New approach to $Q-P$ $(P-Q)$ ordering of quantum mechanical operators and its applications*

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In this paper, we introduce a new way to obtain the $Q-P$ $(P-Q)$ ordering of quantum mechanical operators, i.e., from the classical correspondence of $Q-P$ $(P-Q)$ ordered operators by replacing q and p with coordinate and momentum operators, respectively. Some operator identities are derived concisely. As for its applications, the single (two-) mode squeezed operators and Fresnel operator are examined. It is shown that the classical correspondence of Fresnel operator's $Q-P$ $(P-Q)$ ordering is just the integration kernel of Fresnel transformation. In addition, a new photo-counting formula is constructed by the $Q-P$ ordering of operators.

Keywords: $Q-P$ $(P-Q)$ ordering, Fresnel operator, photo-counting

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1. Introduction

In quantum mechanics and quantum optics theory, operators have been considered as an important characteristic by endowing with the basic commutative relation, such as $[Q,P]=\mathrm{i}\hbar$ for the coordinate and the momentum operators.[1] In order to deal with a lot of problems involved in quantum optics, say calculating some averages of physical quality or various matrix elements, the operator ordering is often fallen back on due to its convenience. For example, the normal ordering is used to calculate the averages in coherent states and the P-representation can be obtained directly by the antinormal ordering operators.[2] Apart from these operator ordering forms, some others are proposed, such as Weyl ordering of operators,[3] s-parameter ordering of operators,[4] and the Q-ordering and P-ordering.[5–7]

Recently, the technique of integration within $Q-P$ ordering and $P-Q$ ordering has been introduced,[8] and a method has been presented for putting operators into their $Q-P$ ordering ($\mathfrak{Q}$-ordering) and $P-Q$ ordering ($\mathfrak{P}$-ordering). Different from this method, in this paper, we shall introduce a new way to obtain the $Q-P$ $(P-Q)$ ordering, i.e., from the classical correspondence of $Q-P$ $(P-Q)$ ordered operators by replacing q and p with coordinate and momentum operators, respectively. In this way, some operator identities can be obtained more concisely. As for its applications, the single (two-) mode squeezed operators and Fresnel operator are examined, and a new photo-counting formula is derived by the $Q-P$ ordering of operators.

This paper is arranged as follows. In Section 2, we shall introduce the new approach for obtaining $\mathfrak{Q}$-($\mathfrak{P}$)-ordering of operators. Section 3 is used to derive the $\mathfrak{Q}$- and $\mathfrak{P}$-ordering of squeezing operators, including the single-mode squeezed operator and the two-mode squeezed operator, as well as the Fresnel operator. It is shown that our approach is convenient. Section 4 is devoted to deriving a new photo-counting formula on the basis of the $\mathfrak{Q}$-ordering. Our conclusions are involved in Section 5.

2. Approach to $\mathfrak{Q}$-($\mathfrak{P}$)-ordering of operators

In this section, we introduce our approach to $\mathfrak{Q}$-($\mathfrak{P}$)-ordering of operators, where the symbol $\mathfrak{Q}(P)$ denotes the ordering when all $Q(P)$ stand on the left of all $P(Q)$, respectively. Using the completeness relation of the coordinate and momentum eigenvector, and $\langle q|\,p\rangle=\mathrm{e}^{\mathrm{i}qp}/\sqrt{2\pi}$, one has

$$\begin{aligned}1 &= \int \mathrm{d}q\,|q\rangle\langle q|\int \mathrm{d}p\,|p\rangle\langle p|\\ &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}q\mathrm{d}p\,|q\rangle\langle p|\,\mathrm{e}^{\mathrm{i}qp}.\end{aligned}\tag{1}$$

Thus for the arbitrary operator $G(Q,P)$, whose $\mathfrak{Q}$-ordering form is denoted by $\mathfrak{Q}[F(Q,P)]$, one can get

$$\begin{aligned}G(Q,P) &= \mathfrak{Q}[F(Q,P)]\\ &= \int \mathrm{d}q'\,|q'\rangle\langle q'|\,\mathfrak{Q}[F(Q,P)]\int \mathrm{d}p'\,|p'\rangle\langle p'|\\ &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}p'\mathrm{d}q'F(q',p')\,|q'\rangle\langle p'|\,\mathrm{e}^{\mathrm{i}q'p'},\end{aligned}\tag{2}$$

which indicates that the matrix element $\langle q|G(Q,P)|\,p\rangle$ of operator $G(Q,P)$ is given by

$$\langle q|G(Q,P)|\,p\rangle=\frac{1}{\sqrt{2\pi}}F(q,p)\,\mathrm{e}^{\mathrm{i}qp},\tag{3}$$

where the relation $\langle q\,|q'\rangle=\delta(q'-q)$ and $\langle p\,|p'\rangle=\delta(p'-p)$ are used. Thus if the matrix element $\langle q|G(Q,P)|\,p\rangle$ of operator $G(Q,P)$ is known, then one can obtain the $\mathfrak{Q}$-ordered

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operator $\mathfrak{Q}[F(Q,P)]$'s classical correspondence $F(q,p)$. In other words, replacing q by Q, and p by P in classical function $F(q,p)$, one can get the $\mathfrak{Q}$-ordered form of operator $G(Q,P)$,

$$G(Q,P) \to F(q,p) = \sqrt{2\pi}\langle q|G(Q,P)|p\rangle\, \mathrm{e}^{-\mathrm{i}qp}. \quad (4)$$

In a similar way, one can obtain the $\mathfrak{P}$-ordered form of operator $G(Q,P)$ by replacing q by Q, p by P in the classical function $F(q,p)$, i.e.,

$$\langle p|G(Q,P)|q\rangle = \frac{1}{\sqrt{2\pi}}F(p,q)\,\mathrm{e}^{-\mathrm{i}pq},$$
$$G(Q,P) \to F(p,q) = \sqrt{2\pi}\langle p|G(Q,P)|q\rangle\,\mathrm{e}^{\mathrm{i}pq}. \quad (5)$$

From Eqs. (3)–(5), we see that it is a very concise and clear way to calculate the $\mathfrak{Q}$- and $\mathfrak{P}$-ordered form for the arbitrary operator.

3. The $\mathfrak{Q}$- and $\mathfrak{P}$-orderings of squeezing operators

In this section, using the above new method, we examine the $\mathfrak{Q}$- and $\mathfrak{P}$-ordering of some squeezing operators, such as the single-mode squeezing operator, the two-mode squeezing operator, and the Fresnel operator (considered as a generalized squeezing operator).

3.1. The $\mathfrak{Q}$- and $\mathfrak{P}$-ordered forms of single-mode squeezing operator

The single-mode squeezing operator is defined as $S_1 = \exp[(\zeta^* a^2 - \zeta a^{\dagger 2})/2]$, where $\zeta = \lambda\,\mathrm{e}^{\mathrm{i}\theta}$ is the complex squeezing parameter, and a, $a^\dagger$ are the Bose annihilation and creation operators satisfying $\left[a, a^\dagger\right] = 1$, respectively.

Noticing that the normal ordering form of the squeezing operator[8] is

$$S_1 = \mathrm{sech}^{1/2}\lambda :\, \exp\left[\frac{\tanh\lambda}{2}\left(a^2\,\mathrm{e}^{-\mathrm{i}\theta} - a^{\dagger 2}\,\mathrm{e}^{\mathrm{i}\theta}\right)\right] \times \exp\left[(\mathrm{sech}\lambda - 1)a^\dagger a\right] :, \quad (6)$$

thus using the completeness relation of coherent states and the overlap between the coherent states and the eigenvectors of coordinate operator $Q = (a + a^\dagger)/\sqrt{2}$ and momentum operator $P = (a - a^\dagger)/(\sqrt{2}\mathrm{i})$, i.e.,

$$\langle q|\alpha\rangle = \pi^{-1/4}\exp\left(-\frac{q^2}{2} + \sqrt{2}q\alpha - \frac{\alpha^2}{2} - \frac{|\alpha|^2}{2}\right),$$
$$\langle \beta|p\rangle = \pi^{-1/4}\exp\left(-\frac{p^2}{2} + \sqrt{2}\mathrm{i}p\beta^* + \frac{\beta^{*2}}{2} - \frac{|\beta|^2}{2}\right), \quad (7)$$

one can deduce the matrix element of S_1 as

$$\langle q|S_1|p\rangle = \sqrt{\frac{A_1}{2\pi}}\exp\left[\mathrm{i}A_2 q^2 + \mathrm{i}A_1 qp - \mathrm{i}A_2 p^2\right], \quad (8)$$

where we have set $A_1 = \mathrm{sech}\lambda/[1 - \cos\theta\tanh\lambda], A_2 = \sin\theta/[2(\cos\theta - \coth\lambda)]$, and used the following integration formula[10]

$$\int \frac{\mathrm{d}^2 z}{\pi}\exp\left\{A|z|^2 + Bz + Cz^* + Dz^2 + Ez^{*2}\right\}$$
$$= \frac{1}{\sqrt{A^2 - 4DE}}\exp\left\{\frac{-ABC + B^2E + C^2D}{A^2 - 4DE}\right\}. \quad (9)$$

Equation (8) implies that the classical function $F_1(q,p)$ of the single-mode squeezing operator S_1 is

$$F_1(q,p) = \sqrt{A_1}\exp\left[\mathrm{i}A_2 q^2 + \mathrm{i}qp(A_1 - 1) - \mathrm{i}A_2 p^2\right], \quad (10)$$

thus the $\mathfrak{Q}$-ordering form of S_1 is

$$S_1 = \sqrt{A_1}\,\mathfrak{Q}\left\{\exp\left[\mathrm{i}A_2 Q^2 + \mathrm{i}QP(A_1 - 1) - \mathrm{i}A_2 P^2\right]\right\}. \quad (11)$$

In a similar way to deriving Eq. (8), we can obtain the $\mathfrak{P}$-ordered form of S_1

$$S_1 = \sqrt{B_1}\,\mathfrak{P}\left\{\exp\left[-\mathrm{i}B_2 Q^2 - \mathrm{i}PQ(B_1 - 1) + \mathrm{i}B_2 P^2\right]\right\}, \quad (12)$$

where $B_1 = \mathrm{sech}\lambda/(1 + \cos\theta\tanh\lambda)$ and $B_2 = \sin\theta/[2(\cos\theta + \coth\lambda)]$. In particular, when $\theta = 0$, $B_1 = \mathrm{sech}\lambda/(1 + \tanh\lambda) = \mathrm{e}^{-\lambda}, B_2 = 0$, equation (12) becomes $S_1 = \mathrm{e}^{-\lambda/2}\mathfrak{P}\left\{\exp\left[-\mathrm{i}PQ\left(\mathrm{e}^{-\lambda} - 1\right)\right]\right\}$, which is the same as the result in Ref. [8].

3.2. The $\mathfrak{Q}$- and $\mathfrak{P}$-ordered forms of two-mode squeezing operator

In this subsection, we consider the two-mode squeezing operator, defined in coordinate representation,[11]

$$S_2 = \int \mathrm{d}q_1\mathrm{d}q_2 \left|\begin{pmatrix}\cosh\lambda & \sinh\lambda\\ \sinh\lambda & \cosh\lambda\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right|, \quad (13)$$

where

$$\left|\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle = |q_1\rangle \otimes |q_2\rangle.$$

Thus the matrix element $\langle q_1, q_2|S_2|p_1, p_2\rangle$ of S_2 can be directly calculated as

$$\langle q_1, q_2|S_2|p_1, p_2\rangle = \frac{1}{2\pi}\exp\left\{\mathrm{i}(p_1 q_1 + p_2 q_2)\cosh\lambda\right\} \times \exp\left\{-\mathrm{i}(p_1 q_2 + p_2 q_1)\sinh\lambda\right\}, \quad (14)$$

which indicates that the classical function $F_2(q,p)$ of the two-mode squeezing operator S_2 is

$$F_2(q,p) = 2\pi\langle q_1, q_2|S_2|p_1, p_2\rangle\exp\left[-\mathrm{i}(p_1 q_2 + p_2 q_1)\right] = \exp\left[-2\mathrm{i}(p_1 q_1 + p_2 q_2)\sinh^2\frac{\lambda}{2} - \mathrm{i}(p_1 q_2 + p_2 q_1)\sinh\lambda\right]. \quad (15)$$

Thus replacing q_1, p_1, q_2, p_2 by Q_1, P_1, Q_2, P_2, respectively, the $\mathfrak{Q}$-ordering form of S_2 is

$$S_2 = \mathfrak{Q}\exp\left[-2\mathrm{i}(P_1 Q_1 + P_2 Q_2)\sinh^2\frac{\lambda}{2} - \mathrm{i}(P_1 Q_2 + P_2 Q_1)\sinh\lambda\right]. \quad (16)$$

Similarly, the $\mathfrak{P}$-ordered form of S_2 can be directly given by

$$S_2 = \mathfrak{P}\exp\left[-2\mathrm{i}(P_1 Q_1 + P_2 Q_2)\cosh^2\frac{\lambda}{2} - \mathrm{i}(P_1 Q_2 + P_2 Q_1)\sinh\lambda\right]. \quad (17)$$

3.3. The $\mathfrak{Q}$- and $\mathfrak{P}$-ordered forms of Fresnel operator

The Fresnel transform is an important one in classical optics. The quantum optical counterpart for this transformation is found[12] by introducing the so-called generalized Fresnel operators (GFOs). For the single-mode case, the GFO can be expressed in the coherent state representation as

$$F(s,r)=\sqrt{s}\int\frac{\mathrm{d}^2z}{\pi}\left|sz-rz^*\right\rangle\langle z|, \tag{18}$$

where $|z\rangle=\exp[-|z|^2/2+za^\dagger]|0\rangle$ is a coherent state and the unitiry condition $ss^*-rr^*=1$ is required.

Using Eq. (18) and Eq. (7), one can derive

$$\langle q|F|p\rangle=\sqrt{\frac{1}{2\pi A}}\exp\left[\frac{\mathrm{i}}{2A}\left(Cq^2+2pq-Bp^2\right)\right], \tag{19}$$

where we have set $A=[(s+s^*)-(r+r^*)]/2$, $B=\mathrm{i}[(s-s^*)+(r-r^*)]/2$, $C=(-\mathrm{i})[(s-s^*)-(r-r^*)]/2$, $D=[(s+s^*)+(r+r^*)]/2$. Thus, the classical function $F'(q,p)$ of the Fresnel operator $F(s,r)$ is

$$\begin{aligned}F'(q,p)&=2\pi\langle q|F(s,r)|p\rangle\,\mathrm{e}^{-\mathrm{i}qp}\\&=\sqrt{\frac{1}{A}}\exp\left[\frac{\mathrm{i}}{2A}\left(Cq^2+2pq-Bp^2\right)-\mathrm{i}qp\right],\end{aligned} \tag{20}$$

and the $\mathfrak{Q}$-ordering form of $F(s,r)$ is given by

$$F(s,r)=\frac{1}{\sqrt{A}}\mathfrak{Q}\left\{\exp\left[\frac{\mathrm{i}}{2A}\left[CQ^2+2(1-A)QP-BP^2\right]\right]\right\}. \tag{21}$$

According to the definition of $\mathfrak{Q}$-ordering and using the operator identity[8]

$$\mathrm{e}^{-\lambda Q(\mathrm{i}P)}=\mathfrak{Q}\left[\exp\left(\mathrm{e}^{-\lambda}-1\right)Q(\mathrm{i}P)\right], \tag{22}$$

one can see that

$$\begin{aligned}F(s,r)&=\exp\left[\frac{\mathrm{i}C}{2A}Q^2\right]\exp\left[-\frac{\mathrm{i}}{2}(QP+PQ)\ln A\right]\\&\quad\times\exp\left[-\frac{\mathrm{i}B}{2A}P^2\right],\quad(A\neq0).\end{aligned} \tag{23}$$

In a similar way, one can also obtain the P-ordered form of $F(s,r)$, i.e.,

$$\begin{aligned}F(s,r)&=\frac{1}{\sqrt{D}}\mathfrak{P}\left\{\exp\left[\frac{\mathrm{i}}{2D}\left[CQ^2+2(D-1)QP-BP^2\right]\right]\right\}\\&=\exp\left[-\frac{\mathrm{i}B}{2D}P^2\right]\exp\left[\frac{\mathrm{i}}{2}(QP+PQ)\ln D\right]\\&\quad\times\exp\left[\frac{\mathrm{i}C}{2D}Q^2\right],\quad(D\neq0).\end{aligned} \tag{24}$$

Equations (23) and (24) are just two decomposition forms of the Fresnel operator,[12] which have been derived by using the matrix decompositions way. It is shown that the classical correspondence of the Fresnel operator $\mathfrak{Q}$ ($\mathfrak{P}$)-ordering is just the integration kernel of Fresnel transformation.

4. Application in photon counting distribution

As another application of $\mathfrak{Q}$-ordering, in this section we examine the photon counting distribution from the classical correspondence of $\mathfrak{Q}$-ordering operator. For the case of the single radiation mode, the probability distribution $\boldsymbol{p}(n)$ of registering n photoelectrons in time interval $\Delta\tau$ is given by[13,14]

$$\boldsymbol{p}(n,\sigma)=tr\left\{\rho:\frac{\left(\sigma a^\dagger a\right)^n}{n!}\mathrm{e}^{-\sigma a^\dagger a}:\right\}, \tag{25}$$

where ρ is a single-mode density operator of the light field concerned, : : stands for normal ordering, and σ is called the quantum efficiency of the detector. Denoting $G=:\left(\sigma a^\dagger a\right)^n\mathrm{e}^{-\sigma a^\dagger a}/n!:$, and using the completeness relations of coordinate and momentum, one can obtain

$$\begin{aligned}\boldsymbol{p}(n,\sigma)&=\int\mathrm{d}p\mathrm{d}q\,\langle q|G|p\rangle\langle p|\rho|q\rangle\\&=\int\frac{\mathrm{d}p\mathrm{d}q}{\sqrt{2\pi}}\mathfrak{F}(q,p)\,\mathrm{e}^{\mathrm{i}qp}\langle p|\rho|q\rangle,\end{aligned} \tag{26}$$

where $\mathfrak{F}(q,p)$ is the classical correspondence of the $\mathfrak{Q}$-ordering of operator G. Equation (26) implies that if we know the classical function of $\mathfrak{F}(q,p)$ or matrix element $\langle q|G|p\rangle$, then we can calculate the photon counting distribution $\boldsymbol{p}(n,\sigma)$ for various different density operators.

Now let us derive firstly the expression of matrix element $\langle q|G|p\rangle$. Using the coherent state completeness relation and Eq. (7) and the integration formula Eq. (9), we can derive

$$\begin{aligned}&\mathfrak{F}(q,p)\\&=\frac{\sigma^n}{n!\sqrt{\pi\tau}}\exp\left[\frac{\tau-2}{2\tau}\left(q^2+p^2\right)-2\mathrm{i}pq\frac{\chi}{\tau}\right]\\&\quad\times\frac{\partial^n}{\partial s^n}\frac{\partial}{\partial t^n}\exp\left[\frac{1}{2\tau}\left(t^2-s^2\right)+\frac{\chi}{\tau}st\right]\\&\quad\times\exp\left\{\frac{\sqrt{2}}{\tau}\left[(q+\mathrm{i}p\chi)s+(\mathrm{i}p-q\chi)t\right]\right\}_{s=t=0},\end{aligned} \tag{27}$$

where we have set $\tau=\sigma^2-2\sigma+2$, $\chi=\sigma-1$.

Further using the generating function of single variable Hermite polynomials

$$\mathrm{H}_n(x)=\left.\frac{\partial^n}{\partial t^n}\exp\left(2xt-t^2\right)\right|_{t=0}, \tag{28}$$

and the recurrence relation of $\mathrm{H}_n(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x^l}\mathrm{H}_n(x)=\frac{2^l n!}{(n-l)!}\mathrm{H}_{n-l}(x), \tag{29}$$

equation (27) can be rewritten as

$$\begin{aligned}\mathfrak{F}(q,p)&=\frac{\sigma^n}{n!\sqrt{\pi\tau}}\exp\left[\frac{\tau-2}{2\tau}\left(q^2+p^2\right)-2\mathrm{i}pq\frac{\chi}{\tau}\right]\\&\quad\times\sum_{j=0}^{n}\frac{(n!)^2\chi^j(-1)^{(n-j)/2}}{j![(n-j)!]^2(2\tau)^{n-j}}\end{aligned}$$

$$\times \mathrm{H}_{n-j}\left[(q+\mathrm{i}p\chi)/\sqrt{\tau}\right]$$
$$\times \mathrm{H}_{n-j}\left[-(p+\mathrm{i}q\chi)/\sqrt{\tau}\right]. \quad (30)$$

Thus we can calculate $\boldsymbol{p}(n,\zeta)$ by substituting Eq. (30) into Eq. (26).

Next we consider two special cases.

(i) For a coherent state, the density operator is $\rho = |z\rangle\langle z|$. Using Eq. (7), we have

$$\langle p|z\rangle\langle z|q\rangle = \pi^{-1/2}\exp\left[-\frac{1}{2}\left(p^2+q^2\right)-|z|^2\right] \times \exp\left[\sqrt{2}\left(qz^*-\mathrm{i}pz\right)+\frac{1}{2}\left(z^2-z^{*2}\right)\right]. \quad (31)$$

Substituting Eq. (31) into Eq. (26), the photon counting distribution $\boldsymbol{p}(n,\sigma)$ of a coherent state can be directly obtained

$$\begin{aligned}\boldsymbol{p}(n,\sigma) &= \frac{\sigma^n}{\pi n!\sqrt{\tau}}\exp\left(-|z|^2+\frac{1}{2}\left(z^2-z^{*2}\right)\right)\\ &\times\frac{\partial^{2n}}{\partial s^n\partial t^n}\exp\left(\frac{1}{2\tau}\left(t^2-s^2\right)+\frac{\chi}{\tau}st\right)\\ &\times\int \mathrm{d}p\mathrm{d}q\exp\left(-\frac{q^2+p^2}{\tau}+\frac{\sqrt{2}}{\tau}\left(\tau z^*-\chi t+s\right)q\right)\\ &\times\exp\left(\frac{\sqrt{2}\mathrm{i}}{\tau}\left(\chi s+t-\tau z-\sqrt{2}q\chi\right)p\right)\Big|_{s=t=0}\\ &= \frac{\sigma^n}{n!}\frac{\partial^{2n}}{\partial s^n\partial t^n}\exp\left[sz^*+tz-\sigma|z|^2\right]_{s=t=0}\\ &= \frac{\sigma^n|z|^{2n}}{n!}\mathrm{e}^{-\sigma|z|^2}, \end{aligned} \quad (32)$$

which is just Eq. (18) in Refs. [15] and [16].

(ii) For the squeezing vacuum state, the density operator is defined as $\rho_{\mathrm{s}} = S_1|0\rangle\langle 0|S_1^{\dagger}$, we takes $S_1 = \exp\left[\lambda\left(a^2-a^{\dagger 2}\right)/2\right]$ as the single-mode squeezing operator. Using the $\mathfrak{Q}$-ordering and $\mathfrak{P}$-ordering of squeezing operator S_1, we have

$$\begin{aligned}&\langle p|S_1|0\rangle\langle 0|S_1^{\dagger}|q\rangle\\ &= \int \mathrm{d}q'\langle p|S_1|q'\rangle\langle q'|0\rangle\left(\int \mathrm{d}p'\langle q|S_1|p'\rangle\langle p'|0\rangle\right)^{\dagger}\\ &= \pi^{-1/2}\exp\left[-p^2\mathrm{e}^{-2\lambda}/2-q^2\mathrm{e}^{2\lambda}/2\right]. \end{aligned} \quad (33)$$

Substituting Eq. (33) into Eq. (26), the photon counting distribution $\boldsymbol{p}(n,\sigma)$ of squeezing vacuum state ρ_{s} is calculated as

$$\begin{aligned}&\boldsymbol{p}(n,\sigma)\\ &= \frac{\sigma^n\operatorname{sech}\lambda}{n!\sqrt{1-\mathfrak{G}^2}}\frac{\partial^{2n}}{\partial s^n\partial t^n}\\ &\times\exp\left(\frac{\tanh\lambda}{2(1-\mathfrak{G}^2)}\left[2st(1-\sigma)\tanh\lambda-s^2-t^2\right]\right)\Big|_{s=t=0}\\ &= \frac{\operatorname{sech}\lambda}{\sqrt{1-\mathfrak{G}^2}}\frac{(-\sigma)^n\tanh^n\lambda}{\left(\mathfrak{G}^2-1\right)^{3n/2}}P_n\left(\frac{\mathfrak{G}}{\mathfrak{G}^2-1}\right), \end{aligned} \quad (34)$$

where we have set $\mathfrak{G} = (1-\sigma)\tanh\lambda$ and we have used the formula[17]

$$\frac{\partial^{2m}}{\partial t^m\partial\tau^m}\exp\left(-t^2-\tau^2+\frac{2t\tau x}{\sqrt{x^2-1}}\right)\Big|_{t=\tau=0} = \frac{2^m m!}{\left(x^2-1\right)^{m/2}}\mathrm{P}_m(x), \quad (35)$$

with $\mathrm{P}_m(x)$ being the m-order Legendre polynomial. Equation (34) seems a new result related to the Legendre polynomial.

5. Conclusions

By using the completeness relation of the coordinate and momentum eigenvector, we identify that the classical correspondence of $\boldsymbol{Q}-P$ $(P-Q)$ ordered operators $G(Q,P)$ correspond to the matrix elements of $F(q,p) = \sqrt{2\pi}\langle q|G(Q,P)|p\rangle\,\mathrm{e}^{-\mathrm{i}qp}$ and $F(p,q) = \sqrt{2\pi}\langle p|G(Q,P)|q\rangle\,\mathrm{e}^{\mathrm{i}pq}$, respectively. Then the $Q-P$ $(P-Q)$ ordered operators can be obtained from the classical correspondence function $F(q,p)$ or $F(p,q)$ by replacing q and p with coordinate and momentum operators, respectively. On the basis of this new approach, some operator identities are derived concisely. As for its applications, the single (two-) mode squeezed operators and Fresnel operator are examined. It is shown that the classical correspondence of the Fresnel operator $Q-P$ $(P-Q)$ ordering is just the integration kernel of Fresnel transformation. In addition, a new photo-counting formula is derived on the basis of the $Q-P$ ordering, which is also used to examine two special examples.

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