Bounded consensus tracking of second-order multi-agent systems with sampling delay under directed networks*

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(Received 6 May 2013; revised manuscript received 31 May 2013)

The bounded consensus tracking problems of second-order multi-agent systems under directed networks with sampling delay are addressed in this paper. When the sampling delay is more than a sampling period, new protocols based on sampled-data control are proposed so that each agent can track the time-varying reference state of the virtual leader. By using the delay decomposition approach, the augmented matrix method, and the frequency domain analysis, necessary and sufficient conditions are obtained, which guarantee that the bounded consensus tracking is realized. Furthermore, some numerical simulations are presented to demonstrate the effectiveness of the theoretical results.

Keywords: second-order multi-agent systems, bounded consensus tracking, sampling delay, directed networks

PACS: 05.65.+b, 02.30.Yy, 02.10.Yn **DOI:** 10.1088/1674-1056/22/11/110505

1. Introduction

As one of the fundamental problems in the distributed control of multi-agent systems, the consensus of multi-agent systems has been an active research area in the past several years, the main idea of which is to design appropriate protocols such that each agent updates its own states through local interaction without a central controller or global communication, and the states of all the agents may reach an agreement on certain quantities of value eventually.

Following the work of Olfati-Saber and Murray,[1] which is looked upon as the typical framework of consensus protocols for continuous-time first-order multi-agent systems, Moreau[2] considered the discrete-time consensus protocols of first-order multi-agent systems employing set-valued Lyapunov theory. Up to now, there have been several research topics on consensus problems of first-order multi-agent systems, such as finite-time consensus,[3,4] asynchronous information consensus,[5,6] consensus with time-delays,[7,8] and so on. Besides, in recent years a lot of research has focused on the consensus problems of second-order multi-agent systems, in which agents such as mobile robots can update their motions directly by their accelerations rather than by their velocities.[9–11]

Among most of the above literature, the final consensus states are usually functions of all the agents' states, which are unknown in advance. However, agents are often required to converge to a desired common value in real applications, which can be classified as the consensus tracking,[12] where all the agents exchange their own state information with their neighbors as the goal of tracking a time-varying reference state available to only a portion of agents. Ren[13] solved the consensus tracking problems by proposing the proportional and derivative like consensus protocols of leader-following multi-agent systems. Other major issues concerning the study of consensus tracking include the controllability,[14] the distributed observer design,[15] and the connection maintenance[16] to name but a few.

With the broad applications of digital communication and control in distributed systems, some problems such as the reliability of information channels, the constraints of the network's bandwidth and the communication cost must be paid attention to. Considering it is difficult to ensure the continuity of full-state information transmission in many cases, the research about consensus of multi-agent systems based on sampled-data control is of great significance, where each agent can obtain the measurements just at sampling instants. Sampled-data consensus of second-order multi-agent systems had been widely developed by then.[17,18] Necessary and sufficient conditions that guarantee second-order multi-agent systems via sampled control to achieve consensus with zero relative positions and a common velocity were obtained in Ref. [19]. In addition, the sampled-data consensus problems of second-order multi-agent systems under undirected and directed topologies were discussed in Refs. [20] and [21], respectively.

In engineering applications, how to choose the sampling period guaranteeing the multi-agent system runs well should be taken into account. Li and Zhang[22] proved that when the sampling size was sufficiently small, the static mean square error between the individual state and the average initial states of all agents was arbitrarily small. Moreover, sampling delays might cause the system to oscillate or diverge, thus the delay effects on the stability of multi-agent systems cannot be

*Project supported by the National Natural Science Foundation of China (Grant Nos. 60874053 and 61034006).

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ignored. The cases of sampling delay less than a sampling period have been considered in Refs. [23] and [24]. However, to the best of our knowledge, the case that sampling delay is more than a sampling period in second-order multi-agent systems under a directed network topology has not been studied.

Different from the aforementioned literature, our main idea is to investigate the bounded consensus tracking problems of second-order multi-agent systems under directed networks, where the control input of the individual can only use the information measured at the sampling instant from its neighbors or the virtual leader with a time-varying reference state, and the measurements are corrupted by sampling delay, which might be more than a sampling period. Then, we will explore necessary and sufficient conditions to guarantee the bounded consensus tracking.

An outline of the rest of this paper is as follows. In Section 2, we review some concepts in algebra graph theory and provide some useful lemmas. Section 3 presents the bounded consensus tracking protocols based on sampled-data control, and Section 4 investigates the convergence of the second-order multi-agent systems under directed networks with sampling delay. Numerical simulation results are provided to illustrate the utility of the theoretical results in Section 5. Finally, Section 6 concludes the whole paper.

Throughout this paper, $\mathbf{0}_n$, $\mathbf{1}_n$, $\mathbf{0}_{n\times n}$ and $\boldsymbol{I}_n$ denote the n-dimensional column vector with all zeros, the n-dimensional column vector with all ones, the $n\times n$ matrix with all zeros, and the $n\times n$ identity matrix, respectively. Let $\lambda\in\mathbb{C}$ be a complex number, $\mathrm{Re}(\lambda)$, $\mathrm{Im}(\lambda)$, and $|\lambda|$ be the real part, the imaginary part, and the modulus of λ, respectively, then we have $\lambda=\mathrm{Re}(\lambda)+\mathbf{i}\,\mathrm{Im}(\lambda)$, where $\mathbf{i}$ is the imaginary unit. For a n-dimensional matrix $\boldsymbol{M}$, $\lambda_i(\boldsymbol{M})$ $(i\in\boldsymbol{I})$ denote all of its eigenvalues. Meanwhile, $\lambda_{\max}(\boldsymbol{M})$ is the maximum eigenvalue of $\boldsymbol{M}$. The norm, the determinant, and the trace of matrix $\boldsymbol{M}$ are denoted by $\|\boldsymbol{M}\|$, $\det(\boldsymbol{M})$, and $\mathrm{Tr}(\boldsymbol{M})$, respectively.

2. Brief review of graph theory and useful lemmas

Graph theory is an effective tool in modeling the interaction topology among agents. A set of nodes $\boldsymbol{V}=\{1,2,\ldots,n\}$, a set of edges $\boldsymbol{E}\subseteq\boldsymbol{V}\times\boldsymbol{V}$, and a weighted adjacency matrix $\boldsymbol{A}=[a_{ij}]\in\boldsymbol{R}^{n\times n}$ compose a weighted directed graph (or digraph for short) $\boldsymbol{G}=(\boldsymbol{V},\boldsymbol{E},\boldsymbol{A})$. An edge of $\boldsymbol{G}$ is denoted by $e_{ij}=(i,j)$, which starts from i and ends on j. The adjacency elements associated with the edges are positive, i.e., $e_{ij}\in\boldsymbol{E}\Leftrightarrow a_{ij}>0$, and we assume $a_{ii}=0$ for all $i\in\boldsymbol{I}$. Thus, the adjacency elements a_{ij} are nonnegative.

The set of neighbors of node i is denoted by $N_i=\{j\in\boldsymbol{V}:e_{ij}\in\boldsymbol{E}\}$. The Laplacian matrix of the digraph G is defined as $\boldsymbol{L}=[l_{ij}]\in\boldsymbol{R}^{n\times n}$, where $l_{ii}=\sum_{j\neq i}a_{ij}$ and $l_{ij}=-a_{ij}$ $(i\neq j)$. An important fact of $\boldsymbol{L}$ is that all the row sums are zero, so $\boldsymbol{L}$ has a right eigenvector $\mathbf{1}_n$ associated with the zero eigenvalue. A directed path between two distinct nodes i and j means a finite ordered sequence of ordered edges whose form is like $(i,k_1),(k_1,k_2),\ldots,(k_l,j)$ in a digraph. If there exists a path from node i to node j, we say that j is reachable from i. A digraph G is strongly connected if there is a path between any two distinct nodes. Moreover, for node i, if there exists a path from any other nodes to node i, the node i is said to be globally reachable, which is much weaker than strong connectedness.

The following lemmas will play important roles in the convergence analysis of bounded consensus tracking.

Lemma 1 (Schur's formula) Let $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$, $\boldsymbol{M}_4\in R^{n\times n}$, where n denotes the dimension of a matrix, then $\det\left(\begin{bmatrix}\boldsymbol{M}_1 & \boldsymbol{M}_2\\ \boldsymbol{M}_3 & \boldsymbol{M}_4\end{bmatrix}\right)=\det(\boldsymbol{M}_1\boldsymbol{M}_4-\boldsymbol{M}_3\boldsymbol{M}_2)$, if $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$, and $\boldsymbol{M}_4$ commute pairwise.

Lemma 2 All eigenvalues of a matrix $\boldsymbol{M}\in\boldsymbol{R}^{n\times n}$ are within the unit circle, if and only if there exists a matrix norm $\|\cdot\|_*$ such that $\|M\|_*<1$.[25]

Lemma 3 (Hermite–Biehler theorem) The polynomial $f(\sigma)=f(\mathbf{i}\omega)=p(\omega)+\mathbf{i}q(\omega)$ is Hurwitz stable if and only if the related pair $p(\omega)$ and $q(\omega)$ are interlaced, and $p(0)q'(0)-p'(0)q(0)>0$, where $\omega\in R$.[26]

Lemma 4 Given a digraph $\boldsymbol{G}$ composed of n agents, the matrix $\boldsymbol{H}=\boldsymbol{B}+\boldsymbol{L}$ must be positively stable if and only if the virtual leader is globally reachable, i.e., all the eigenvalues of $\boldsymbol{H}$ have positive real parts, where the matrices $\boldsymbol{B}$ and $\boldsymbol{L}$ are the leader adjacency matrix and the Laplacian matrix associated with $\boldsymbol{G}$, respectively.[27]

3. Consensus protocols based on sampled-data control

Consider the second-order multi-agent systems modeled by

$$\begin{cases}\dot{x}_i(t)=y_i(t),\\ \dot{y}_i(t)=u_i(t),\end{cases}\tag{1}$$

where $i\in\boldsymbol{I}$, $x_i\in\boldsymbol{R}$, and $y_i\in\boldsymbol{R}$ are the position state and the velocity state of agent i, respectively, and $u_i\in\boldsymbol{R}$ is the associated control input, which is called the protocol. Note that x_i and y_i may also represent other physical quantities, for example, x_i may play the role of altitude or heading angle, while y_i may mean climb rate or angular velocity of agent i.

For the continuous-time multi-agent systems (1), a classical consensus protocol was proposed as[28]

$$u_i(t)=-\sum_{j\in N_i}a_{ij}\left[(x_j(t)-x_i(t))+\alpha(y_j(t)-y_i(t))\right],\tag{2}$$

where $i\in\boldsymbol{I}$ and $\alpha>0$ is a constant feedback gain. The consensus protocol (2) is said to solve a consensus problem asymptotically if $x_i(t)\to x_j(t)$ and $y_i(t)\to y_j(t)$, $\forall i\neq j$, as $t\to\infty$ for any $x_i(0)$ and $y_j(0)$.

Furthermore, for the leader-following multi-agent systems, Ren[29] supposed all the agents with access to the virtual leader, and proposed the following continuous-time consensus tracking protocol:

$$u_i(t) = \dot{y}^0(t) - \beta\left[(x_i(t)-x^0(t)) + \alpha(y_i(t)-y^0(t))\right] + \sum_{j\in N_i} a_{ij}\left[(x_j(t)-x_i(t)) + \alpha(y_j(t)-y_i(t))\right], \quad (3)$$

where $i \in \boldsymbol{I}$, $\alpha > 0$, and $\beta > 0$ are constant feedback gains. If all the agents track the virtual leader with the desired reference state $(x^0(t), y^0(t))$, i.e., $x_i(t) \to x^0(t)$, $y_i(t) \to y^0(t)$, $\forall i \in I$, as $t \to \infty$ for all $x_i(0)$ and $y_i(0)$, then the consensus tracking is reached under the protocol (3).

By using the direct discretization in Ref. [30], the continuous-time system (1) becomes the following discrete-time sampled-data model:

$$\begin{cases} x_i(kT+T) = x_i(kT) + Ty_i(kT) + 0.5T^2u_i(kT), \\ y_i(kT+T) = y_i(kT) + Tu_i(kT), \end{cases} \quad (4)$$

where $k = 0, 1, \ldots$, $i \in \boldsymbol{I}$, $T > 0$ represents the sampling period, and k represents the discrete-time index.

Denote

$$\tau_s = mT + \varepsilon, \quad (5)$$

where $\tau_s > 0$ represents the sampling delay, m is a nonnegative integer, and $\varepsilon \in [0, T)$.

Assume that only a portion of agents have access to the virtual leader, then the following consensus tracking protocol based on sampled-data with the sampling delay is proposed, which is induced from the continuous-time consensus tracking protocol (3):

$$u_i(t) = \begin{cases} b_i\Big\{\frac{1}{T}[y^0(kT-mT-T) - y^0(kT-mT-2T)] \\ -\{[x_i(kT-mT-T) - x^0(kT-mT-T)] \\ +\alpha[y_i(kT-mT-T) - y^0(kT-mT-T)]\}\Big\} \\ + \sum\limits_{j\in N_i} a_{ij}\{[x_j(kT-mT-T) - x_i(kT-mT-T)] \\ +\alpha[y_j(kT-mT-T) - y_i(kT-mT-T)]\}, \\ t \in [kT, kT+\varepsilon), \\ b_i\Big\{\frac{1}{T}[y^0(kT-mT) - y^0(kT-mT-T)] \\ -\{[x_i(kT-mT) - x^0(kT-mT)] \\ +\alpha[y_i(kT-mT) - y^0(kT-mT)]\}\Big\} \\ + \sum\limits_{j\in N_i} a_{ij}\{[x_j(kT-mT) - x_i(kT-mT)] \\ +\alpha[y_j(kT-mT) - y_i(kT-mT)]\}, \\ t \in [kT+\varepsilon, kT+T), \end{cases}$$
$$k = 0, 1, 2, \ldots;\ i \in I. \quad (6)$$

Here, the constant feedback gain $b_i > 0$, if the virtual leader is a neighbor of the agent i, otherwise b_i is equal to zero. Thus, the leader adjacency matrix $\boldsymbol{B} = \mathrm{diag}(b_1, b_2, \ldots, b_n)$.

Definition 1 The protocol (6) is called a bounded consensus tracking protocol, if the second-order multi-agent systems (4) using the protocol (6) has the property of

$$\begin{cases} \lim_{k\to\infty} \left\| x_i(kT) - x^0(kT) \right\| \le C < \infty, \\ \lim_{k\to\infty} \left\| y_i(kT) - y^0(kT) \right\| \le C < \infty, \end{cases} \quad \forall i \in \boldsymbol{I},$$

where C is a bounded positive constant independent of k.

4. Bounded consensus tracking analysis

The convergence analysis of consensus problems under directed network topologies is more challenging than that under undirected topologies due to the complexity of directed topologies. In addition, the research on undirected topologies may be looked upon as a special case of directed topologies, for the strong connected directed topology is equivalent to the undirected topology. In this section, we will find the necessary and sufficient conditions guaranteeing the system (4) using the protocol (6) to achieve the bounded consensus tracking.

Employing the augmented matrix method, some other mathematical technologies and stability criterion, we obtain the following main results.

Theorem 1 Consider a directed network topology $\boldsymbol{G}$ composed of n agents, and the virtual leader is globally reachable. Then, the second-order multi-agent system (4), applying the consensus tracking protocol (6), achieves bounded consensus tracking if and only if all the roots of the following equation:

$$s^{m+3} - 2s^{m+2} + s^{m+1} + (0.5T+\alpha)(T-\varepsilon)\lambda_i(H)s^2 + (0.5T^2 - \alpha T + 2\alpha\varepsilon)\lambda_i(H)s + (0.5T-\alpha)\varepsilon\lambda_i(H) = 0 \quad (7)$$

are within the unit circle.

Proof Substituting the piecewise protocol (6) into the system (4) leads to

$$\begin{aligned} &\boldsymbol{x}(kT+T) \\ &= x(kT) + Ty(kT) + 0.5T\varepsilon\Big\{\frac{B}{T}[y^0(kT-mT-T) \\ &\quad - y^0(kT-mT-2T)]1_n \\ &\quad - B\{[x(kT-mT-T) - x^0(kT-mT-T)1_n] \\ &\quad + \alpha[y(kT-mT-T) - y^0(kT-mT-T)1_n]\} \\ &\quad - L[x(kT-mT-T) + \alpha y(kT-mT-T)]\Big\} \\ &\quad + 0.5T(T-\varepsilon)\Big\{\frac{B}{T}[y^0(kT-mT) \\ &\quad - y^0(kT-mT-T)]1_n - B\{[x(kT-mT) \\ &\quad - x^0(kT-mT)1_n] + \alpha[y(kT-mT) - y^0(kT-mT)1_n]\} \\ &\quad - L[x(kT-mT) + \alpha y(kT-mT)]\}, \\ &\boldsymbol{y}(kT+T) \\ &= y(kT) + \varepsilon\Big\{\frac{B}{T}[y^0(kT-mT-T) \\ &\quad - y^0(kT-mT-2T)]1_n \end{aligned}$$

$$
\begin{aligned}
&-B\{[x(kT-mT-T)-x^0(kT-mT-T)1_n]\\
&+\alpha[y(kT-mT-T)-y^0(kT-mT-T)1_n]\}\\
&-L[x(kT-mT-T)+\alpha y(kT-mT-T)]\Big\}\\
&+(T-\varepsilon)\Big\{\frac{B}{T}[y^0(kT-mT)-y^0(kT-mT-T)1_n]\\
&-B\{[x(kT-mT)-x^0(kT-mT)1_n]\\
&+\alpha[y(kT-mT)-y^0(kT-mT)1_n]\}\\
&-L[x(kT-mT)+\alpha y(kT-mT)]\Big\},\\
&k=0,1,2,\ldots;\ i\in \boldsymbol{I}, \qquad (8)
\end{aligned}
$$

where $\boldsymbol{x}(kT)=[x_1(kT),x_2(kT),\ldots,x_n(kT)]^{\mathrm{T}}$ and $y(kT)=[y_1(kT),y_2(kT),\ldots,y_n(kT)]^{\mathrm{T}}$.

Set $\Delta x(kT)=[x_1(kT)-x^0(kT),\ x_2(kT)-x^0(kT),\ \ldots,\ x_n(kT)-x^0(kT)]^{\mathrm{T}}$, $\Delta y(kT)=[y_1(kT)-y^0(kT),\ y_2(kT)-y^0(kT),\ \ldots,\ y_n(kT)-y^0(kT)]^{\mathrm{T}}$ and $\Delta z(kT)=[\Delta x(kT),\Delta y(kT)]^{\mathrm{T}}$. Define the tracking error vector as $\boldsymbol{\delta}(kT)=[\Delta \boldsymbol{z}(kT),\ \Delta \boldsymbol{z}(kT-T),\ \ldots,\ \Delta \boldsymbol{z}(kT-mT-T)]^{\mathrm{T}}$. Note that $\boldsymbol{L}1_n=\boldsymbol{0}_n$. Combining (8) and the following $m+1$ auxiliary equations:

$$
\begin{cases}
\Delta \boldsymbol{z}(kT)=\Delta \boldsymbol{z}(kT),\\
\Delta \boldsymbol{z}(kT-T)=\Delta \boldsymbol{z}(kT-T),\\
\quad\vdots\\
\Delta \boldsymbol{z}(kT-mT)=\Delta \boldsymbol{z}(kT-mT),
\end{cases}
$$

we can obtain

$$
\begin{aligned}
\boldsymbol{\delta}(NT) &= \boldsymbol{P}^N\boldsymbol{\delta}(0)\\
&\quad+\sum_{k=0}^{N-1}\boldsymbol{P}^{N-k-1}\left[\boldsymbol{U}^0(kT),\boldsymbol{V}^0(kT),\boldsymbol{0}_{2(m+1)n}\right]^{\mathrm{T}}, \qquad (9)
\end{aligned}
$$

where $\boldsymbol{\delta}(0)$ means the initial state of the tracking error vector $\boldsymbol{\delta}(kT)$,

$$
\begin{aligned}
\boldsymbol{U}^0(kT) &= [-x^0(kT+T)+x^0(kT)+Ty^0(kT)]1_n\\
&\quad+0.5B\{\varepsilon[y^0(kT-mT-T)\\
&\quad-y^0(kT-mT-2T)]+(T-\varepsilon)\\
&\quad\times[y^0(kT-mT)-y^0(kT-mT-T)]\}1_n,\\
\boldsymbol{V}^0(kT) &= [-x^0(kT+T)+x^0(kT)+Ty^0(kT)]1_n\\
&\quad+\frac{B}{T}\{\varepsilon[y^0(kT-mT-T)\\
&\quad-y^0(kT-mT-2T)]+(T-\varepsilon)\\
&\quad\times[y^0(kT-mT)-y^0(kT-mT-T)]\}1_n,
\end{aligned}
$$

and

$$
\boldsymbol{P}=\begin{bmatrix}
\boldsymbol{Q}_0 & \boldsymbol{0}_{2n\times 2n} & \cdots & \boldsymbol{0}_{2n\times 2n} & -\boldsymbol{Q}_1 & -\boldsymbol{Q}_2\\
\boldsymbol{I}_{2n} & \boldsymbol{0}_{2n\times 2n} & \cdots & \boldsymbol{0}_{2n\times 2n} & \boldsymbol{0}_{2n\times 2n} & \boldsymbol{0}_{2n\times 2n}\\
\boldsymbol{0}_{2n\times 2n} & \boldsymbol{I}_{2n}\boldsymbol{0}_{2n\times 2n} & \cdots & \boldsymbol{0}_{2n\times 2n} & \boldsymbol{0}_{2n\times 2n} & \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\
\boldsymbol{0}_{2n\times 2n} & \cdots & \boldsymbol{0}_{2n\times 2n} & \boldsymbol{I}_{2n}\boldsymbol{0}_{2n\times 2n} & \boldsymbol{0}_{2n\times 2n} & \\
\boldsymbol{0}_{2n\times 2n} & \boldsymbol{0}_{2n\times 2n} & \cdots & \boldsymbol{0}_{2n\times 2n} & \boldsymbol{I}_{2n} & \boldsymbol{0}_{2n\times 2n}
\end{bmatrix},
$$

with

$$
\begin{aligned}
\boldsymbol{Q}_0 &= \begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{T}\boldsymbol{I}_n\\ \boldsymbol{0}_{n\times n} & \boldsymbol{I}_n\end{bmatrix},\\
\boldsymbol{Q}_1 &= \begin{bmatrix}0.5T(T-\varepsilon)H & 0.5\alpha T(T-\varepsilon)H\\ (T-\varepsilon)H & \alpha(T-\varepsilon)H\end{bmatrix},\\
\boldsymbol{Q}_2 &= \begin{bmatrix}0.5T\varepsilon H & 0.5T\varepsilon\alpha H\\ \varepsilon H & \varepsilon\alpha H\end{bmatrix}.
\end{aligned}
$$

According to Lemma 1, the characteristic polynomial of $\boldsymbol{P}$ is

$$
\begin{aligned}
F(s) &= \det\left(s\boldsymbol{I}_{2(m+2)n}-\boldsymbol{P}\right)\\
&= \det\left(\begin{bmatrix}s^{m+1}\boldsymbol{I}_{2n}-s^m\boldsymbol{Q}_0 & s\boldsymbol{Q}_1+\boldsymbol{Q}_2\\ -\boldsymbol{I}_{2n} & s\boldsymbol{I}_{2n}\end{bmatrix}\right)\\
&= \det\left(s^{m+2}\boldsymbol{I}_{2n}-s^{m+1}\boldsymbol{Q}_0+s\boldsymbol{Q}_1+\boldsymbol{Q}_2\right)\\
&= s^{(m+1)n}\Pi_{i=1}^{n}\{s^{m+3}-2s^{m+2}+s^{m+1}\\
&\quad+[0.5T(T-\varepsilon)+\alpha(T-\varepsilon)]\lambda_i(\boldsymbol{H})s^2\\
&\quad+(0.5T^2-\alpha T+2\alpha\varepsilon)\lambda_i(\boldsymbol{H})s\\
&\quad+(0.5T\varepsilon-\alpha\varepsilon)\lambda_i(\boldsymbol{H})\}.
\end{aligned}
$$

Obviously, the characteristic equation $F(s)=0$ has $(m+1)n$ roots equal to zero. Therefore, $\rho(P)<1$ if and only if all the roots of Eq. (7) are within the unit circle, where $\rho(P)$ denotes the spectral radius of P.

From Lemma 2, which is necessary and sufficient, we can draw the conclusion that when all the roots of Eq. (7) are within the unit circle, i.e., $\rho(P)<1$, there exists a matrix norm $\|\cdot\|_*$ such that

$$
\|P\|_*<1. \qquad (10)
$$

Based on the equivalence of matrix norms, there exists a constant $C_*>0$ such that $\|\cdot\|_*\le C_2\|\cdot\|_2$. In addition, we assume that the changing rate of the velocity state of the virtual leader is bounded, i.e.,

$$
\left\|\frac{1}{T}[\boldsymbol{y}^0(kT+T)-\boldsymbol{y}^0(kT)]\right\|\le\xi^0<\infty,\ \forall T>0,\ k=0,1,2,\ldots
$$

Then, from (9), we obtain

$$
\begin{aligned}
&\|\boldsymbol{\delta}(NT)\|_*\\
&\le \left\|\boldsymbol{P}^N\right\|_*\|\boldsymbol{\delta}(0)\|_*\\
&\quad+\left\|\sum_{k=0}^{N-1}\boldsymbol{P}^{N-k-1}\left[\boldsymbol{U}^0(kT),\boldsymbol{V}^0(kT),\boldsymbol{0}_{2(m+1)n}\right]^{\mathrm{T}}\right\|_*\\
&\le C_2\|\boldsymbol{\delta}(0)\|_2\|\boldsymbol{P}\|_*^N\\
&\quad+C_2T\xi^0\sqrt{T^2(\sqrt{n}+0.5\sqrt{\mathrm{Tr}(\boldsymbol{B})})^2+(\sqrt{n}+\sqrt{\mathrm{Tr}(\boldsymbol{B})})^2}\\
&\quad\times\frac{1-\|\boldsymbol{P}\|_*^{N-1}}{1-\|\boldsymbol{P}\|_*}. \qquad (11)
\end{aligned}
$$

Combining (10) and (11) leads to

$$
\limsup_{N\to\infty}\|\boldsymbol{\delta}(NT)\|_*
$$

$$\leq \frac{C_2 T\xi^0}{1-\|\boldsymbol{P}\|_*}\sqrt{T^2(\sqrt{n}+0.5\sqrt{\mathrm{Tr}(\boldsymbol{B})})^2+(\sqrt{n}+\sqrt{\mathrm{Tr}(\boldsymbol{B})})^2},$$

which further yields

$$\begin{aligned}&\limsup_{N\to\infty}\left\|\begin{pmatrix}\boldsymbol{x}(NT)-\boldsymbol{x}^0(NT)\mathbf{1}_n\\ \boldsymbol{y}(NT)-\boldsymbol{y}^0(NT)\mathbf{1}_n\end{pmatrix}\right\|_*\\ &\leq \frac{C_2 T\xi^0}{(m+2)(1-\|P\|_*)}\\ &\quad\times\sqrt{T^2(\sqrt{n}+0.5\sqrt{\mathrm{Tr}(\boldsymbol{B})})^2+(\sqrt{n}+\sqrt{\mathrm{Tr}(\boldsymbol{B})})^2}<\infty.\end{aligned}$$

According to Definition 1, under the given conditions, the second-order multi-agent system (4), applying the consensus tracking protocol (6), can achieve the bounded consensus tracking. Thus, Theorem 1 is proved.

Remark 1 In the above analysis, we assume $(\boldsymbol{x}_i(-lT), \boldsymbol{y}_i(-lT)) = (\boldsymbol{x}_i(0), \boldsymbol{y}_i(0))$, $i \in \boldsymbol{I}$, and $(\boldsymbol{x}^0(-lT), \boldsymbol{y}^0(-lT)) = (x^0(0), y^0(0))$, $l = 1,2,\ldots,m+1$, which does not affect the results obtained in Theorem 1.

Remark 2 In the delay decomposition expression (5), for given sampling delay τ_{s}, m increases with the decrease in T. When m is large, it is not easy to derive the explicit allowable scope of the sampling period T and the sampling delay τ_{s} from the condition (7).

Employing the particular delay decomposition technology (5), we have the following corollaries from Theorem 1.

Corollary 1 Consider a directed network topology $\boldsymbol{G}$ composed of n agents, and the virtual leader is globally reachable. Then the multi-agent system (4), applying the consensus tracking protocol (6) with $m = 0$ and $Im(\lambda) = 0$, achieves the bounded consensus tracking if and only if the following three conditions

$$\alpha^2 < \frac{1}{\mathrm{Re}(\lambda_{\max})}, \tag{12}$$

$$T > \max\left\{\frac{2\sqrt{[\sqrt{\mathrm{Re}(\lambda)}\alpha-1]^2+1}+2}{\sqrt{Re(\lambda)}}\right\}, \tag{13}$$

and

$$\begin{aligned}&\max\left\{\frac{\mathrm{Re}(\lambda)T^2-4}{2\mathrm{Re}(\lambda)(T-2\alpha)}, \frac{T}{2}-\frac{1}{\alpha\mathrm{Re}(\lambda)},\right.\\ &\left.\frac{[\mathrm{Re}(\lambda)(T^2-4\alpha^2)-4]-\sqrt{\Delta_{f_1(\tau_{\mathrm{s}})}}}{2\mathrm{Re}(\lambda)(T-2\alpha)}\right\}<\tau_{\mathrm{s}}\\ &<\min\left\{\frac{[\mathrm{Re}(\lambda)(T^2-4\alpha^2)-4]+\sqrt{\Delta_{f_1(\tau_{\mathrm{s}})}}}{2\mathrm{Re}(\lambda)(T-2\alpha)}, \alpha, T\right\}\end{aligned} \tag{14}$$

hold simultaneously.

Proof Obviously, Corollary 1 is one of the special cases of Theorem 1. When $m = 0$, i.e., $0 \leq \tau_{\mathrm{s}} = \varepsilon < T$, Eq. (7) becomes the following equation with complex coefficients:

$$\begin{aligned}s^3 &+ [(0.5T+\alpha)(T-\tau_{\mathrm{s}})\lambda_i(H)-2]s^2+[(0.5T^2-\alpha T\\ &+ 2\alpha\tau_{\mathrm{s}})\lambda_i(H)+1]s+(0.5T-\alpha)\tau_{\mathrm{s}}\lambda_i(H)=0.\end{aligned} \tag{15}$$

Applying the bilinear transformation $\sigma = \phi(s) = \frac{s+1}{s-1}$, we obtain a new polynomial associated with Eq. (15) as follows:

$$\begin{aligned}f(\sigma) &= (\sigma-1)^3\left\{\frac{(\sigma+1)^3}{(\sigma-1)^3}+[(0.5T+\alpha)(T-\tau_{\mathrm{s}})\lambda_i(H)-2]\right.\\ &\quad\times\frac{(\sigma+1)^2}{(\sigma-1)^2}+[(0.5T^2-\alpha T+2\alpha\tau_{\mathrm{s}})\lambda_i(H)+1]\\ &\quad\left.\times\frac{\sigma+1}{\sigma-1}+(0.5T-\alpha)\tau_{\mathrm{s}}\lambda_i(H)\right\}\\ &= T^2\lambda_i(H)\sigma^3+2T\lambda_i(H)(\alpha-\tau_{\mathrm{s}})\sigma^2\\ &\quad+[(2T\tau_{\mathrm{s}}-T^2-4\alpha\tau_{\mathrm{s}})\lambda_i(H)+4]\sigma\\ &\quad+2\alpha\lambda_i(H)(2\tau_{\mathrm{s}}-T)+4.\end{aligned}$$

Note $\lambda = \mathrm{Re}(\lambda)+\mathbf{i}\mathrm{Im}(\lambda)$, the above equation is rewritten as

$$\begin{aligned}&f_1(\sigma)\\ &= \sigma^3+\frac{2(\alpha-\tau_{\mathrm{s}})}{T}\sigma^2\\ &\quad+\frac{(2T\tau_{\mathrm{s}}-T^2-4\alpha\tau_{\mathrm{s}})|\lambda_i(H)|^2+4\mathrm{Re}(\lambda)-4\mathrm{Im}(\lambda)\mathbf{i}}{T^2|\lambda_i(H)|^2}\sigma\\ &\quad+\frac{2\alpha|\lambda_i(H)|^2(2\tau_{\mathrm{s}}-T)+4\mathrm{Re}(\lambda)-4\mathrm{Im}(\lambda)\mathbf{i}}{T^2|\lambda_i(H)|^2}.\end{aligned} \tag{16}$$

Let $\sigma = \mathbf{i}\omega$, and $f_1(\sigma) = f_1(i\omega) = p_1(\omega)+\mathbf{i}q_1(\omega)$, then we obtain

$$\begin{aligned}p_1(\omega) &= -\frac{2(\alpha-\tau_{\mathrm{s}})}{T}\omega^2+\frac{4\mathrm{Im}(\lambda)\omega}{T^2|\lambda_i(H)|^2}\\ &\quad+\frac{2\alpha|\lambda_i(H)|^2(2\tau_{\mathrm{s}}-T)+4\mathrm{Re}(\lambda)}{T^2|\lambda_i(H)|^2},\\ q_1(\omega) &= -\omega^3+\frac{[(2T\tau_{\mathrm{s}}-T^2-4\alpha\tau_{\mathrm{s}})|\lambda_i(H)|^2+4\mathrm{Re}(\lambda)]\omega}{T^2|\lambda_i(H)|^2}\\ &\quad-\frac{4\mathrm{Im}(\lambda)}{T^2|\lambda_i(H)|^2}.\end{aligned}$$

Denote the roots of $p_1(\omega)$ and $q_1(\omega)$ as p_1, p_2 and q_1, q_2, q_3, respectively. From Lemma 3, the polynomial (16) is Hurwitz stable if and only if the following conditions hold simultaneously:

$$q_1 < p_1 < q_2 < p_2 < q_3, \tag{17}$$

$$p_1(0)q_1'(0) - p_1'(0)q_1(0) > 0. \tag{18}$$

Under the special condition that $\mathrm{Im}(\lambda) = 0$, we can obtain $\lambda_i(H) = \mathrm{Re}(\lambda) > 0$ (invoking Lemma 4). The quadratic polynomial

$$p_1(\omega) = -\frac{2(\alpha-\tau_{\mathrm{s}})}{T}\omega^2+\frac{2\alpha\mathrm{Re}(\lambda)(2\tau_{\mathrm{s}}-T)+4}{T^2\mathrm{Re}(\lambda)}$$

has two different real roots if and only if

$$\frac{\alpha\mathrm{Re}(\lambda)(2\tau_{\mathrm{s}}-T)+2}{T\mathrm{Re}(\lambda)(\alpha-\tau_{\mathrm{s}})} > 0,$$

and the roots of $p_1(\omega)$ are respectively given by

$$p_1 = -\sqrt{\frac{\alpha \mathrm{Re}(\lambda)(2\tau_s - T)+2}{T\,\mathrm{Re}(\lambda)(\alpha-\tau_s)}},$$
$$p_2 = \sqrt{\frac{\alpha \mathrm{Re}(\lambda)(2\tau_s - T)+2}{T\,\mathrm{Re}(\lambda)(\alpha-\tau_s)}}.$$

As a result, all the roots of Eq. (15) are within the unit circle, we have

$$(0.5T-\alpha)\tau_s|\lambda_i(H)|<1 \Leftrightarrow T-2\alpha<\frac{2}{\tau_s|\lambda_i(H)|},$$

then we have the condition

$$\frac{\alpha \mathrm{Re}(\lambda)(2\tau_s - T)+2}{T\,\mathrm{Re}(\lambda)(\alpha-\tau_s)}>0 \Leftrightarrow \frac{T}{2}-\frac{1}{\alpha\,\mathrm{Re}(\lambda)}<\tau_s<\alpha. \quad (19)$$

If the inequality

$$(2T\tau_s - 4\alpha\tau_s - T^2)\mathrm{Re}(\lambda)+4>0$$
$$\Leftrightarrow\ \tau_s > \frac{\mathrm{Re}(\lambda)T^2-4}{2\,\mathrm{Re}(\lambda)(T-2\alpha)} \quad (20)$$

holds, the quadratic polynomial

$$q_1(\omega) = -\omega^3 + \frac{[(2T\tau_s - T^2 - 4\alpha\tau_s)\mathrm{Re}(\lambda)+4]\omega}{T^2\,\mathrm{Re}(\lambda)}$$

has three different real roots as follows:

$$q_1 = -\sqrt{\frac{(2T\tau_s - 4\alpha\tau_s - T^2)\mathrm{Re}(\lambda)+4}{T^2\,\mathrm{Re}(\lambda)}},\quad q_2 = 0,$$
$$q_3 = \sqrt{\frac{(2T\tau_s - 4\alpha\tau_s - T^2)\mathrm{Re}(\lambda)+4}{T^2\,\mathrm{Re}(\lambda)}}.$$

According to the condition (17), we can obtain

$$\frac{(2T\tau_s - 4\alpha\tau_s - T^2)\mathrm{Re}(\lambda)+4}{T^2\,\mathrm{Re}(\lambda)} > \frac{\alpha \mathrm{Re}(\lambda)(2\tau_s - T)+2}{T\,\mathrm{Re}(\lambda)(\alpha-\tau_s)}$$
$$\Leftrightarrow\ f_1(\tau_s) = 2\mathrm{Re}(\lambda)(2\alpha - T)\tau_s^2$$
$$-[4+\mathrm{Re}(\lambda)(4\alpha^2 - T^2)]\tau_s + 2(2\alpha - T) > 0. \quad (21)$$

By simple calculation, we obtain

$$p_1(0) = \frac{2\alpha|\lambda_i(H)|^2(2\tau_s - T)+4\mathrm{Re}(\lambda)}{T^2|\lambda_i(H)|^2},$$
$$p_1'(0) = \frac{4\,\mathrm{Im}(\lambda)}{T^2|\lambda_i(H)|^2},$$
$$q_1(0) = -\frac{4\,\mathrm{Im}(\lambda)}{T^2|\lambda_i(H)|^2},$$
$$q_1'(0) = \frac{[(2T\tau_s - T^2 - 4\alpha\tau_s)|\lambda_i(H)|^2+4\mathrm{Re}(\lambda)]}{T^2|\lambda_i(H)|^2}.$$

Then, the inequality (18) is simplified into the following inequality:

$$\frac{2\alpha \mathrm{Re}(\lambda)(2\tau_s - T)+4}{T^2\,\mathrm{Re}(\lambda)}$$
$$\times \frac{(2T\tau_s - T^2 - 4\alpha\tau_s)\mathrm{Re}(\lambda)+4}{T^2\,\mathrm{Re}(\lambda)} > 0$$
$$\Leftrightarrow\ f_2(\tau_s) = 4\alpha\,\mathrm{Re}^2(\lambda)(T-2\alpha)\tau_s^2$$
$$+4\mathrm{Re}(\lambda)T[1+\alpha\,\mathrm{Re}(\lambda)(\alpha-T)]\tau_s$$
$$+[\alpha\,\mathrm{Re}(\lambda)T-2][\mathrm{Re}(\lambda)T^2-4]>0, \quad (22)$$

with $\Delta_{f_2(\tau_s)} = 16\mathrm{Re}^2(\lambda)\left\{[1+\alpha^2\,\mathrm{Re}(\lambda)]T - 4\alpha\right\}^2 \geq 0$, and the two roots of $f_2(\tau_s)=0$ are as follows:

$$\tau_{s_1} = \frac{4-\mathrm{Re}(\lambda)T^2}{2\,\mathrm{Re}(\lambda)(2\alpha - T)},\quad \tau_{s_2} = \frac{\alpha\,\mathrm{Re}(\lambda)T-2}{2\alpha\,\mathrm{Re}(\lambda)}.$$

When $T<2\alpha$, combining the conditions (19) and (20), the condition (22) is equivalent to the empty set. When $T>2\alpha$, from the conditions (19) and (20), we can have

$$\frac{\mathrm{Re}(\lambda)T^2-4}{\mathrm{Re}(\lambda)(2T-4\alpha)} < \alpha \Rightarrow 4\alpha^2 < (T-\alpha)^2+3\alpha^2$$
$$< \frac{4}{\mathrm{Re}(\lambda)} \Rightarrow \alpha^2 < \frac{1}{\mathrm{Re}(\lambda)}. \quad (23)$$

The condition (21) is equivalent to the following inequalities:

$$\frac{4+\mathrm{Re}(\lambda)(4\alpha^2 - T^2)}{4\,\mathrm{Re}(\lambda)(2\alpha - T)} > 0 \Rightarrow T > 2\sqrt{\alpha^2+\frac{1}{\mathrm{Re}(\lambda)}}, \quad (24)$$
$$\Delta_{f_1(\tau_s)} = [4+\mathrm{Re}(\lambda)(4\alpha^2 - T^2)]^2$$
$$-16\mathrm{Re}(\lambda)(2\alpha - T)^2 > 0$$
$$\Rightarrow\ \sqrt{\mathrm{Re}(\lambda)}T > 2\sqrt{[\sqrt{\mathrm{Re}(\lambda)}\alpha - 1]^2+1}+2, \quad (25)$$
$$\frac{[\mathrm{Re}(\lambda)(T^2-4\alpha^2)-4]-\sqrt{\Delta_{f_1(\tau_s)}}}{2\,\mathrm{Re}(\lambda)(T-2\alpha)} < \tau_s$$
$$< \frac{[\mathrm{Re}(\lambda)(T^2-4\alpha^2)-4]+\sqrt{\Delta_{f_1(\tau_s)}}}{2\,\mathrm{Re}(\lambda)(T-2\alpha)}. \quad (26)$$

By simple calculation, the condition (25) implies the inequality (24).

Combining the conditions (19) and (20), thus the condition (22) is equivalent to the inequality

$$\max\left\{\frac{\mathrm{Re}(\lambda)T^2-4}{2\,\mathrm{Re}(\lambda)(T-2\alpha)}, \frac{T}{2}-\frac{1}{\alpha\,\mathrm{Re}(\lambda)}\right\} < \tau_s < \alpha. \quad (27)$$

Synthesizing the condition (23), (25), (26), and (27), the necessary and sufficient conditions (12), (13) and (14) to guarantee the bounded consensus tracking are obtained.

Corollary 2 Consider a directed network topology G composed of n agents, and the virtual leader is globally reachable. Then the multi-agent system (4), applying the consensus tracking protocol (6) with $m=0$ and $\mathrm{Im}(\lambda)\neq 0$, achieves the bounded consensus tracking if and only if the following conditions

$$\alpha^2|\lambda_i(H)|^4T^3 - 4\alpha|\lambda_i(H)|^2[\alpha^2|\lambda_i(H)|^2+\mathrm{Re}(\lambda)]T^2$$
$$+4[\alpha^2|\lambda_i(H)|^2+\mathrm{Re}(\lambda)]^2T + 32\alpha\,\mathrm{Im}^2(\lambda)>0, \quad (28)$$

$$(2T-4\alpha)\tau_s > T^2 - \frac{4\mathrm{Re}(\lambda)}{|\lambda_i(H)|^2}, \tag{29}$$

$$6\sqrt{3}T\,|\mathrm{Im}(\lambda)|\,|\lambda_i(H)| < \sqrt{[(2T\tau_s - T^2 - 4\alpha\tau_s)\,|\lambda_i(H)|^2 + 4\mathrm{Re}(\lambda)]^3}, \tag{30}$$

$$\min\{q_1,q_2,q_3\} > \min\{p_1,p_2\}, \tag{31}$$

$$\max\{\max\{q_1,q_2,q_3\} - \arg\max\{q_1,q_2,q_3\}\} < \max\{p_1,p_2\} < \max\{q_1,q_2,q_3\}, \tag{32}$$

$$\begin{aligned} &4\alpha\,|\lambda_i(H)|^2\,(T-2\alpha)\tau_s^2 \\ &+4T[\alpha\,|\lambda_i(H)|^2\,(\alpha-T)+\mathrm{Re}(\lambda)]\tau_s \\ &+\alpha\,|\lambda_i(H)|^2\,T^3 - 2\mathrm{Re}(\lambda)T^2 \\ &-4\alpha\,\mathrm{Re}(\lambda)T+8>0 \end{aligned} \tag{33}$$

hold simultaneously, where the expressions of p_1, p_2 and q_1, q_2, q_3 are shown as Eqs. (34), (35), and (36), respectively.

Proof Corollary 1 may be regarded as a simplified form of Corollary 2. The quadratic polynomial $p_1(\omega)$ has two different real roots if and only if

$$\begin{aligned} \Delta_{p_1(\omega)} &= 4\mathrm{Im}^2(\lambda) + T(\alpha-\tau_s)\,|\lambda_i(H)|^2 \\ &\quad\times[\alpha\,|\lambda_i(H)|^2\,(2\tau_s - T) + 2\mathrm{Re}(\lambda)] \\ &= -2\alpha\,|\lambda_i(H)|^4\,T\tau_s^2 + [2\alpha^2\,|\lambda_i(H)|^2 \\ &\quad+\alpha\,|\lambda_i(H)|^2\,T - 2\mathrm{Re}(\lambda)]\,|\lambda_i(H)|^2\,T\tau_s \\ &\quad-\alpha^2\,|\lambda_i(H)|^4\,T^2 + 2\alpha\,\mathrm{Re}(\lambda)\,|\lambda_i(H)|^2\,T \\ &\quad+4\mathrm{Im}^2(\lambda) > 0, \end{aligned}$$

noticing that $-2\alpha T\,|\lambda_i(H)|^4 < 0$, then $\Delta_{p_1(\omega)} > 0$ is equivalent to the condition (28). The two roots of $p_1(\omega) = 0$ are given as follows:

$$p_{1,2} = \frac{2\,\mathrm{Im}(\lambda) \pm \sqrt{\Delta_{p_1(\omega)}}}{2(\alpha-\tau_s)T\,|\lambda_i(H)|^2}. \tag{34}$$

The quadratic polynomial $q_1(\omega) = 0$ has three different real roots if and only if the following two conditions hold simultaneously:

$$\begin{aligned} &(2T\tau_s - T^2 - 4\alpha\tau_s)\,|\lambda_i(H)|^2 + 4\mathrm{Re}(\lambda) > 0, \\ &\Delta_{q_1(\omega)} = 108T^2\,|\lambda_i(H)|^2\,\mathrm{Im}^2(\lambda) - [(2T\tau_s - T^2 \\ &\qquad - 4\alpha\tau_s)\,|\lambda_i(H)|^2 + 4\mathrm{Re}(\lambda)]^3 < 0, \end{aligned}$$

which is equivalent to the condition (29) and (30), respectively. Then the three different real roots of $q_1(\omega) = 0$ are given as follows:

$$q_1 = \frac{2\sqrt{3[(2T\tau_s - T^2 - 4\alpha\tau_s)\,|\lambda_i(H)|^2 + 4\mathrm{Re}(\lambda)]}\cos\frac{\theta}{3}}{3T\,|\lambda_i(H)|}, \tag{35}$$

$$q_{2,3} = \frac{-2\sqrt{3[(2T\tau_s - T^2 - 4\alpha\tau_s)\,|\lambda_i(H)|^2 + 4\mathrm{Re}(\lambda)]}\cos\left(\frac{\theta}{3} \pm \frac{\pi}{3}\right)}{3T\,|\lambda_i(H)|}, \tag{36}$$

where

$$\theta = \arccos\left(\frac{T\,|\lambda_i(H)|\,(27T^2\,|\lambda_i(H)|^2 - 4\mathrm{Im}(\lambda))}{6\sqrt{3[(2T\tau_s - T^2 - 4\alpha\tau_s)\,|\lambda_i(H)|^2 + 4\mathrm{Re}(\lambda)]^3}}\right).$$

According to the condition (17), the condition (31) and (32) can be obtained.

The inequality (18) is simplified into the following inequality:

$$\begin{aligned} &[2\alpha\,|\lambda_i(H)|^2\,(2\tau_s - T) + 4\mathrm{Re}(\lambda)][(2T\tau_s - T^2 - 4\alpha\tau_s) \\ &|\lambda_i(H)|^2 + 4\mathrm{Re}(\lambda)] + 16\mathrm{Im}^2(\lambda) > 0, \end{aligned}$$

which is equivalent to the condition (33). Hence, the Corollary 2 is proved.

Remark 3 Without loss of generality, let the control gain $\alpha = 1$, which may simplify the necessary and sufficient conditions in Corollaries 1 and 2.

Remark 4 If we employ the Lyapunov functions to study the convergence of the bounded consensus tracking problems, the conditions will be simple but just sufficient. Compared with the method of Lyapunov functions, the conditions obtained by frequency domain analysis are more complex but necessary and sufficient.

5. Simulation results

In this section, we present numerical simulations to illustrate the theoretical results obtained in the above sections. The nodes and weighted edges are shown in Fig. 1, where the nodes 1–5 compose the multi-agent system and the node 0 represents the virtual leader agent.

By calculation, the eigenvalues of $\boldsymbol{H} = \boldsymbol{B} + \boldsymbol{L}$ are $\lambda_1 = 1.9760, \lambda_{2,3} = 0.9374 \pm 0.4516\mathbf{i}$, $\lambda_4 = 1.0137$ and $\lambda_5 = 0.1356$, then $|\lambda_{\max}(H)| = 1.9760$. Assume the initial states of five agents are $(x_1(0), y_1(0)) = (-5,2)$, $(x_2(0), y_2(0)) = (-3,-3)$, $(x_3(0), y_3(0)) = (2,1)$, $(x_4(0), y_4(0)) = (4,-2)$, $(x_5(0), y_5(0)) = (6,4)$, and the time-varying reference state of the virtual leader is $x^0(kT) = (kT)^3$, $y^0(kT) = 3(kT)^2$.

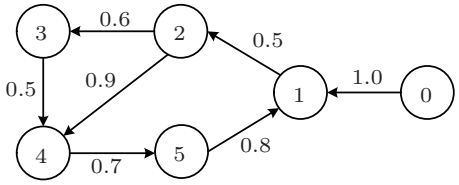


Fig. 1. The directed network topology of the system.

When the second-order multi-agent systems (4) apply the consensus tracking protocol (6), each agent can track time-varying reference state with $\tau_s = 0.1, T = 0.3, \alpha = 1$ in Fig. 2. Figure 3 shows the divergence curves under the conditions of $\tau_s = 0.5$, $T = 1$, $\alpha = 1$, which means that the system cannot achieve the consensus tracking.

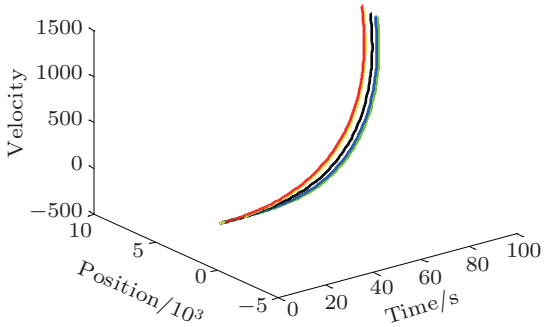


Fig. 2. (color online) The states of agents with $\tau_s = 0.1$, $T = 0.3$, and $\alpha = 1$.

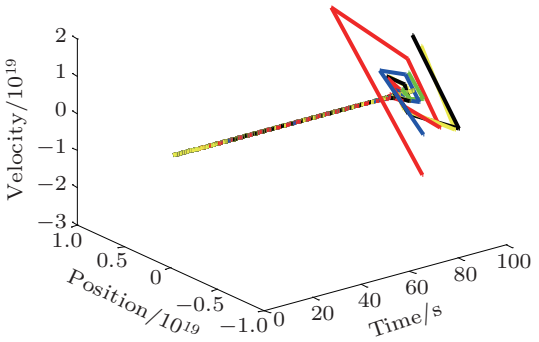


Fig. 3. (color online) The states of agents with $\tau_s = 0.5$, $T = 1$, and $\alpha = 1$.

Consider a special case, if the given reference state of virtual leader is $(x^0(kT), y^0(kT)) = (3,0)$, then the consensus tracking problem is simplified to the aggregating problem. From Fig. 4, it is easy to see that when $\tau_s = 0.1, T = 0.2$, $\alpha = 1$, which satisfy the given conditions in Corollary 2, the consensus position states of agents 1–5 are three, and the speed states are zero. However, in the case of the multi-agent system (4), applying the consensus tracking protocol (6) with $m = 0$ and $\tau_s = 0.5$, $T = 1.2$, and $\alpha = 1$, cannot achieve the aggregating is shown in Fig. 5. Then based on the above numerical results, the results of Theorem 1 are numerically verified.

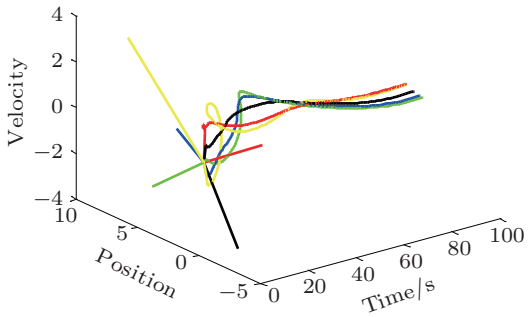


Fig. 4. (color online) The states of agents with $\tau_s = 0.1$, $T = 0.2$, and $\alpha = 1$.

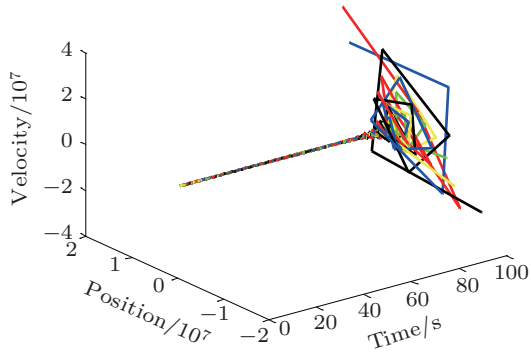


Fig. 5. (color online) The states of agents with $\tau_s = 0.5$, $T = 1.2$, and $\alpha = 1$.

6. Conclusions

The bounded consensus tracking problems of second-order multi-agent systems based on sampled-data with sampling delay under directed networks have been studied. Sufficient and necessary conditions, which depend on the topology of communication network, the control parameters, the sampling delay, and the sampling period, are derived to guarantee the bounded consensus tracking. Using the particular delay decomposition technology, we also have obtained the explicit allowable scope of the sampling period and the sampling delay in the special cases. The future work includes the leaderless consensus case, the switching network topology case and the time-varying delay case.

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