Generalized 2D problem of icosahedral quasicrystals containing an elliptic hole*

Li Lian-He(李联和)†

College of Mathematics Science, Inner Mongolia Normal University, Huhhot 010022, *China*
Key Laboratory of Nanomagnetic and Functional Material, College of Physical Science and Technology, Inner Mongolia University, Hohhot 010021, *China*

(Received 15 November 2012; revised manuscript received 17 April 2013)

The generalized 2D problem of icosahedral quasicrystals containing an elliptic hole is considered by using the extended Stroh formalism. The closed-form solutions for the displacements and stresses are obtained under general loading conditions. The solution of the Griffith crack problem as a special case of the results is also observed. The stress intensity factor and strain energy release rate are given. The effect of the phonon–phason coupling elastic constant on the mechanical behavior is also discussed.

Keywords: icosahedral quasicrystal, plane elasticity, Stroh formulism, elliptic hole

PACS: 61.44.Br, 62.20.D–, 02.30.Em **DOI:** 10.1088/1674-1056/22/11/116101

1. Introduction

Quasiperiodic structures with icosahedral symmetry that are incompatible with the extended periodicity of the crystal were discovered by Israeli scientist Dan Shechtman in 1984.[1] Quasiperiodicity induces new degrees of freedom and phason modes, which can be understood in the high-dimensional description of quasicrystals. In the case of the icosahedral symmetry, the quasicrystalline structure can be embedded in a six-dimensional (6D) space which decomposes into two orthogonal 3D subspaces: the physical space and the perpendicular or complementary space.[2,3]

Quasicrystals are fascinating substances that form a family of specific structures with strange physical and mechanical properties. The physical properties, such as the structural, electronic, magnetic, optical, and thermal properties, of quasicrystals have been investigated extensively.[3–5] These properties lay a foundation for its further application to technology. A Swedish company found quasicrystals in one of the most durable kinds of steel, which is now used in products such as razor blades and thin needles made specifically for eye surgery. Quasicrystals used in the manufacture of frying pans and heat insulation in engines as coating.

It is well known that an elliptic hole problem is the basis of a crack problem in elastic analysis. The mechanics of elliptic holes or cracks in planar decagonal quasicrystals has already been investigated.[6–8] The analytical solutions for the Griffith crack in a decagonal quasicrystal were given by exploiting the displacement potential function method and Fourier transform.[9,10]

Icosahedral quasicrystals play a central role in the study of quasicrystalline solids, which are more complicated than one- and two-dimensional quasicrystals. Due to their complexity, only a few results have been obtained. Analytical expressions were derived as a first-order approximation for the elastic displacement fields induced by spherical inclusions in icosahedral quasicrystals.[11] The final governing equations of plane elasticity of icosahedral quasicrystals were reduced to the single sextuple harmonic equation under a few suitable hypotheses by introducing the displacement potential function or stress potential function.[12]

Both the Lekhnitskii and the Stroh formalisms are for the analysis of two-dimensional deformation of an anisotropic linear elastic solid. The difference is that the Lekhnitskii formalism begins with the two-dimensional stresses, whereas the Stroh formalism starts with the two-dimensional displacements. The two methods have been generalized successfully to quasicrystal materials. Lekhnitskii's formalism is extended to one-dimensional quasicrystals and cubic quasicrystal solids.[13,14] Radi and Mariano considered the straight cracks in two-dimensional quasicrystals by using the Stroh formalism.[7] The closed-form expressions for the elastic displacement and stress fields induced by a dislocation in an icosahedral quasicrystal were obtained using the extended Stroh formalism.[15] Wang *et al.* derived the detailed structures of the fundamental elasticity matrix in the Stroh formalism of anisotropic elasticity for quasicrystals.[16] Elastic fields

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11072104, 1272053, and 11262017), the Key Project of Chinese Ministry of Education (Grant No. 212029), the Natural Science Foundation of Inner Mongolia Autonomous Region, China (Grant No. 2013MS0114), the Natural Science Foundation of Inner Mongolia Department of Public Education, China (Grant No. NJZZ13037), the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region, China (Grant No. NJYT-13-B07), and the Program of Higher-level Talents of Inner Mongolia University, China (Grant No. 125125).

†Corresponding author. E-mail: nmglilianhe@163.com

© 2013 Chinese Physical Society and IOP Publishing Ltd

of icosahedral quasicrystals containing an elliptic hole are discussed by the extended Stroh formulism in this paper.

2. Basic equations and Stroh formalism

According to the elastic theory of icosahedral quasicrystals,[3] we have the generalized Hooke's law

$$\begin{aligned}\sigma_{ij} &= C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl},\\ H_{ij} &= R_{klij}\varepsilon_{kl} + K_{ijkl}w_{kl},\end{aligned} \tag{1}$$

and the equilibrium equations (if the body force is neglected)

$$\begin{aligned}C_{ijkl}u_{k,jl} + R_{ijkl}w_{k,jl} &= 0,\\ R_{klij}u_{k,jl} + K_{ijkl}w_{k,jl} &= 0,\end{aligned} \tag{2}$$

in which a comma denotes differentiation, repeated indices imply summation, σ_{ij}, ε_{ij}, u_i and C_{ijkl} are stress, strain, displacement, and elastic constants of phonon fields, respectively. H_{ij}, w_{ij}, w_i, and K_{ijkl} are the stress, strain, displacement, and elastic constants of phason fields. R_{ijkl} are the phonon–phason coupling elastic constants. The coordinate system A2P-A2-A5, i.e., with the x_1 axis parallel to a pseudo-twofold axis, the x_2 axis parallel to a two-fold axis, and the x_3 axis parallel to a fivefold axis, is adopted here, as used in Ref. [17].

Consider a generalized two-dimensional problem of icosahedral quasicrystals in which all the field variables are independent of x_3. Adopting the Stroh formalism for icosahedral quasicrystal elasticity,[15] the displacement vector $\boldsymbol{u}$ can be expressed as

$$\boldsymbol{u} = [u_1, u_2, u_3, w_1, w_2, w_3]^{\mathrm{T}} = \boldsymbol{a}\boldsymbol{f}(z), \tag{3}$$

where

$$z = x_1 + px_2,$$

the superscript T represents the transpose, $\boldsymbol{f}(\boldsymbol{z})$ is an analytic function, p is a complex number, and $\boldsymbol{a}$ is a constant six-element column vector. Equation (2) can be satisfied by Eq. (3) for arbitrary $\boldsymbol{f}(z)$ if

$$[\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p^2\boldsymbol{T}]\boldsymbol{a} = \boldsymbol{0}, \tag{4}$$

where the 6×6 matrices $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{T}$ are given by

$$\begin{aligned}\boldsymbol{Q} &= \begin{bmatrix} C_{i1k1} & R_{i1k1} \\ R^{\mathrm{T}}_{i1k1} & K_{i1k1}\end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} C_{i1k2} & R_{i2k1} \\ R^{\mathrm{T}}_{i1k2} & K_{i1k2}\end{bmatrix},\\ \boldsymbol{T} &= \begin{bmatrix} C_{i2k2} & R_{i2k2} \\ R^{\mathrm{T}}_{i2k2} & K_{i2k2}\end{bmatrix}, \quad (i,k = 1,2,3).\end{aligned} \tag{5}$$

For a nontrivial solution of $\boldsymbol{a}$, we must have

$$|\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p^2\boldsymbol{T}| = 0, \tag{6}$$

which gives twelve roots p_α and $\overline{p}_\alpha$ $(\alpha = 1,2,\ldots,6)$, where p_α are assumed to be the six distinct complex roots.[18] The associated eigenvector $\boldsymbol{a}$ is determined by Eq. (4). Then, the general solution of Eqs. (1) and (2) can be expressed as

$$\boldsymbol{u} = 2\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(\boldsymbol{z})\}, \tag{7}$$

in which

$$\begin{aligned}\boldsymbol{A} &= [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3\ \boldsymbol{a}_4\ \boldsymbol{a}_5\ \boldsymbol{a}_6],\\ \boldsymbol{f}(\boldsymbol{z}) &= [f_1(z_1)\ f_2(z_2)\ f_3(z_3)\ f_4(z_4)\ f_5(z_5)\ f_6(z_6)]^{\mathrm{T}},\end{aligned} \tag{8}$$

and $z_\alpha = x_1 + p_\alpha x_2$. Without loss of generality, we let $\mathrm{Im}\, p_\alpha > 0$.

Making use of Eqs. (1), (3), and (5), the phonon stress and phason stress can be rewritten as

$$\begin{aligned}\sigma_{i1} &= -pb_i f'(z),\ \sigma_{i2} = b_i f'(z),\\ H_{i1} &= -pb_{i+3} f'(z),\ H_{i2} = b_{i+3} f'(z),\end{aligned} \tag{9}$$

where $i = 1,2,3$. In matrix notation,

$$\boldsymbol{b} = (\boldsymbol{R}^{\mathrm{T}} + p\boldsymbol{T})\boldsymbol{a}. \tag{10}$$

Introducing the stress function ϕ, the phonon stress σ_{ij} and phason stress H_{ij} can be expressed as

$$\begin{aligned}\sigma_{i1} &= -\phi_{i,2},\ \sigma_{i2} = \phi_{i,1},\\ H_{i1} &= -\phi_{i+3,2},\ H_{i2} = \phi_{i+3,1},\end{aligned} \tag{11}$$

where $i = 1,2,3$. Then, the general solution for stress function ϕ is obtained as

$$\phi = 2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(\boldsymbol{z})\}, \tag{12}$$

in which

$$\boldsymbol{B} = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4\ \boldsymbol{b}_5\ \boldsymbol{b}_6].$$

The appropriate form of $\boldsymbol{f}(\boldsymbol{z})$ in Eqs. (7) and (12) depends on the boundary conditions of the problems considered.

3. Elliptic hole problem

Consider an elliptic hole L, which is described by the equation $x_1^2/a^2 + x_2^2/b^2 = 1$, in the infinite icosahedral quasicrystal medium subjected to a uniform mechanical loading $\boldsymbol{t}_2^\infty = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, H_{21}^\infty, H_{22}^\infty, H_{23}^\infty]^{\mathrm{T}}$ at infinity, as shown in Fig. 1.

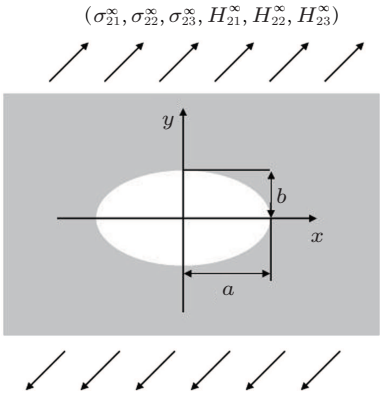


Fig. 1. An elliptic hole in an icosahedral quasicrystal solid subjected to far-field mechanical loading.

The boundary conditions in this problem are

$$\phi = 0 \text{ along the hole boundary } \boldsymbol{L}, \tag{13}$$

$$\sigma_{ij} \to \sigma_{ij}^{\infty},\ H_{ij} \to H_{ij}^{\infty},\ (x_1^2 + x_2^2)^{1/2} \to \infty. \tag{14}$$

By using the perturbation technique, the complex potential vectors for the current problem can be expressed as

$$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{f}_0(\boldsymbol{z}) + \boldsymbol{f}_1(\boldsymbol{z}), \tag{15}$$

where the homogenous solution $\boldsymbol{f}_0(\boldsymbol{z})$ is obtained by assuming that the elliptic hole is absent from the infinite quasicrystal medium subjected to uniform loading $\boldsymbol{t}_2^{\infty}$. Using Eqs. (12) and (14), one obtains

$$\boldsymbol{f}_0(\boldsymbol{z}) = \frac{1}{2}\langle z_\alpha \rangle \boldsymbol{B}^{-1}\boldsymbol{t}_2^{\infty}, \tag{16}$$

where $\langle\ \rangle$ indicates the diagonal matrix. The function $\boldsymbol{f}_1(\boldsymbol{z})$ corresponds to the perturbed field due to the elliptic hole. It is an unknown function to be determined according to the boundary conditions.

Introducing the conformal mapping functions[19]

$$z_\alpha(\zeta_\alpha) = c_\alpha \zeta_\alpha + d_\alpha \zeta_\alpha^{-1}, \tag{17}$$

where $c_\alpha = (a - \mathrm{i}p_\alpha b)/2$ and $d_\alpha = (a + \mathrm{i}p_\alpha b)/2$ map the exterior of the elliptic L_α in the z_α plane into the exterior of a unit circle γ that can be described by $\sigma = \mathrm{e}^{\mathrm{i}\theta}(0 \le \theta \le 2\pi)$ in the ζ_α plane.

Substituting Eq. (15) into Eqs. (12) and (14) results in

$$\phi = \boldsymbol{t}_2^{\infty} x_1 + 2\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}_1(\boldsymbol{z})\}. \tag{18}$$

Then, using Eqs. (13) and (18), we obtain

$$2\mathrm{Re}\{\boldsymbol{B}\widehat{\boldsymbol{f}}_1(\sigma)\} = -\frac{1}{2}a\boldsymbol{t}_2^{\infty}\left(\frac{1}{\sigma} + \sigma\right), \tag{19}$$

where $\widehat{\boldsymbol{f}}_1(\sigma) = \boldsymbol{f}_1(\boldsymbol{z}(\sigma))$. Multiplying both sides of Eq. (19) by $\int_\gamma \mathrm{d}\sigma/(\sigma - \zeta)$, and according to Cauchy's formula for the infinite region,[20] we obtain

$$\widehat{\boldsymbol{f}}_1(\zeta) = -\frac{a}{2}\langle \zeta_\alpha^{-1}(z_\alpha)\rangle \boldsymbol{B}^{-1}\boldsymbol{t}_2^{\infty}, \tag{20}$$

where

$$\zeta_\alpha^{-1}(z_\alpha) = \frac{z_\alpha - \sqrt{z_\alpha^2 - (a^2 + p_\alpha^2 b^2)}}{a + \mathrm{i}p_\alpha b}.$$

Utilizing Eqs. (12), (15), (16), and (20), the stresses of the phonon and phason fields can be determined at the z_α plane as

$$\begin{aligned}
\boldsymbol{t}_1 &= \{\sigma_{1j}, H_{1j}\}^{\mathrm{T}} \\
&= -a\mathrm{Re}\left[\boldsymbol{B}\left\langle \frac{p_\alpha z_\alpha}{(a + ip_\alpha b)\sqrt{z_\alpha^2 - (a^2 + p_\alpha^2 b^2)}}\right\rangle \boldsymbol{B}^{-1}\right]\boldsymbol{t}_2^{\infty}, \\
\boldsymbol{t}_2 &= \{\sigma_{2j}, H_{2j}\}^{\mathrm{T}} \\
&= a\mathrm{Re}\left[\boldsymbol{B}\left\langle \frac{z_\alpha}{(a + ip_\alpha b)\sqrt{z_\alpha^2 - (a^2 + p_\alpha^2 b^2)}}\right\rangle \boldsymbol{B}^{-1}\right]\boldsymbol{t}_2^{\infty}.
\end{aligned} \tag{21}$$

4. Solution to a crack

As a special case, we consider an infinite icosahedral quasicrystal medium containing a Griffith crack with length of $2a$. The medium is subjected to remote uniform loads given by $\boldsymbol{t}_2^{\infty}$. Then, the complex potential function $\boldsymbol{f}(\boldsymbol{z})$ can be expressed as

$$\boldsymbol{f}(\boldsymbol{z}) = \frac{1}{2}\left\langle \sqrt{z_\alpha^2 - a^2}\right\rangle \boldsymbol{B}^{-1}\boldsymbol{t}_2^{\infty}. \tag{22}$$

Substituting Eq. (22) into Eqs. (11) and (12) yields

$$\begin{aligned}
\boldsymbol{t}_1 &= \{\sigma_{1j}, H_{1j}\}^{\mathrm{T}} = -\mathrm{Re}[\boldsymbol{B}\langle p_\alpha H_0(z_\alpha)\rangle \boldsymbol{B}^{-1}]\boldsymbol{t}_2^{\infty}, \\
\boldsymbol{t}_2 &= \{\sigma_{2j}, H_{2j}\}^{\mathrm{T}} = \mathrm{Re}[\boldsymbol{B}\langle H_0(z_\alpha)\rangle \boldsymbol{B}^{-1}]\boldsymbol{t}_2^{\infty},
\end{aligned} \tag{23}$$

where

$$H_0(z_\alpha) = \frac{z_\alpha}{\sqrt{z_\alpha^2 - a^2}}. \tag{24}$$

The field intensity factors at the right crack tip ($x_1 = a$) are defined as

$$\begin{aligned}
\boldsymbol{K} &= \{\mathrm{K_{II}}, \mathrm{K_I}, \mathrm{K_{III}}\}^{\mathrm{T}} \\
&= \lim_{x_1 \to a}[\sqrt{2\pi(x_1 - a)}\{\sigma_{21}, \sigma_{22}, \sigma_{23}\}^{\mathrm{T}}].
\end{aligned} \tag{25}$$

Inserting Eq. (23) into Eq. (25) leads to

$$\boldsymbol{K} = \sqrt{\pi a}\,\boldsymbol{t}_2^{\infty}. \tag{26}$$

Consider the special case in which a uniform tension σ_{22}^{∞} is applied at infinity and there is no coupling between the phonon and phason fields (i.e., $R = 0$). The analytic expressions for both the stresses and the displacements are obtained (see appendix A and B), which show that the phonon stresses are in accord with the classical solutions of isotropic materials.

It is obvious that the distributions of the elastic fields near the crack tip are of great interest to us. By introducing a polar coordinate system (r, θ) with the origin at the crack right tip, we have

$$z_\alpha = a + r(\cos\theta + p_\alpha \sin\theta). \tag{27}$$

When r is small compared to the half length a of the crack, equation (23) for the stresses can be rewritten as

$$\begin{aligned}
\boldsymbol{t}_1 &= -\sqrt{\frac{a}{2r}}\,\mathrm{Re}\left[\boldsymbol{B}\left\langle \frac{p_\alpha}{\Lambda_\alpha(\theta)}\right\rangle \boldsymbol{B}^{-1}\right]\boldsymbol{t}_2^{\infty}, \\
\boldsymbol{t}_2 &= \sqrt{\frac{a}{2r}}\,\mathrm{Re}\left[\boldsymbol{B}\left\langle \frac{1}{\Lambda_\alpha(\theta)}\right\rangle \boldsymbol{B}^{-1}\right]\boldsymbol{t}_2^{\infty},
\end{aligned} \tag{28}$$

where

$$\Lambda_\alpha(\theta) = \sqrt{\cos\theta + p_\alpha \sin\theta}.$$

In order to have a clear and direct picture on how the phonon–phason coupling elastic constant R affects the stresses near the crack tip, we use the icosahedral Al–Pd–Mn as an example to numerically demonstrate the normalized phonon stress σ_{22} near the crack tip, as shown in Fig. 2. The material properties[21] are given by

$\lambda = 74.9$ GPa, $\mu = 72.4$ GPa, $K_1 = 125$ MPa, $K_2 = -50$ MPa.

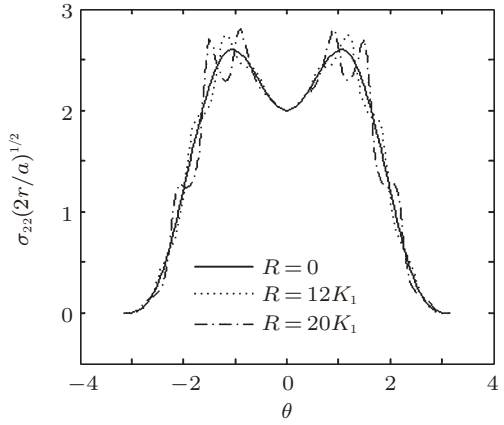


Fig. 2. Variation of the normalized phonon stress σ_{22} near the crack tip with the coupling parameter R ($\sigma_{22}^{\infty}=2$ MPa).

From Fig. 2, it can be seen that the phonon stress σ_{22} near the crack tip is significantly affected by the phonon–phason coupling elastic constants R. With the increase in the coupling parameter R, the effects on the stress is more and more obvious. Therefore, the phonon–phason coupling effect on the stresses is negligible.

The energy release rate can be obtained by considering the work done in closing the crack tip over an infinitesimal distance $\triangle x$ as

$$G = \lim_{\triangle x\to 0}\frac{1}{2\triangle x}\int_0^{\triangle x}\boldsymbol{t}_2^{\mathrm{T}}(x_1)\cdot\triangle\boldsymbol{u}(x_1-\triangle x)\,\mathrm{d}x_1 \quad (29)$$

$$= \frac{\pi a}{2}\boldsymbol{t}_2^{\infty\mathrm{T}}\cdot\mathrm{Re}\{\mathrm{i}AB^{-1}\}\cdot\boldsymbol{t}_2^{\infty}. \quad (30)$$

The normalized energy release rate ratio G/G_0 is the energy release rate for $R=0$ is plotted in Fig. 3, where

$$G_0=\frac{\pi a}{4}\left(\frac{1}{\mu}+\frac{1}{\lambda+\mu}\right)\sigma_{22}^{\infty 2}.$$

It can be found that the energy release rate increases with the increase in the coupling parameter R, i.e., the fracture toughness will be large for the large coupling parameter R. This conclusion is consistent with the results of Mariano[7] and Mikulla.[22]

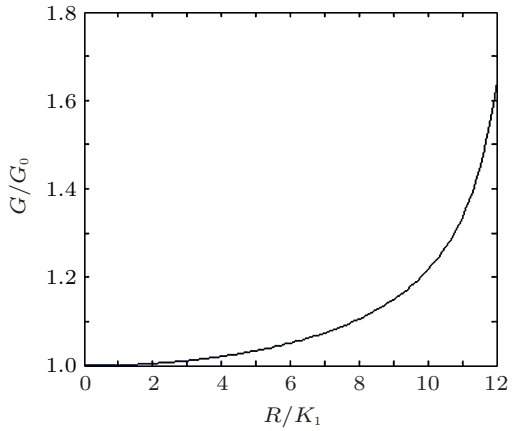


Fig. 3. Variation of the normalized energy release rate G/G_0 with the coupling parameter R.

5. Conclusion and discussion

We have studied the generalized 2D problem of an elliptic hole in icosahedral quasicrystal media by means of the extended Stroh formulism. The influences of the phonon–phason coupling elastic constants R on the stresses are numerically demonstrated. The results are reduced to the classical ones when the phason field is neglected. These results show that the extended Stroh formulism is powerful for the complicated boundary value problem of icosahedral quasicrystals.

Appendix A

The phonon and phason stress fields ($R=0$) of the crack can be given by

$$\sigma_{11} = \sigma_{22}^{\infty}\mathrm{Re}\left(\frac{z}{\sqrt{z^2-a^2}}-\frac{\mathrm{i}a^2y}{\sqrt{(z^2-a^2)^3}}-1\right),$$

$$\sigma_{12} = \sigma_{22}^{\infty}\mathrm{Re}\frac{a^2y}{\sqrt{(z^2-a^2)^3}},$$

$$\sigma_{22} = \sigma_{22}^{\infty}\mathrm{Re}\left(\frac{z}{\sqrt{z^2-a^2}}+\frac{\mathrm{i}a^2y}{\sqrt{(z^2-a^2)^3}}-1\right),$$

$$\sigma_{31} = \sigma_{22}^{\infty}a^2y\mathrm{Re}\left[\frac{\lambda\mathrm{i}}{2(\lambda+\mu)}\frac{1}{\sqrt{(z^2-a^2)^3}}+\frac{3}{4}\frac{yz}{\sqrt{(z^2-a^2)^5}}\right],$$

$$\sigma_{32} = \sigma_{22}^{\infty}a^2y\mathrm{Re}\left[-\frac{\lambda}{2(\lambda+\mu)}\frac{1}{\sqrt{(z^2-a^2)^3}}+\frac{3\mathrm{i}}{4}\frac{yz}{\sqrt{(z^2-a^2)^5}}\right],$$

$$H_{11} = \frac{\sigma_{22}^{\infty}a^2}{2}\left\{\frac{1}{\mu(\lambda+\mu)K_2^2}\mathrm{Re}\left[\mathrm{i}\lambda K_1^2(K_1-K_2)\frac{y}{\sqrt{(z^2-a^2)^3}}+\frac{3}{2}K_1(2\lambda K_2^2+K_1(2\lambda+\mu)(K_1-K_2))\frac{y^2z}{\sqrt{(z^2-a^2)^5}}-\frac{\mathrm{i}}{2}K_1(K_1(\lambda+\mu)(K_1-K_2)+K_2^2(5\lambda+2\mu))\frac{y^3(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}}\right]-\mathrm{Re}\left[\frac{15}{24}\frac{K_1}{\mu}\frac{(4\lambda+3\mu)}{(\lambda+\mu)}\frac{y^4z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}}-\frac{45\mathrm{i}}{120}\frac{K_1}{\mu}\frac{y^5(a^4+12a^2z^2+8z^4)}{\sqrt{(z^2-a^2)^{11}}}\right]\right\},$$

$$H_{12} = \frac{\sigma_{22}^{\infty}a^2}{2}\left\{\frac{1}{\mu(\lambda+\mu)K_2^2}\mathrm{Re}[-\lambda K_1(K_1^2-K_1K_2-K_2^2)\times\frac{y}{\sqrt{(z^2-a^2)^3}}+\frac{3\mathrm{i}}{2}K_1(K_1(2\lambda+\mu)(K_1-K_2)-K_2^2(\lambda+\mu))\frac{y^2z}{\sqrt{(z^2-a^2)^5}}+\frac{1}{2}K_1((2\lambda K_2^2+K_1(\lambda+\mu)(K_1-K_2))\frac{y^3(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}}\right]-\mathrm{Re}\left[\frac{15\mathrm{i}}{24}\frac{K_1}{\mu}\frac{(3\lambda+2\mu)}{(\lambda+\mu)}\frac{y^4z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}}\right.$$

$$+\frac{45}{120}\frac{K_1}{\mu}\frac{y^5(a^4+12a^2z^2+8z^4)}{\sqrt{(z^2-a^2)^{11}}}\Bigg]\Bigg\},$$

$$H_{21} = \frac{\sigma_{22}^{\infty}a^2}{4}\Bigg\{\frac{K_1}{\mu(\lambda+\mu)}\mathrm{Re}\Bigg[3\lambda\frac{y}{\sqrt{(z^2-a^2)^3}} - \frac{3\mathrm{i}}{2}(10\lambda+3\mu)\frac{y^2z}{\sqrt{(z^2-a^2)^5}}\Bigg] - \frac{1}{2}(13\lambda+7\mu) \times \frac{y^3(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}} + \frac{5\mathrm{i}}{4}(4\lambda+3\mu)\frac{y^4z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}} + \mathrm{Re}\Bigg[\frac{3}{4}\frac{K_1}{\mu}\frac{y^5(a^4+12a^2z^2+8z^4)}{\sqrt{(z^2-a^2)^{11}}}\Bigg]\Bigg\},$$

$$H_{22} = \frac{\sigma_{22}^{\infty}a^2}{4}\Bigg\{\frac{K_1}{\mu(\lambda+\mu)}\mathrm{Re}\Bigg[\mathrm{i}\lambda\frac{y}{\sqrt{(z^2-a^2)^3}} + \frac{3}{2}(4\lambda+\mu)\frac{y^2z}{\sqrt{(z^2-a^2)^5}}\Bigg] - \frac{\mathrm{i}}{2}(7\lambda+3\mu) \times \frac{y^3(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}} - \frac{5}{4}(3\lambda+2\mu)\frac{y^4z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}} + \mathrm{Re}\Bigg[\frac{3\mathrm{i}}{4}\frac{K_1}{\mu}\frac{y^5(a^4+12a^2z^2+8z^4)}{\sqrt{(z^2-a^2)^{11}}}\Bigg]\Bigg\},$$

$$H_{31} = \frac{\sigma_{22}^{\infty}a^2}{4}\frac{K_2^2+K_1K_2-K_1^2}{\mu K_2(\lambda+\mu)}\mathrm{Re}\Bigg[-3\mathrm{i}\lambda\frac{y}{\sqrt{(z^2-a^2)^3}} - \frac{3}{2}(8\lambda+3\mu)\frac{y^2z}{\sqrt{(z^2-a^2)^5}} + \frac{\mathrm{i}}{2}(7\lambda+5\mu) \times \frac{y^3(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}} + \frac{5}{4}(\lambda+\mu)\frac{y^4z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}}\Bigg],$$

$$H_{32} = \frac{\sigma_{22}^{\infty}a^2}{4}\frac{K_2^2+K_1K_2-K_1^2}{\mu K_2(\lambda+\mu)}\mathrm{Re}\Bigg[\lambda\frac{y}{\sqrt{(z^2-a^2)^3}} - \frac{3\mathrm{i}}{2}(4\lambda+\mu)\frac{y^2z}{\sqrt{(z^2-a^2)^5}} - \frac{1}{2}(5\lambda+3\mu) \times \frac{y^3(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}} + \frac{5\mathrm{i}}{4}(\lambda+\mu)\frac{y^4z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}}\Bigg].$$

Appendix B

The phonon and phason displacement fields ($R=0$) of the crack can be given by

$$u_1 = \frac{\sigma_{22}^{\infty}}{2}\mathrm{Re}\Bigg[\frac{1}{\lambda+\mu}(\sqrt{z^2-a^2}-z)+\frac{\mathrm{i}y}{\mu}\Bigg(\frac{z}{\sqrt{z^2-a^2}}-1\Bigg)\Bigg],$$

$$u_2 = -\frac{\sigma_{22}^{\infty}}{2}\mathrm{Re}\Bigg[\frac{\mathrm{i}(\lambda+2\mu)}{\mu(\lambda+\mu)}(\sqrt{z^2-a^2}-z) + \frac{y}{\mu}\Bigg(\frac{z}{\sqrt{z^2-a^2}}-1\Bigg)\Bigg],$$

$$u_3 = -\frac{\sigma_{22}^{\infty}y}{2}\mathrm{Re}\Bigg[\frac{y}{2\mu}\frac{a^2}{\sqrt{(z^2-a^2)^3}} + \frac{\mathrm{i}\lambda}{\mu(\lambda+\mu)}\Bigg(\frac{z}{\sqrt{z^2-a^2}}-1\Bigg)\Bigg].$$

$$w_1 = \frac{\sigma_{22}^{\infty}y}{4K_2^2\mu(\lambda+\mu)}\mathrm{Re}\Bigg[-\mathrm{i}\lambda(2K_1^2-2K_1K_2-K_2^2) \times \Bigg(\frac{z}{\sqrt{z^2-a^2}}-1\Bigg) - \frac{y}{2}(2K_1(2\lambda+\mu)(K_1-K_2) - \mu K_2^2)\frac{a^2}{\sqrt{(z^2-a^2)^3}} + \frac{\mathrm{i}y^2}{2}(2K_1(\lambda+\mu)(K_1-K_2) + K_2^2(5\lambda+\mu))\frac{a^2z}{\sqrt{(z^2-a^2)^5}} + \frac{K_2^2y^3}{4}(3\lambda+2\mu) \times \frac{a^2(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}} - \frac{\mathrm{i}K_2^2y^4}{4}(\lambda+\mu)\frac{a^2z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}}\Bigg],$$

$$w_2 = -\frac{\sigma_{22}^{\infty}y^2}{4\mu(\lambda+\mu)}\mathrm{Re}\Bigg[-\mathrm{i}\lambda\frac{a^2}{\sqrt{(z^2-a^2)^3}} - y(3\lambda+\mu) \times \frac{a^2z}{\sqrt{(z^2-a^2)^5}} + \frac{\mathrm{i}y^2}{4}(3\lambda+2\mu)\frac{a^2(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}} + \frac{y^3}{4}(\lambda+\mu)\frac{a^2z(3a^2+4z^2)}{\sqrt{(z^2-a^2)^9}}\Bigg],$$

$$w_3 = -\frac{\sigma_{22}^{\infty}K_1y}{4\mu(\lambda+\mu)K_2}\mathrm{Re}\Bigg[\mathrm{i}\lambda\Bigg(\frac{z}{\sqrt{z^2-a^2}}-1\Bigg) + \frac{y}{2}(4\lambda+\mu)\frac{a^2}{\sqrt{(z^2-a^2)^3}} - \frac{\mathrm{i}y^2}{2}(5\lambda+3\mu) \times \frac{a^2z}{\sqrt{(z^2-a^2)^5}} - \frac{y^3}{4}(\lambda+\mu)\frac{a^2(4z^2+a^2)}{\sqrt{(z^2-a^2)^7}}\Bigg].$$

References

[1] Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
[2] Bak P 1985 *Phys. Rev. Lett.* **54** 1517
[3] Hu C Z, Wang R H and Ding D H 2000 *Rep. Prog. Phys.* **63** 1
[4] Fu H, Liao L G and Fu X J 2009 *Chin. Phys. Lett.* **26** 056103
[5] Wen Z B, Hou Z L and Fu X J 2011 *Chin. Phys. Lett.* **28** 046102
[6] Li L H and Fan T Y 2006 *J. Phys.: Condens. Matter* **18** 10631
[7] Radi E and Mariano P M 2010 *Int. J. Fract.* **166** 105
[8] Liu G T, He Q L and Guo R P 2009 *Acta Phys. Sin.* **58** 118 (in Chinese)
[9] Li X F and Fan T Y 1998 *Chin. Phys. Lett.* **15** 278
[10] Fan T Y and Mai Y W 2004 *Appl. Mech. Rev.* **57** 325
[11] Wang J B, Mancini L, Wang R H and Gastaldi J 2003 *J. Phys.: Condens. Matter* **15** L363
[12] Fan T Y 2011 *Mathematical Theory of Elasticity of Quasicrystals and Its Applications* (Berlin: Springer) pp. 120–128
[13] Gao Y 2012 *Pramana-J. Phys.* **79** 311
[14] Gao Y, Ricoeur A and Zhang L 2011 *Phys. Lett. A* **375** 2775
[15] Li L H and Liu G T 2012 *Phys. Lett. A* **3769** 987
[16] Wang X and Schiavone P 2013 *Math. Mech. Complex Syst.* **1** 1
[17] Levine D, Lubensky T C, Ostlund S, Ramaswamy S, Steinhardt P J and Toner J 1985 *Phys. Rev. Lett.* **54** 1520
[18] Gao Y, Zhao B S and Xu S P 2008 *J. Elast.* **93** 263
[19] Ting T C T 1996 *Anisotropic Elasticity: Theory and Applications* (New York: Oxford Science Publications)
[20] Muskhelishvili N I 1963 *Some Basic Problems of the Mathematical Theory of Elasticity* (Noordhoff: Groningen)
[21] Edagawa K and Takeuchi S 2007 *Dislocation in Solids* **13** 365
[22] Mikulla R, Gumbsch P and Trebin H R 1998 *Phil. Mag. Lett.* **78** 369