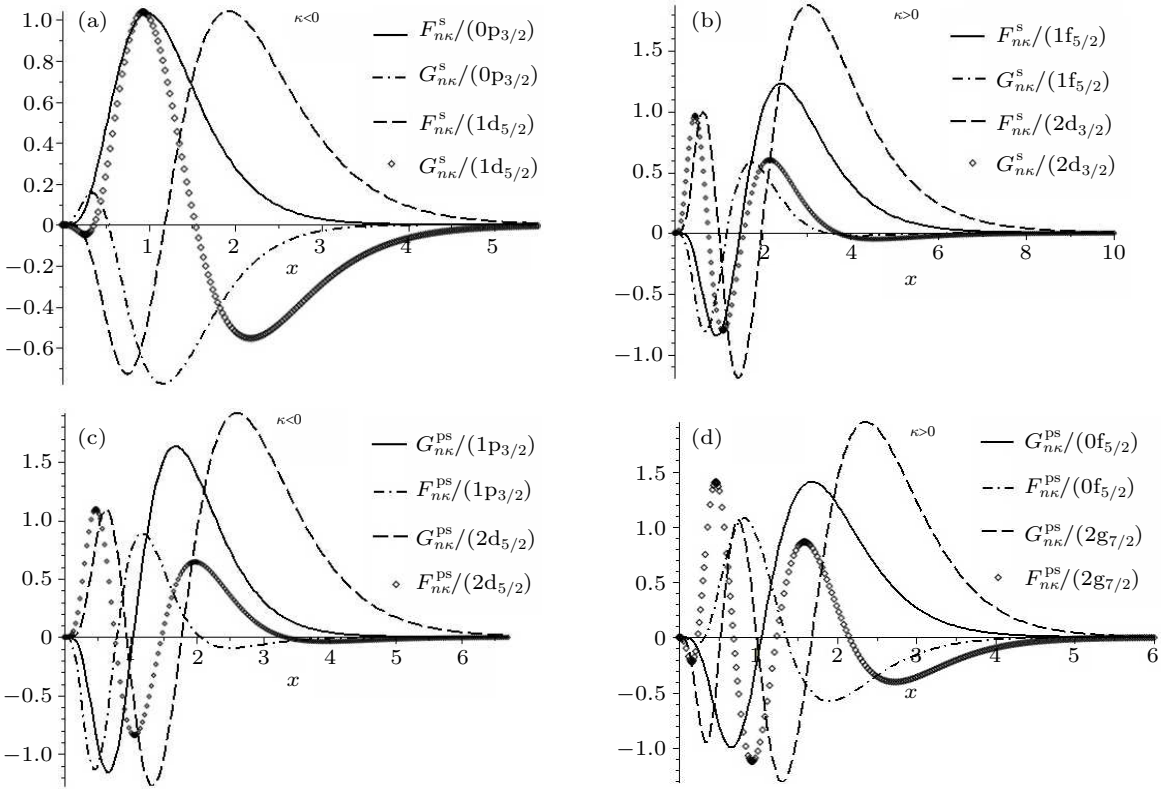


Fig. 8. (a) and (b) Lower and upper radial wave functions in view of the spin symmetry for $M=5$, $V_0=-5$, $V_1=2$, $C_{\rm s}=2$, $b=1.01$, and $d_0=0.1$. (c) and (d) Lower and upper radial wave functions in view of the pseudospin symmetry for $M=5$, $V_0=5$, $V_1=-2$, $C_{\rm ps}=-2$, $b=1.01$, and $d_0=0.1$.

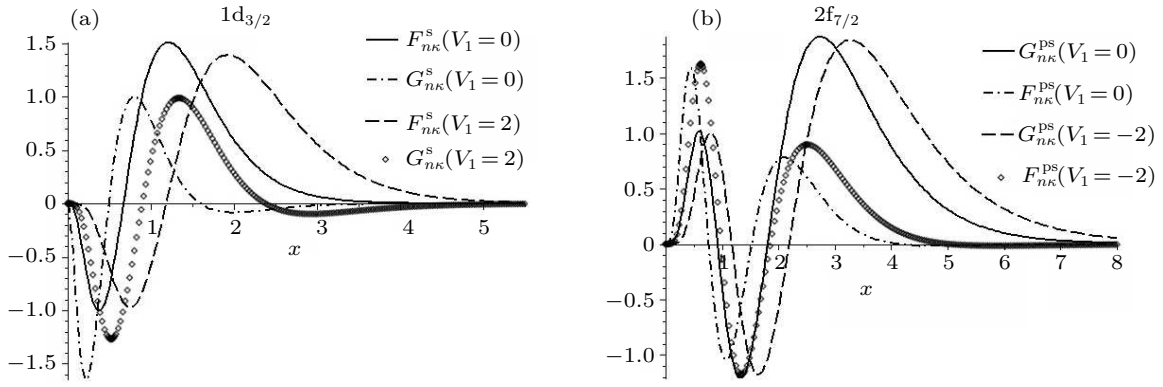


Fig. 9. (a) Lower and upper radial wave functions in view of the spin symmetry for $M=5$, $V_0=-5$, $C_{\rm s}=2$, $b=1.01$, and $d_0=0.1$. (b) Lower and upper radial wave functions in view of the pseudospin symmetry for $M=5$, $V_0=5$, $C_{\rm ps}=-2$, $b=1.01$, and $d_0=0.1$.

5. Conclusions

In this study, we investigated the approximate analytical spin and pseudospin symmetric solutions of the Dirac theory with the shifted Hulthén potential which includes the ad-hoc inverse square term. To obtain any l-state solution of the problem, we apply a proper approximation to the centrifugal term. We use the powerful supersymmetry quantum mechanics and the shape invariance condition in the calculations. Therefore, the wave functions are obtained in terms of the hypergeometric function.

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