The periodic oscillations in the ENSO recharge–discharge oscillator model*

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(Received 9 April 2013; revised manuscript received 17 May 2013)

A class of recharge–discharge oscillator model for the El Niño/Southern Oscillation (ENSO) is considered. A stable limit cycle is obtained by transforming the ENSO model into the van der Pol-Duffing equation. We proved that there exists periodic oscillations in the ENSO recharge–discharge oscillator model.

Keywords: ENSO model, van der Pol-Duffing equation, limit cycle

PACS: 02.30.Jr **DOI:** 10.1088/1674-1056/22/12/120201

1. Introduction

The El Niño/Southern Oscillation (ENSO) is the dominant interannual mode of the tropical coupled ocean–atmosphere system and a key seasonal-to-interannual predictability of the global climate system, it has worldwide impacts not only on the Earth system but also in areas as diverse as ecosystems and socioeconomic benefit.[1–4] To understand ENSO many conceptual oscillator models, such as the delayed oscillator, the western Pacific oscillator, the recharge–discharge oscillator, the stratosphere-reflection oscillator, and the unified oscillator, have been proposed to explore the ENSO mechanism, see Refs. [5] and [6] for details. Recently, a variety of methods to study these ENSO oscillator models have been suggested by some scholars (see, e.g., Refs. [7]–[12]). In this paper, we continue to study the following recharge–discharge oscillator of the ENSO model[6,10,12,13]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = CT + Dh - \varepsilon T^3, \tag{1a}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -ET - R_h h, \tag{1b}$$

where T is the sea-surface temperature anomalies (SSTA) in equatorial eastern Pacific, h is the thermoclinic depth anomalies in the equatorial western Pacific, C, D, E, R_h, and ε are all positive model parameters, defined in Ref. [13]. By using Mawhin's continuation theorem, a result on the existence of periodic solutions for ENSO model is obtained by Li.[10] In this study, by transforming (1) into the van der Pol-Duffing equation, a stable limit cycle is obtained through qualitative analysis of the van der Pol-Duffing equation.[14–16] We strictly proved that there exists periodic oscillations in the ENSO model (1).

2. The transformed van der Pol-Duffing equation

Equation (1) can be rewritten as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - C + \varepsilon T^2\right) T = Dh, \tag{2a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + R_h\right) h = -ET. \tag{2b}$$

Multiplying Eq. (2a) by $(\mathrm{d}/\mathrm{d}t + R_h)$ and eliminating h with Eq. (2b), we get

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} + \omega^2 T + \varepsilon\left[(\mu + 3T^2)\frac{\mathrm{d}T}{\mathrm{d}t} + R_h T^3\right] = 0. \tag{3}$$

Equation (3) is the van der Pol-Duffing equation for a weakly nonlinear system, where

$$\mu\varepsilon = R_h - C, \quad \omega^2 = DE - CR_h, \tag{4}$$

and

$$0 < \varepsilon < 1, \quad CR_h < DE. \tag{5}$$

Now we will obtain the approximate solution of Eq. (3) by the multi-scale method[17] and introduce a new time variable in the form of

$$\tau = \varepsilon t \tag{6}$$

and the variable SSTA T can be expressed by

$$T(t,\varepsilon) = T_0(t,\tau) + \varepsilon T_1(t,\tau) + \cdots \tag{7}$$

and the corresponding differential operators are

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau} = D_0 + \varepsilon D_1, \tag{8a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 D_1^2, \tag{8b}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 40975028 and 41175052).

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Substituting Eqs. (6), (7), (8) into Eq. (3), and the coefficients of the same power of ε on two sides of the equation are supposed to be equal, we have,

$$D_0^2 T_0 + \omega^2 T_0 = 0, \tag{9}$$

$$D_0^2 T_1 + \omega^2 T_1 = -2D_0 D_1 T_0 - (\mu + 3T_0^2) D_0 T_0 - R_h T_0^3, \tag{10}$$

the solution of Eq. (9) is assumed as

$$T_0 = A_1(\tau)\,\mathrm{e}^{\mathrm{i}\omega t} + A_2(\tau)\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{11}$$

where A_1 and A_2 are complex conjugate. The substitution of Eq. (11) into Eq. (10) yields

$$\begin{aligned} & D_0^2 T_1 + \omega^2 T_1 \\ = & -2\mathrm{i}\omega[D_1 A_1\,\mathrm{e}^{\mathrm{i}\omega t} - D_1 A_2\,\mathrm{e}^{-\mathrm{i}\omega t}] \\ & - \mathrm{i}\omega(A_1\,\mathrm{e}^{\mathrm{i}\omega t} - A_2\,\mathrm{e}^{-\mathrm{i}\omega t}) \\ & \times [\mu + 3(A_1^2\,\mathrm{e}^{2\mathrm{i}\omega t} + A_2^2\,\mathrm{e}^{-2\mathrm{i}\omega t} + 2A_1 A_2)] \\ & - R_h[A_1^3\,\mathrm{e}^{3\mathrm{i}\omega t} + A_2^3\,\mathrm{e}^{-3\mathrm{i}\omega t} \\ & + 3A_1^2 A_2\,\mathrm{e}^{\mathrm{i}\omega t} + 3A_1 A_2^2\,\mathrm{e}^{-\mathrm{i}\omega t}]. \end{aligned} \tag{12}$$

To eliminate the secular term by requiring the coefficients of $\mathrm{e}^{\mathrm{i}\omega t}$ and $\mathrm{e}^{-\mathrm{i}\omega t}$ to be zero, we have

$$D_1 A_1 = -\frac{1}{2}(\mu + 3A_1 A_2)A_1 + \mathrm{i}\frac{3R_h}{2\omega} A_1^2 A_2. \tag{13}$$

Assuming $A_1 = a\,\mathrm{e}^{\mathrm{i}\varphi}/2, A_2 = a\,\mathrm{e}^{-\mathrm{i}\varphi}/2$ in Eq. (13), where $a = a(\tau)$ and $\varphi = \varphi(\tau)$ are all real functions. Substituting them into Eq. (13) yields

$$\begin{aligned} & a'\,\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{i}\varphi' a\,\mathrm{e}^{\mathrm{i}\varphi} \\ = & -\frac{1}{2}\left(\mu + \frac{3a^2}{4}\right) a\,\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{i}\frac{3R_h a^2}{8\omega} a^2\,\mathrm{e}^{\mathrm{i}\varphi}, \end{aligned} \tag{14}$$

where $a' = \mathrm{d}a/\mathrm{d}\tau$, $\varphi' = \mathrm{d}\varphi/\mathrm{d}\tau$. By separating the real and imaginary parts, we obtain the first-order approximation equation in polar coordinate as follows:

$$a' = -\frac{1}{2}\left(\mu + \frac{3a^2}{4}\right) a, \tag{15a}$$

$$\varphi' a = \frac{3R_h}{8\omega} a^3, \tag{15b}$$

when $a' = 0$, the corresponding bifurcation equation can be derived as follows:

$$-\frac{1}{2}\left(\mu + \frac{3a^2}{4}\right) a = 0 \tag{16}$$

and

$$a_1 = 0, \tag{17a}$$

$$a_2 = 2\sqrt{-\frac{\mu}{3}}, \quad \mu < 0. \tag{17b}$$

According to Eqs. (17), the system has only one solution of $a_1 = 0$ when $\mu > 0$, while it has two solutions of $a_1 = 0$ and $a_2 = 2\sqrt{-\mu/3}$ when $\mu < 0$. That means a limit cycle exists with a radius of $2\sqrt{-\mu/3}$ ($\mu < 0$). Then we will discuss the stabilities of the equilibrium states.

Assuming $A_1 = u + \mathrm{i}v$ in Eq. (13), where $u = u(\tau)$ and $v = v(\tau)$ are both real functions, then we have

$$\begin{aligned} u' + \mathrm{i}v' = & -\frac{1}{2}[\mu + 3(u^2 + v^2)](u + \mathrm{i}v) \\ & + \mathrm{i}\frac{3R_h}{2\omega}(u^2 + v^2)(u + \mathrm{i}v), \end{aligned} \tag{18}$$

where $u' = \mathrm{d}u/\mathrm{d}\tau$, $v' = \mathrm{d}v/\mathrm{d}\tau$. By separating the real and imaginary parts, we obtain

$$u' = -\frac{1}{2}\mu u - \left(\frac{3}{2}u + \frac{3R_h}{2\omega}v\right)(u^2 + v^2), \tag{19a}$$

$$v' = -\frac{1}{2}\mu v - \left(\frac{3}{2}v - \frac{3R_h}{2\omega}u\right)(u^2 + v^2). \tag{19b}$$

The Jacobi matrix of Eqs. (19) is as follows:

$$P = \begin{bmatrix} -\dfrac{\mu}{2} - \dfrac{3(3u^2 + v^2)}{2} - \dfrac{3R_h}{\omega}uv & -\dfrac{3R_h}{2\omega}(u^2 + v^2) - \left(3u + \dfrac{3R_h v}{\omega}\right)v \\ \dfrac{3R_h}{2\omega}(u^2 + v^2) - \left(3v - \dfrac{3R_h u}{\omega}\right)u & -\dfrac{\mu}{2} - \dfrac{3(u^2 + 3v^2)}{2} + \dfrac{3R_h}{\omega}uv \end{bmatrix}. \tag{20}$$

For $A_1 = 0$, $u = v = 0$, we have

$$P_{(0,0)} = \begin{bmatrix} -\dfrac{\mu}{2} & 0 \\ 0 & -\dfrac{\mu}{2} \end{bmatrix}. \tag{21}$$

The characteristic equation at point $(0,0)$ is

$$\lambda^2 + \mu\lambda + \frac{\mu^2}{4} = 0. \tag{22}$$

There are two same eigenvalues for Eq. (22)

$$\lambda_1 = \lambda_2 = \lambda = -\frac{\mu}{2}. \tag{23}$$

When λ is a positive real number with $\mu < 0$, the solution $(0,0)$ is an unstable critical node. In this case, when a slight deviation occurs, the system may easily get into instability and move away from the mean state of climate. When λ is a negative real number with $\mu > 0$, the point $(0,0)$ is a stable critical node and deviations would never cause the instability of the system. If $\mu = 0$, we will discuss the stability of the point $(0,0)$ by using the Lyapunov's second method.

When $\mu = 0$, equation (19) can be rewritten as

$$u' = f(u,v), \quad f(u,v) = -\left(\frac{3}{2}u + \frac{3R_h}{2\omega}v\right)(u^2+v^2), \tag{24a}$$

$$v' = g(u,v), \quad g(u,v) = -\left(\frac{3}{2}v - \frac{3R_h}{2\omega}u\right)(u^2+v^2). \tag{24b}$$

Define $V = (u^2+v^2)/2$, and V is satisfied by the following two conditions:

(i) $V(0,0) = 0$, and $V(u,v) > 0$ when $(u,v) \neq (0,0)$.

(ii) When $0 < u^2+v^2 < 1$,

$$\begin{aligned} V' &= \frac{\partial V}{\partial u} f(u,v) + \frac{\partial V}{\partial v} g(u,v) \\ &= -\frac{3(u^2+v^2)^2}{2} < 0. \end{aligned}$$

Thus we conclude that the point (0,0) is asymptotically stable when $\mu = 0$. The limit cycle only exists when $\mu < 0$ and the system

$$u' = P(u,v),$$

$$P(u,v) = -\frac{1}{2}\mu u - \left(\frac{3}{2}u + \frac{3R_h}{2\omega}v\right)(u^2+v^2), \tag{25a}$$

$$v' = Q(u,v),$$

$$Q(u,v) = -\frac{1}{2}\mu v - \left(\frac{3}{2}v - \frac{3R_h}{2\omega}u\right)(u^2+v^2) \tag{25b}$$

satisfies

$$\int_0^\delta \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right) \mathrm{d}t = 7\mu\delta < 0,$$

where Γ is along the limit cycle, δ is the cycle time of Γ. Obviously, the limit cycle is stable.

A bifurcation occurs when $\mu = 0$, and a stable limit cycle appears at the same time. In fact, the stable limit cycle is an isolated periodic wave solution under the weak damping condition, also known as the periodic attractor. As the above analysis indicates: when a small perturbation occurs, the ENSO system may easily depart from the mean climate state, and enter into another cycled oscillation. We have rigorously proved that a stable limit cycle does exist in the ENSO model. In other words, there exists periodic oscillations in the ENSO model. Our conclusions are consistent with the results of Zhao *et al.*,[9] Li,[10] and Wang *et al.*[11]

To check the accuracy of the above analysis, we solved the equation (3) numerically by taking $\varepsilon = 0.1$, $R_h = 1$, $\omega = \pi/2$ (/yr). The bifurcation occurs as $\mu = 0$. When $\mu < 0$ the equilibrium (0,0) is unstable, and the bifurcating periodic solutions are unstable. Numerical simulations of system (3) with $\mu = -2$ are shown in Fig. 1, here the parameter $\mu = -2$ is chosen to satisfy the condition of the stable limit cycle existing. If $\mu > 0$, the corresponding bifurcating periodic solutions are asymptotically stable, which is shown in Fig. 2.

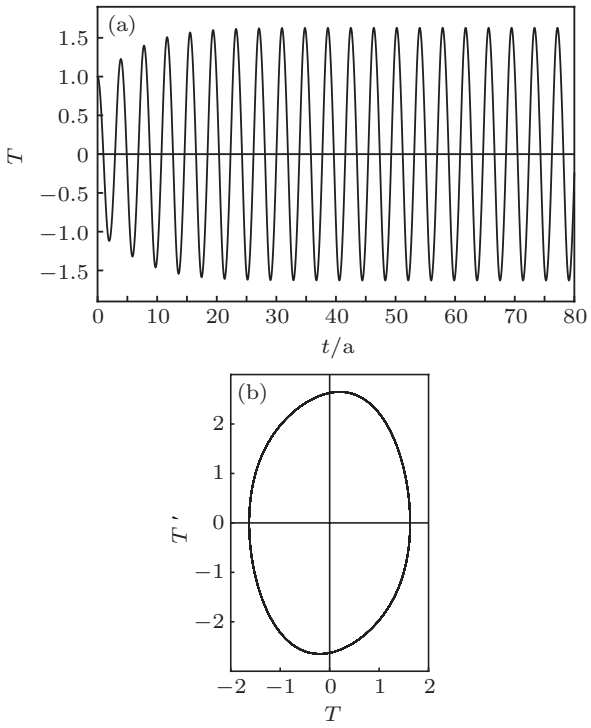


Fig. 1. (a) The time history; (b) the phase portrait, showing a stable limit cycle.

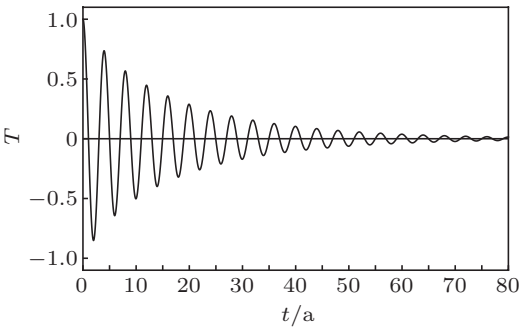


Fig. 2. Numerical simulations of system (3) with $\mu = 1$.

3. Conclusions

The ENSO is a very complicated coupled ocean–atmosphere system, and it describes an interannual climate variability. However, our limited understanding of the physical mechanisms determining its oscillatory nature greatly hinder ENSO simulations or predictions. The interesting result in this study is that we have strictly proved that there exists periodic oscillations of the ENSO recharge–discharge oscillator model in mathematical–physical theory.

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