$\mathcal{H}_\infty$ synchronization of chaotic neural networks with time-varying delays*

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In this paper, we investigate the problem of $\mathcal{H}_\infty$ synchronization for chaotic neural networks with time-varying delays. A new model of the networks with disturbances in both master and slave systems is presented. By constructing a suitable Lyapunov–Krasovskii functional and using a reciprocally convex approach, a novel $\mathcal{H}_\infty$ synchronization criterion for the networks concerned is established in terms of linear matrix inequalities (LMIs) which can be easily solved by various effective optimization algorithms. Two numerical examples are given to illustrate the effectiveness of the proposed method.

Keywords: chaotic neural networks, time-varying delays, $\mathcal{H}_\infty$ synchronization, Lyapunov method

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1. Introduction

In the early 1990s, the concept of synchronization was first introduced by Pecora and Carroll.[1,2] Since then, chaos synchronization has received a great deal of attention from many researchers due to its extensive applications such as biology, chemistry, secure communication, image processing, harmonic oscillation generation, and so on.[3–9] In Refs. [10] and [11], it has been verified that neural network models can exhibit a chaotic behavior that shows some specific features such as being extremely sensitive to a small perturbation in initial conditions, having bounded state trajectories, and so on.[12] It should be noted that time delays occur in many practical systems because of the finite information computing speed, and it is well known that the time delay can cause oscillation, poor performance, or even instability of dynamic systems. Therefore, until now, there have been so many papers[13–19] aiming to investigate the stability of neural networks with time delays because it is a pre-requisite and an important job to check whether the equilibrium points of the designed network is stable or not.

In this regard, the synchronization problem for chaotic neural networks with time delays has become a common topic of some researchers.[20–24] In Ref. [20], the synchronization problem for chaotic neural networks with both discrete time-varying and distributed time-varying delays was investigated by introducing an improved Lyapunov–Krasovskii functional and employing definition on a convex set. Furthermore, extended results of Ref. [20] were reported in Ref. [21]. Cui and Lou[22] proposed a new method to synchronize two identical chaotic recurrent neural networks by designing a nonlinear feedback control law. In Ref. [23], by employing the Lyapunov stability method and Hermitian matrices theory, a sufficient condition for the exponential synchronization of a class of chaotic nonlinear neural networks with time-varying delays was exploited. Recently, by utilizing the delay-decomposition method which is well known in reducing the conservatism of stability criteria for dynamic systems with time delays, some new synchronization criteria for a class of chaotic nonlinear neural networks with time-varying delays were proposed in Ref. [24] with the framework of LMIs which are dependent on the size of the time delay.

On the other hand, in practical control systems, it can be confirmed that disturbance inputs exist widely. Thus, the real control problem would be complicated. Since the $\mathcal{H}_\infty$ control problem was first presented by Zames,[25] a great number of studies on this topic have been carried out[26–28] for the $\mathcal{H}_\infty$ control theory that minimizes the effects of the external disturbances is an efficient approach to deal with such problems. It is the aim of this theory to design the controller such that the closed-loop system is internally stable and its $\mathcal{H}_\infty$-norm of the transfer function between the controlled output and the disturbances will not exceed a given $\mathcal{H}_\infty$ performance level. It should be stressed that some noises or disturbances always occur, perhaps leading to instability and poor performance. Therefore, the effects of the noises or distur-

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bances should also be reduced in synchronization for chaotic systems. In this regard, Park *et al.*[29] first investigated an $\mathcal{H}_\infty$ synchronization problem for time-delayed chaotic systems by using a dynamic feedback control scheme. In Ref. [30], the $\mathcal{H}_\infty$ synchronization for chaotic systems with external disturbances was introduced. In Ref. [31], the $\mathcal{H}_\infty$ synchronization method for a class of uncertain master and slave neural networks (MSNNs) with mixed time delays were proposed. Qi *et al.*[32] studied an $\mathcal{H}_\infty$ exponential synchronization problem for general discrete-time chaotic neural networks with or without time delays. Recently, a sliding mode approach was proposed in Ref. [33] to solve the problem of an exponential $\mathcal{H}_\infty$ synchronization for a class of master–slave time-delay systems with both discrete and distributed time delays, norm-bounded nonlinear uncertainties, and Markovian switching parameters. Wang *et al.*[34] considered the problem of the multi-dimensional and the single-dimensional robust $\mathcal{H}_\infty$ synchronization controllers for a hyper-chaotic system with disturbance input. Very recently, a novel concept of $\mathcal{H}_\infty$ cluster synchronization was proposed to quantify against the exogenous disturbance of the complex networks in Ref. [35]. From the previous work,[29–35] the issue of $\mathcal{H}_\infty$ synchronization has been widely applied in various systems. However, to the best of our knowledge, an $\mathcal{H}_\infty$ synchronization problem for a class of chaotic nonlinear neural networks with time-varying delays has not been tackled in any other literature.

Stimulated by the motivation above, in this paper, the problem of $\mathcal{H}_\infty$ synchronization for a class of chaotic nonlinear neural networks with time-varying delays is investigated. In Refs. [29]–[34], disturbances were constructed in only slave parts. However, in the practical point of view, it is worth investigating the $\mathcal{H}_\infty$ control problem for chaotic neural networks with the disturbances in both master and slave parts since disturbances can also exist in the master part. Thus, in this paper, a new model of chaotic neural networks with the disturbances in both master and slave parts is constructed and its $\mathcal{H}_\infty$ synchronization criterion is proposed. First, by constructing a suitable Lyapunov–Krasovskii functional and utilizing a reciprocally convex optimization approach with some added decision variables,[36,37] a new synchronization criterion for a class of chaotic nonlinear neural networks without disturbances will be derived in Theorem 1 by the framework of LMIs. Second, based on the results of Theorem 1, an $\mathcal{H}_\infty$ synchronization criterion for a class of chaotic nonlinear neural networks with the disturbances in both master and slave parts will be introduced in Theorem 2. Via two numerical examples, the superiority and effectiveness of Theorems 1 and 2 are demonstrated.

2. Notation

$\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices. For real symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, $\boldsymbol{X}>\boldsymbol{Y}$ (resp., $\boldsymbol{X}\geq\boldsymbol{Y}$) means that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive (resp., nonnegative) definite. $\boldsymbol{X}^\perp$ denotes a basis for the null-space of $\boldsymbol{X}$. $\boldsymbol{I}_n$, 0_n, and $0_{m\cdot n}$ denote $n\times n$ identity matrix, $n\times n$ and $m\times n$ zero matrices, respectively. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix norm. diag$\{\cdots\}$ denotes the block diagonal matrix. For square matrix $\boldsymbol{X}$, sym$\{\boldsymbol{X}\}$ means the sum of $\boldsymbol{X}$ and its symmetric matrix $\boldsymbol{X}^{\mathrm{T}}$; i.e., sym$\{\boldsymbol{X}\}=\boldsymbol{X}+\boldsymbol{X}^{\mathrm{T}}$. $\star$ represents the elements below the main diagonal of a symmetric matrix.

3. Problem statement

Based on the master-slave concept, consider the following unidirectional coupled neural networks with time-varying delays and disturbances.

The master system is given by

$$\begin{aligned}\dot{\boldsymbol{x}}_{\mathrm{m}}(t) &= -\boldsymbol{\alpha}(\boldsymbol{x}_{\mathrm{m}}(t))+\boldsymbol{W}_0\boldsymbol{f}(\boldsymbol{x}_{\mathrm{m}}(t))\\ &\quad+\boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}_{\mathrm{m}}(t-h(t)))+\boldsymbol{b}+\boldsymbol{D}_{\mathrm{m}}\boldsymbol{w}_{\mathrm{m}}(t),\\ \boldsymbol{z}_{\mathrm{m}}(t) &= \boldsymbol{F}\boldsymbol{x}_{\mathrm{m}}(t).\end{aligned}\tag{1}$$

The slave system is given by

$$\begin{aligned}\dot{\boldsymbol{x}}_{\mathrm{s}}(t) &= -\boldsymbol{\alpha}(\boldsymbol{x}_{\mathrm{s}}(t))+\boldsymbol{W}_0\boldsymbol{f}(\boldsymbol{x}_{\mathrm{s}}(t))\\ &\quad+\boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}_{\mathrm{s}}(t-h(t)))+\boldsymbol{b}+\boldsymbol{D}_s\boldsymbol{w}_{\mathrm{s}}(t)+\boldsymbol{u}(t),\\ \boldsymbol{z}_{\mathrm{s}}(t) &= \boldsymbol{F}\boldsymbol{x}_{\mathrm{s}}(t),\end{aligned}\tag{2}$$

where the subscripts m and s represent, respectively, the master and slave system, n denotes the number of neurons in the neural networks, $\boldsymbol{x}(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^{\mathrm{T}}\in\mathbb{R}^n$ is the neuron state vector, $\boldsymbol{f}(\boldsymbol{x}(\cdot))=[f(\boldsymbol{x}_1(\cdot)),f(\boldsymbol{x}_2(\cdot)),\ldots,f(\boldsymbol{x}_n(\cdot))]^{\mathrm{T}}\in\mathbb{R}^n$ means the neuron activation functions, $\boldsymbol{w}_{\mathrm{m}}(t)\in\mathbb{R}^m$ and $\boldsymbol{w}_{\mathrm{s}}(t)\in\mathbb{R}^m$ are the external disturbances, $z(t)\in\mathbb{R}^l$ is the output vector, $\boldsymbol{W}_0=[w_{ij}^0]_{n\times n}\in\mathbb{R}^{n\times n}$ and $\boldsymbol{W}_1=[w_{ij}^1]_{n\times n}\in\mathbb{R}^{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons, $\boldsymbol{b}=[b_1,b_2,\ldots,b_n]^{\mathrm{T}}\in\mathbb{R}^n$ represents a constant input vector, $\boldsymbol{D}_{\mathrm{m}}\in\mathbb{R}^{n\times m}$, $\boldsymbol{D}_{\mathrm{s}}\in\mathbb{R}^{n\times m}$ and $\boldsymbol{F}\in\mathbb{R}^{l\times n}$ are constant matrices, and $u(t)$ is a unidirectional coupled term, which is considered as control input and will be appropriately designed to obtain a certain control objective.

The delay $h(t)$ is a time-varying continuous function that satisfies $0\leq h(t)\leq h_{\mathrm{U}}$ where h_{U} is a known constant scalar.

The nonlinear functions $\boldsymbol{\alpha}(\boldsymbol{x}(t))$ and $\boldsymbol{f}(\boldsymbol{x}(\cdot))$ satisfy the following assumptions.

Assumption 1 Each function $\boldsymbol{\alpha}_i(\cdot):\mathbb{R}\to\mathbb{R}$ $(i=1,2,\ldots,n)$ is nondecreasing and locally Lipschitz, i.e., there exists a positive real d_i such that

$$\boldsymbol{\alpha}_i'(\boldsymbol{w})=d_i$$

for any $\boldsymbol{w}\in\mathbb{R}$ at which $\boldsymbol{\alpha}_i(\cdot)$ is a differentiable function.

Assumption 2 Each function $f_i(\cdot) : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,\ldots,n)$ with $f(0) = 0$ is monotonic nondecreasing and globally Lipschitz, i.e., there exists a positive real k_i such that

$$0 \leq \frac{\boldsymbol{f}_i(u) - \boldsymbol{f}_i(v)}{u - v} \leq k_i$$

for any $u, v \in \mathbb{R}$ with $u \neq v$. Let us define $\boldsymbol{\eta}(t) = \boldsymbol{x}_{\mathrm{m}}(t) - \boldsymbol{x}_{\mathrm{s}}(t)$. This means that the synchronization error is between master and slave systems. Thus, the error dynamic system can be written as follows:

$$\begin{aligned} \dot{\boldsymbol{\eta}}(t) &= -\tilde{\boldsymbol{\alpha}}(\boldsymbol{\eta}(t)) + \boldsymbol{W}_0 \boldsymbol{g}(\boldsymbol{\eta}(t)) \\ &\quad + \boldsymbol{W}_1 g(\boldsymbol{\eta}(t - h(t))) + \boldsymbol{D}\boldsymbol{w}(t) - \boldsymbol{u}(t), \\ \boldsymbol{z}(t) &= \boldsymbol{F}\boldsymbol{\eta}(t), \end{aligned} \tag{3}$$

where

$$\begin{aligned} &\tilde{\boldsymbol{\alpha}}(\boldsymbol{\eta}(t)) = \boldsymbol{\alpha}(\boldsymbol{x}_{\mathrm{m}}(t)) - \boldsymbol{\alpha}(\boldsymbol{x}_{\mathrm{s}}(t)), \\ &g(\boldsymbol{\eta}(\cdot)) = \boldsymbol{f}(\boldsymbol{\eta}(\cdot) + \boldsymbol{x}_{\mathrm{s}}(\cdot)) - \boldsymbol{f}(\boldsymbol{x}_{\mathrm{s}}(\cdot)), \\ &\boldsymbol{D} = [\boldsymbol{D}_{\mathrm{m}}, -\boldsymbol{D}_{\mathrm{s}}] \in \mathbb{R}^{n \times 2m}, \\ &\boldsymbol{w}(t) = [\boldsymbol{w}_{\mathrm{m}}^{\mathrm{T}}(t), \boldsymbol{w}_{\mathrm{s}}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{2m}. \end{aligned}$$

Here, from Assumption 1 and Lebourg theorem,[38] there exist $a_i \geq d_i$ $(i = 1,2,\ldots,n)$ such that

$$\tilde{\boldsymbol{\alpha}}(\boldsymbol{\eta}(t)) = \boldsymbol{A}\boldsymbol{\eta}(t),$$

where $\boldsymbol{A} = \mathrm{diag}\{a_i\}$. Also, from Assumption 2, $\boldsymbol{g}_i(\boldsymbol{\eta}_i(\cdot))$ $(i = 1,2,\ldots,n)$ is the nonlinear function vector satisfying

$$0 \leq \boldsymbol{\eta}_i(\cdot)\boldsymbol{g}_i(\boldsymbol{\eta}_i(\cdot)) \leq \boldsymbol{k}_i \boldsymbol{\eta}_i^2(\cdot).$$

In order to synchronize the coupled neural networks, the control input $u(t)$ is designed as follows:

$$u(t) = K\boldsymbol{\eta}(t), \tag{4}$$

where $\boldsymbol{K} = [k_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}$ is the state feedback control gain matrix to be determined for synchronizing both master and slave systems.

Therefore, under the control law (4), an error dynamic model of coupled neural networks with time-varying delay and disturbances can be represented as

$$\begin{aligned} \dot{\boldsymbol{\eta}}(t) &= -(\boldsymbol{A} + \boldsymbol{K})\boldsymbol{\eta}(t) + \boldsymbol{W}_0 \boldsymbol{g}(\boldsymbol{\eta}(t)) \\ &\quad + \boldsymbol{W}_1 \boldsymbol{g}(\boldsymbol{\eta}(t - h(t))) + \boldsymbol{D}\boldsymbol{w}(t), \\ \boldsymbol{z}(t) &= \boldsymbol{F}\boldsymbol{\eta}(t). \end{aligned} \tag{5}$$

Before describing the objective of this paper, we first need the following definition.

Definition 1 ($\mathcal{H}_\infty$ control problem[39]) The coupled loop systems (5) will achieve $\mathcal{H}_\infty$ synchronization with the disturbance attenuation γ if the following conditions are satisfied:

(i) With zero disturbance, the system (5) with control input $u(t)$ is asymptotically stable.

(ii) With zero initial condition and a given constant $\gamma > 0$, the following condition holds:

$$J = \int_0^\infty \left\{ \boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2 \boldsymbol{w}^{\mathrm{T}}(t)\boldsymbol{w}(t) \right\} \mathrm{d}t \leq 0,$$
$$\left(\text{i.e.,} \sup_{w \neq 0, w \in \mathcal{L}_2[0,\infty)} \frac{||\boldsymbol{z}(t)||_2}{||\boldsymbol{w}(t)||_2} \leq \gamma \right).$$

Then, the controller $u(t)$ is said to be the $\mathcal{H}_\infty$ synchronization controller with the disturbance attenuation γ. The parameter γ is called the $\mathcal{H}_\infty$-norm bound of the system under the control $u(t)$.

The aim of this paper is to design an $\mathcal{H}_\infty$ synchronization controller $\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{x}(t)$ with the disturbance attenuation γ for system (5). In other words, to design a gain $\boldsymbol{K}$ which achieves the synchronization between the master system (1) and the slave system (2) with the disturbance attenuation γ.

4. Main results

In this section, two theorems will be introduced. In Theorem 1, an improved synchronization criterion for system (5) without disturbances ($w(t) = 0_{2m \cdot 1}$) will be introduced. Then, based on the result of Theorem 1, an $\mathcal{H}_\infty$ synchronization criterion for system (5) will be introduced in Theorem 2. Before introducing Theorem 1, some notations will be defined for simplicity of matrix representation. $\boldsymbol{e}_i \in \mathbb{R}^{12n \times n}$ are defined as block entry matrices, e.g., $\boldsymbol{e}_3 = [0_{n \cdot 2n}, \boldsymbol{I}_n, 0_{n \cdot 9n}]^{\mathrm{T}}$ and $\boldsymbol{e}_{12} = [0_{n \cdot 11n}, \boldsymbol{I}_n]^{\mathrm{T}}$. The notations of several matrices are defined as

$$\begin{aligned} \boldsymbol{\zeta}(t) &= \Big[\boldsymbol{\eta}^{\mathrm{T}}(t), \boldsymbol{\eta}^{\mathrm{T}}(t - \boldsymbol{h}(t)), \boldsymbol{\eta}^{\mathrm{T}}(t - h_{\mathrm{U}}), \\ &\quad \dot{\boldsymbol{\eta}}^{\mathrm{T}}(t), \dot{\boldsymbol{\eta}}^{\mathrm{T}}(t - h_{\mathrm{U}}), \int_{t - h(t)}^{t} \boldsymbol{\eta}^{\mathrm{T}}(s)\mathrm{d}s, \\ &\quad \int_{t - h_{\mathrm{U}}}^{t - h(t)} \boldsymbol{\eta}^{\mathrm{T}}(s)\mathrm{d}s, \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\eta}(t)), \\ &\quad \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\eta}(t - h(t))), \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\eta}(t - h_{\mathrm{U}})), \\ &\quad \int_{t - h(t)}^{t} \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\eta}(s))\mathrm{d}s, \int_{t - h_{\mathrm{U}}}^{t - h(t)} \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\eta}(s))\mathrm{d}s \Big]^{\mathrm{T}}, \\ \boldsymbol{\Pi}_1 &= [\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_6 + \boldsymbol{e}_7, \boldsymbol{e}_{11} + \boldsymbol{e}_{12}], \\ \boldsymbol{\Pi}_2 &= [\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_1 - \boldsymbol{e}_3, \boldsymbol{e}_8 - \boldsymbol{e}_{10}], \\ \boldsymbol{\Pi}_3 &= [\boldsymbol{e}_1, \boldsymbol{e}_4, \boldsymbol{e}_8], \quad \boldsymbol{\Pi}_4 = [\boldsymbol{e}_3, \boldsymbol{e}_5, \boldsymbol{e}_{10}], \\ \boldsymbol{\Pi}_5 &= [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3], \\ \boldsymbol{\Pi}_6 &= [\boldsymbol{e}_6, \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_{11}, \boldsymbol{e}_7, \boldsymbol{e}_2 - \boldsymbol{e}_3, \boldsymbol{e}_{12}], \\ \boldsymbol{\Omega}_i &= \mathcal{G} + \begin{bmatrix} 0_n & P_i & 0_n \\ P_i & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}, \quad (i = 1,2), \\ \boldsymbol{\Omega}_3 &= \left[\begin{array}{c|c} \boldsymbol{\Omega}_1 & \mathcal{S} \\ \hline \mathcal{S}^{\mathrm{T}} & \boldsymbol{\Omega}_2 \end{array} \right], \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\Xi}_1 &= \mathrm{sym}\{\boldsymbol{\Pi}_1\mathcal{R}\boldsymbol{\Pi}_2^{\mathrm{T}}\},\\
\boldsymbol{\Xi}_2 &= \boldsymbol{\Pi}_3\mathcal{N}\boldsymbol{\Pi}_3^{\mathrm{T}}-\boldsymbol{\Pi}_4\mathcal{N}\boldsymbol{\Pi}_4^{\mathrm{T}}\\
&\quad+\mathrm{sym}\left\{[\boldsymbol{e}_8,\boldsymbol{e}_1]\begin{bmatrix}\boldsymbol{\Lambda}-\boldsymbol{\Delta}\\ \boldsymbol{K}_{\mathrm{p}}\boldsymbol{\Delta}\end{bmatrix}\boldsymbol{e}_4^{\mathrm{T}}\right\},\\
\boldsymbol{\Xi}_3 &= h_{\mathrm{U}}^2\boldsymbol{\Pi}_3\mathcal{G}\boldsymbol{\Pi}_3^{\mathrm{T}}+h_{\mathrm{U}}\boldsymbol{\Pi}_5\,\mathrm{diag}\{\boldsymbol{P}_1,-\boldsymbol{P}_1+\boldsymbol{P}_2,-\boldsymbol{P}_2\}\\
&\quad\times\boldsymbol{\Pi}_5^{\mathrm{T}}-\boldsymbol{\Pi}_6\boldsymbol{\Omega}_3\boldsymbol{\Pi}_6^{\mathrm{T}},\\
\boldsymbol{\Xi}_4 &= (h_{\mathrm{U}}^2/2)^2\boldsymbol{e}_4\boldsymbol{Q}\boldsymbol{e}_4^{\mathrm{T}}-(h_{\mathrm{U}}\boldsymbol{e}_1-\boldsymbol{e}_6-\boldsymbol{e}_7)\\
&\quad\times\boldsymbol{Q}(h_{\mathrm{U}}\boldsymbol{e}_1-\boldsymbol{e}_6-\boldsymbol{e}_7)^{\mathrm{T}},\\
\boldsymbol{\Theta} &= \sum_{i=1}^{3}\mathrm{sym}\left\{\boldsymbol{e}_i\boldsymbol{K}_{\mathrm{p}}\boldsymbol{H}_i\boldsymbol{e}_{i+7}^{\mathrm{T}}-\boldsymbol{e}_{i+7}H_i\boldsymbol{e}_{i+7}^{\mathrm{T}}\right\},\\
\boldsymbol{\Psi} &= \boldsymbol{e}_1+\boldsymbol{e}_4,\ \boldsymbol{\Gamma}=\boldsymbol{X}(-\boldsymbol{A}\boldsymbol{e}_1^{\mathrm{T}}-\boldsymbol{e}_4^{\mathrm{T}}+\boldsymbol{W}_0\boldsymbol{e}_8^{\mathrm{T}}\\
&\quad+\boldsymbol{W}_1\boldsymbol{e}_9^{\mathrm{T}})-\boldsymbol{Y}\boldsymbol{e}_1^{\mathrm{T}},\\
\boldsymbol{\Phi} &= \sum_{i=1}^{4}\boldsymbol{\Xi}_i+\boldsymbol{\Theta}+\mathrm{sym}\{\boldsymbol{\Psi}\boldsymbol{\Gamma}\}. \qquad (6)
\end{aligned}
$$

Now, the first result is given by Theorem 1.

Theorem 1 For a given positive scalar h_{U}, the master system (1) with $\boldsymbol{w}_{\mathrm{m}}(t)=0_{m\cdot1}$ and the slave system (2) with $\boldsymbol{w}_{\mathrm{s}}(t)=0_{m\cdot1}$ are synchronized for $0\le h(t)\le h_{\mathrm{U}}$ if there exist positive definite matrices $\mathcal{R}\in\mathbb{R}^{4n\times4n}$, $\mathcal{N}\in\mathbb{R}^{3n\times3n}$, $\mathcal{G}\in\mathbb{R}^{3n\times3n}$, $\boldsymbol{Q}\in\mathbb{R}^{n\times n}$, positive diagonal matrices $\boldsymbol{\Lambda}\in\mathbb{R}^{n\times n}$, $\boldsymbol{\Delta}\in\mathbb{R}^{n\times n}$, $\boldsymbol{H}_i\in\mathbb{R}^{3n\times3n}$ $(i=1,2,3)$, any symmetric matrices $\boldsymbol{P}_i\in\mathbb{R}^{n\times n}$ $(i=1,2)$ and any matrices $\mathcal{S}\in\mathbb{R}^{3n\times3n}$, $\boldsymbol{X}\in\mathbb{R}^{n\times n}$, $\boldsymbol{Y}\in\mathbb{R}^{n\times n}$ satisfying the following LMIs:

$$\boldsymbol{\Phi}<0, \qquad (7)$$

$$\boldsymbol{\Omega}_3\ge0, \qquad (8)$$

where $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}_3$ are defined by Eq. (6). Then, the controller gain $\boldsymbol{K}$ can be calculated as $\boldsymbol{K}=\boldsymbol{X}^{-1}\boldsymbol{Y}$.

Proof Let us consider the following Lyapunov–Krasovskii functional candidate:

$$\boldsymbol{V}(\boldsymbol{\eta}(t))=\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_3+\boldsymbol{V}_4, \qquad (9)$$

where

$$
\begin{aligned}
\boldsymbol{V}_1 &= \begin{bmatrix}\boldsymbol{\eta}(t)\\ \boldsymbol{\eta}(t-h_{\mathrm{U}})\\ \int_{h_{\mathrm{U}}}^{t}\boldsymbol{\eta}(s)\mathrm{d}s\\ \int_{h_{\mathrm{U}}}^{t}\boldsymbol{f}(\boldsymbol{\eta}(s))\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\mathcal{R}\begin{bmatrix}\boldsymbol{\eta}(t)\\ \boldsymbol{\eta}(t-h_{\mathrm{U}})\\ \int_{h_{\mathrm{U}}}^{t}\boldsymbol{\eta}(s)\mathrm{d}s\\ \int_{h_{\mathrm{U}}}^{t}\boldsymbol{f}(\boldsymbol{\eta}(s))\mathrm{d}s\end{bmatrix},\\
\boldsymbol{V}_2 &= \int_{h_{\mathrm{U}}}^{t}\begin{bmatrix}\boldsymbol{\eta}(s)\\ \dot{\boldsymbol{\eta}}(s)\\ \boldsymbol{f}(\boldsymbol{\eta}(s))\end{bmatrix}^{\mathrm{T}}\mathcal{N}\begin{bmatrix}\boldsymbol{\eta}(s)\\ \dot{\boldsymbol{\eta}}(s)\\ \boldsymbol{f}(\boldsymbol{\eta}(s))\end{bmatrix}\mathrm{d}s\\
&\quad+2\sum_{i=1}^{n}\left(\lambda_i\int_0^{\eta_i(t)}f_i(s)\mathrm{d}s+\delta_i\int_0^{\eta_i(t)}(k_is-f_i(s))\mathrm{d}s\right),\\
\boldsymbol{V}_3 &= h_{\mathrm{U}}\int_{h_{\mathrm{U}}}^{t}\int_{s}^{t}\begin{bmatrix}\eta(u)\\ \dot{\boldsymbol{\eta}}(u)\\ f(\boldsymbol{\eta}(u))\end{bmatrix}^{\mathrm{T}}\mathcal{G}\begin{bmatrix}\eta(u)\\ \dot{\boldsymbol{\eta}}(u)\\ f(\boldsymbol{\eta}(u))\end{bmatrix}\mathrm{d}u\mathrm{d}s,\\
\boldsymbol{V}_4 &= (h_{\mathrm{U}}^2/2)\int_{h_{\mathrm{U}}}^{t}\int_{s}^{t}\int_{u}^{t}\dot{\boldsymbol{\eta}}^{\mathrm{T}}(v)\boldsymbol{Q}\dot{\boldsymbol{\eta}}(v)\mathrm{d}v\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

$\dot{\boldsymbol{V}}_1$ and $\dot{\boldsymbol{V}}_2$ are calculated by

$$
\begin{aligned}
\dot{\boldsymbol{V}}_1 &= 2\begin{bmatrix}\boldsymbol{\eta}(t)\\ \boldsymbol{\eta}(t-h_{\mathrm{U}})\\ \int_{t-h(t)}^{t}\boldsymbol{\eta}(s)\mathrm{d}s+\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{\eta}(s)\mathrm{d}s\\ \int_{t-h(t)}^{t}\boldsymbol{f}(\boldsymbol{\eta}(s))\mathrm{d}s+\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{f}(\boldsymbol{\eta}(s))\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\\
&\quad\times\mathcal{R}\begin{bmatrix}\dot{\boldsymbol{\eta}}(t)\\ \dot{\boldsymbol{\eta}}(t-h_{\mathrm{U}})\\ \boldsymbol{\eta}(t)-\boldsymbol{\eta}(t-h_{\mathrm{U}})\\ \boldsymbol{f}(\boldsymbol{\eta}(t))-\boldsymbol{f}(\boldsymbol{\eta}(t-h_{\mathrm{U}}))\end{bmatrix}\\
&= \zeta^{\mathrm{T}}(t)\boldsymbol{\Xi}_1\zeta(t),\\
\dot{\boldsymbol{V}}_2 &\le \begin{bmatrix}\boldsymbol{\eta}(t)\\ \dot{\boldsymbol{\eta}}(t)\\ \boldsymbol{f}(\boldsymbol{\eta}(t))\end{bmatrix}^{\mathrm{T}}\mathcal{N}\begin{bmatrix}\boldsymbol{\eta}(t)\\ \dot{\boldsymbol{\eta}}(t)\\ \boldsymbol{f}(\boldsymbol{\eta}(t))\end{bmatrix}\\
&\quad-\begin{bmatrix}\boldsymbol{\eta}(t-h_{\mathrm{U}})\\ \dot{\boldsymbol{\eta}}(t-h_{\mathrm{U}})\\ \boldsymbol{f}(\boldsymbol{\eta}(t-h_{\mathrm{U}}))\end{bmatrix}^{\mathrm{T}}\mathcal{N}\begin{bmatrix}\boldsymbol{\eta}(t-h_{\mathrm{U}})\\ \dot{\boldsymbol{\eta}}(t-h_{\mathrm{U}})\\ \boldsymbol{f}(\boldsymbol{\eta}(t-h_{\mathrm{U}}))\end{bmatrix}\\
&\quad+2\boldsymbol{f}(\boldsymbol{\eta}(t))\boldsymbol{\Lambda}\dot{\boldsymbol{\eta}}(t)+2[K_{\mathrm{p}}\boldsymbol{\eta}(t)-\boldsymbol{f}(\boldsymbol{\eta}(t))]\boldsymbol{\Delta}\dot{\boldsymbol{\eta}}(t)\\
&= \zeta^{\mathrm{T}}(t)\boldsymbol{\Xi}_2\zeta(t). \qquad (10)
\end{aligned}
$$

Inspired by Ref. [36], the following two zero equalities with any symmetric matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are considered:

$$
\begin{aligned}
0 &= \boldsymbol{\eta}^{\mathrm{T}}(t)(h_{\mathrm{U}}P_1)\boldsymbol{\eta}(t)-\boldsymbol{\eta}^{\mathrm{T}}(t-h(t))(h_{\mathrm{U}}P_1)\boldsymbol{\eta}(t-h(t))\\
&\quad-2h_{\mathrm{U}}\int_{t-h(t)}^{t}\boldsymbol{\eta}(s)P_1\boldsymbol{\eta}(s)\mathrm{d}s,\\
0 &= \boldsymbol{\eta}^{\mathrm{T}}(t-h(t))(h_{\mathrm{U}}P_2)\boldsymbol{\eta}(t-h(t))\\
&\quad-\boldsymbol{\eta}^{\mathrm{T}}(t-h_{\mathrm{U}})(h_{\mathrm{U}}P_2)\boldsymbol{\eta}(t-h_{\mathrm{U}})\\
&\quad-2h_{\mathrm{U}}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{\eta}(s)P_2\boldsymbol{\eta}(s)\mathrm{d}s. \qquad (11)
\end{aligned}
$$

By utilizing the well-known Jensen inequality[41] and adding the zero equalities (11) into $\dot{\boldsymbol{V}}_3$, the following inequality can be obtained:

$$
\begin{aligned}
\dot{\boldsymbol{V}}_3 &= h_{\mathrm{U}}^2\begin{bmatrix}\boldsymbol{\eta}(t)\\ \dot{\boldsymbol{\eta}}(t)\\ \boldsymbol{f}(\boldsymbol{\eta}(t))\end{bmatrix}^{\mathrm{T}}\mathcal{G}\begin{bmatrix}\boldsymbol{\eta}(t)\\ \dot{\boldsymbol{\eta}}(t)\\ \boldsymbol{f}(\boldsymbol{\eta}(t))\end{bmatrix}\\
&\quad+\boldsymbol{\eta}^{\mathrm{T}}(t)(h_{\mathrm{U}}P_1)\boldsymbol{\eta}(t)-\boldsymbol{\eta}^{\mathrm{T}}(t-h(t))(h_{\mathrm{U}}P_1)\boldsymbol{\eta}(t-h(t))\\
&\quad+\boldsymbol{\eta}^{\mathrm{T}}(t-h(t))(h_{\mathrm{U}}P_2)\boldsymbol{\eta}(t-h(t))\\
&\quad-\boldsymbol{\eta}^{\mathrm{T}}(t-h_{\mathrm{U}})(h_{\mathrm{U}}P_2)\boldsymbol{\eta}(t-h_{\mathrm{U}})\\
&\quad-h_{\mathrm{U}}\int_{t-h(t)}^{t}\begin{bmatrix}\boldsymbol{\eta}(s)\\ \boldsymbol{\eta}(s)\\ \boldsymbol{f}(\boldsymbol{\eta}(s))\end{bmatrix}^{\mathrm{T}}\\
&\quad\times\underbrace{\left(\mathcal{G}+\begin{bmatrix}0_n & \boldsymbol{P}_1 & 0_n\\ \boldsymbol{P}_1 & 0_n & 0_n\\ 0_n & 0_n & 0_n\end{bmatrix}\right)}_{\boldsymbol{\Omega}_1}\begin{bmatrix}\boldsymbol{\eta}(s)\\ \boldsymbol{\eta}(s)\\ \boldsymbol{f}(\boldsymbol{\eta}(s))\end{bmatrix}\mathrm{d}s\\
&\quad-h_{\mathrm{U}}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\begin{bmatrix}\boldsymbol{\eta}(s)\\ \boldsymbol{\eta}(s)\\ \boldsymbol{f}(\boldsymbol{\eta}(s))\end{bmatrix}^{\mathrm{T}}
\end{aligned}
$$

$$\times \underbrace{\left(\mathcal{G}+\begin{bmatrix} 0_n & \boldsymbol{P}_2 & 0_n \\ \boldsymbol{P}_2 & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}\right)}_{\boldsymbol{\Omega}_2} \begin{bmatrix} \boldsymbol{\eta}(s) \\ \boldsymbol{\eta}(s) \\ \boldsymbol{f}(\boldsymbol{\eta}(s)) \end{bmatrix} \mathrm{d}s$$

$$\leq h_{\mathrm{U}}^2 \begin{bmatrix} \boldsymbol{\eta}(t) \\ \dot{\boldsymbol{\eta}}(t) \\ \boldsymbol{f}(\boldsymbol{\eta}(t)) \end{bmatrix}^{\mathrm{T}} \mathcal{G} \begin{bmatrix} \boldsymbol{\eta}(t) \\ \dot{\boldsymbol{\eta}}(t) \\ \boldsymbol{f}(\boldsymbol{\eta}(t)) \end{bmatrix}$$

$$+h_{\mathrm{U}} \begin{bmatrix} \boldsymbol{\eta}(t) \\ \boldsymbol{\eta}(t-h(t)) \\ \boldsymbol{\eta}(t-h_{\mathrm{U}}) \end{bmatrix}^{\mathrm{T}} \mathrm{diag}\{\boldsymbol{P}_1, -\boldsymbol{P}_1+\boldsymbol{P}_2, -\boldsymbol{P}_2\}$$

$$\times \begin{bmatrix} \boldsymbol{\eta}(t) \\ \boldsymbol{\eta}(t-h(t)) \\ \boldsymbol{\eta}(t-h_{\mathrm{U}}) \end{bmatrix}$$

$$- \begin{bmatrix} \boldsymbol{\varphi}_1(t) \\ \boldsymbol{\varphi}_2(t) \end{bmatrix}^{\mathrm{T}} \underbrace{\left[\begin{array}{c|c} \frac{1}{\alpha(t)}\boldsymbol{\Omega}_1 & 0_{3n} \\ \hline 0_{3n} & \frac{1}{1-\alpha(t)}\boldsymbol{\Omega}_2 \end{array}\right]}_{\hat{\boldsymbol{\Omega}}} \begin{bmatrix} \boldsymbol{\varphi}_1(t) \\ \boldsymbol{\varphi}_2(t) \end{bmatrix}, \quad (12)$$

where $\alpha(t) = h(t)h_{\mathrm{U}}^{-1}$ and

$$\boldsymbol{\varphi}_1(t) = \begin{bmatrix} \int_{t-h(t)}^{t} \boldsymbol{\eta}(s)\mathrm{d}s \\ \boldsymbol{\eta}(t)-\boldsymbol{\eta}(t-h(t)) \\ \int_{t-h(t)}^{t} \boldsymbol{f}(\boldsymbol{\eta}(s))\mathrm{d}s \end{bmatrix},$$

$$\boldsymbol{\varphi}_2(t) = \begin{bmatrix} \int_{t-h_{\mathrm{U}}}^{t-h(t)} \boldsymbol{\eta}(s)\mathrm{d}s \\ \boldsymbol{\eta}(t-h(t))-\boldsymbol{\eta}(t-h_{\mathrm{U}}) \\ \int_{t-h_{\mathrm{U}}}^{t-h(t)} \boldsymbol{f}(\boldsymbol{\eta}(s))\mathrm{d}s \end{bmatrix}.$$

Here, when $0 < h(t) < h_{\mathrm{U}}$, $\alpha(t)$ satisfies $0 < \alpha(t) < 1$. Then, by reciprocally convex approach,[37] the following inequality for any matrix S holds:

$$\underbrace{\left[\begin{array}{c|c} -\sqrt{\frac{\alpha(t)}{1-\alpha(t)}}\boldsymbol{I}_{3n} & 0_{3n} \\ \hline 0_{3n} & \sqrt{\frac{1-\alpha(t)}{\alpha(t)}}\boldsymbol{I}_{3n} \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{\Omega}_1 & \mathcal{S} \\ \hline \mathcal{S}^{\mathrm{T}} & \boldsymbol{\Omega}_2 \end{array}\right]}_{\boldsymbol{\Omega}_3}$$

$$\times \left[\begin{array}{c|c} -\sqrt{\frac{\alpha(t)}{1-\alpha(t)}}\boldsymbol{I}_{3n} & 0_{3n} \\ \hline 0_{3n} & \sqrt{\frac{1-\alpha(t)}{\alpha(t)}}\boldsymbol{I}_{3n} \end{array}\right] > 0,$$

which implies $\hat{\boldsymbol{\Omega}} > \boldsymbol{\Omega}_3$. Also, when $h(t) = 0$ or $h(t) = h_{\mathrm{U}}$, we get $\boldsymbol{\varphi}_1(t) = 0_{3n\times 1}$ or $\boldsymbol{\varphi}_2(t) = 0_{3n\times 1}$, respectively. Thus, if the inequality (8) holds, then an upper bound of $\dot{\boldsymbol{V}}_3$ can be rebounded as

$$\dot{\boldsymbol{V}}_3 \leq \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Xi}_3\boldsymbol{\zeta}(t). \quad (13)$$

Lastly, by Lemma 1, $\dot{\boldsymbol{V}}_4$ is calculated by

$$\begin{aligned} \dot{\boldsymbol{V}}_4 &= (h_{\mathrm{U}}^2/2)^2\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{\eta}}(t) - (h_{\mathrm{U}}^2/2)\int_{h_{\mathrm{U}}}^{t}\int_{s}^{t}\boldsymbol{\eta}^{\mathrm{T}}(s)\boldsymbol{Q}\boldsymbol{\eta}(s)\mathrm{d}s \\ &\leq (h_{\mathrm{U}}^2/2)^2\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{\eta}}(t) - \left(\int_{h_{\mathrm{U}}}^{t}\int_{s}^{t}\boldsymbol{\eta}(s)\mathrm{d}s\right)^{\mathrm{T}} \\ &\quad \times \boldsymbol{Q}\left(\int_{h_{\mathrm{U}}}^{t}\int_{s}^{t}\boldsymbol{\eta}(s)\mathrm{d}s\right) \\ &= (h_{\mathrm{U}}^2/2)^2\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{\eta}}(t) \\ &\quad - \left(h_{\mathrm{U}}\boldsymbol{\eta}(t) - \int_{t-h(t)}^{t}\boldsymbol{\eta}(s)\mathrm{d}s - \int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{\eta}(s)\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{Q} \\ &\quad \times \left(h_{\mathrm{U}}\boldsymbol{\eta}(t) - \int_{t-h(t)}^{t}\boldsymbol{\eta}(s)\mathrm{d}s - \int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{\eta}(s)\mathrm{d}s\right) \\ &= \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Xi}_4\boldsymbol{\zeta}(t). \end{aligned} \quad (14)$$

From Assumption 2, the following inequality holds for any positive diagonal matrices $\boldsymbol{H}_i = \mathrm{diag}\{h_{i1}, \ldots, h_{in}\} \in \mathbb{R}^{n\times n}$ $(i = 1,2,3)$:

$$\begin{aligned} \boldsymbol{0} \leq\ & -2\sum_{i=1}^{n} h_{1i}[\boldsymbol{g}_i(\boldsymbol{\eta}_i(t)) - \boldsymbol{k}_i\boldsymbol{\eta}_i(t)]\,\boldsymbol{g}_i(\boldsymbol{\eta}_i(t)) \\ & -2\sum_{i=1}^{n} h_{2i}[\boldsymbol{g}_i(\boldsymbol{\eta}_i(t-h(t))) \\ & -\boldsymbol{k}_i\boldsymbol{\eta}_i(t-h(t))]\boldsymbol{g}_i(\boldsymbol{\eta}_i(t-h(t))) \\ & -2\sum_{i=1}^{n} h_{3i}[\boldsymbol{g}_i(\boldsymbol{\eta}_i(t-h_{\mathrm{U}})) - \boldsymbol{k}_i\boldsymbol{\eta}_i(t-h_{\mathrm{U}})]\,\boldsymbol{g}_i(\boldsymbol{\eta}_i(t-h_{\mathrm{U}})) \\ =\ & \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Theta}\boldsymbol{\zeta}(t). \end{aligned} \quad (15)$$

The following zero equality with any matrix $\boldsymbol{X}$ is considered:

$$\begin{aligned} \boldsymbol{0} =\ & 2\boldsymbol{\zeta}^{\mathrm{T}}(t)[\boldsymbol{e}_1\boldsymbol{X} + \boldsymbol{e}_4\boldsymbol{X}][-(\boldsymbol{A}+\boldsymbol{K})\boldsymbol{e}_1^{\mathrm{T}} \\ & -\boldsymbol{I}_n\boldsymbol{e}_4^{\mathrm{T}} + \boldsymbol{W}_0\boldsymbol{e}_8^{\mathrm{T}} + \boldsymbol{W}_1\boldsymbol{e}_9^{\mathrm{T}}]\boldsymbol{\zeta}(t). \end{aligned} \quad (16)$$

Then, by defining $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{K}$, the inequality (16) leads to

$$\begin{aligned} \boldsymbol{0} =\ & 2\boldsymbol{\zeta}^{\mathrm{T}}(t)[\boldsymbol{e}_1 + \boldsymbol{e}_4][-(\boldsymbol{X}\boldsymbol{A}+\boldsymbol{Y})\boldsymbol{e}_1^{\mathrm{T}} \\ & -\boldsymbol{X}\boldsymbol{e}_4^{\mathrm{T}} + \boldsymbol{X}\boldsymbol{W}_0\boldsymbol{e}_8^{\mathrm{T}} + \boldsymbol{X}\boldsymbol{W}_1\boldsymbol{e}_9^{\mathrm{T}}]\boldsymbol{\zeta}(t) \\ =\ & \boldsymbol{\zeta}^{\mathrm{T}}(t)\mathrm{sym}\{\boldsymbol{\Psi}\boldsymbol{\Gamma}\}\boldsymbol{\zeta}(t). \end{aligned} \quad (17)$$

Therefore, from Eqs. (10)–(15), and by consideration of zero equality (17) and application of the S-procedure,[40] a sufficient condition guaranteeing synchronization for system (5) can be

$$\dot{\boldsymbol{V}}(x(t)) \leq \boldsymbol{\zeta}^{\mathrm{T}}(t)\left(\sum_{l=1}^{4}\boldsymbol{\Xi}_l + \boldsymbol{\Theta} + \mathrm{sym}\{\boldsymbol{\Psi}\boldsymbol{\Gamma}\}\right)\boldsymbol{\zeta}(t) < 0. \quad (18)$$

In conclusion, the condition (18) is equivalent to the LMIs (7). This completes the proof of Theorem 1.

Remark 1 In Ref. [24], delay-dependent synchronization analysis for continuous time nonlinear complex neural networks with time-varying delay was conducted by utilizing delay decomposition approach. As mentioned in Section 1, the delay decomposition method is well utilized in reducing the feasible region of stability criteria for dynamic systems with time delays. This approach has the advantage of getting larger maximum delay bounds for guaranteeing the asymptotic stability as the delay-partitioning number increases. However, when the delay-partitioning number increases, the matrix formulation becomes more complex and the dimension of stability condition grows bigger because the dimension of an augmented vector increases. In this paper, to reduce the conservatism of synchronization criterion, the reciprocally convex optimization approach[37]

is utilized. Furthermore, inspired by the work of (36), two zero equalities (11) are added in synchronization criterion. As presented in (11), the terms $\boldsymbol{\eta}^{\mathrm{T}}(t)(h_{\mathrm{U}}P_1)\boldsymbol{\eta}(t) - \boldsymbol{\eta}^{\mathrm{T}}(t-h(t))(h_{\mathrm{U}}\boldsymbol{P}_1)\boldsymbol{\eta}(t-h(t))$ and $\boldsymbol{\eta}^{\mathrm{T}}(t-h(t))(h_{\mathrm{U}}\boldsymbol{P}_2)\boldsymbol{\eta}(t-h(t)) - \boldsymbol{\eta}^{\mathrm{T}}(t-h_{\mathrm{U}})(h_{\mathrm{U}}\boldsymbol{P}_2)\boldsymbol{\eta}(t-h_{\mathrm{U}})$ provide the enhanced feasible region of synchronization criterion. As shown in (12), the two integral terms such as $-2h_{\mathrm{U}}\int_{t-h(t)}^{t}\boldsymbol{\eta}^{\mathrm{T}}(s)P_1\boldsymbol{\eta}(s)\,\mathrm{d}s$ and $-2h_{\mathrm{U}}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{\eta}^{\mathrm{T}}(s)P_2\boldsymbol{\eta}(s)\,\mathrm{d}s$ presented in (11) are combined with the integral terms $-h_{\mathrm{U}}\int_{t-h(t)}^{t}\boldsymbol{\eta}^{\mathrm{T}}(s)\mathcal{G}\boldsymbol{\eta}(s)\,\mathrm{d}s$ and $-h_{\mathrm{U}}\int_{t-h_{\mathrm{U}}}^{t-h(t)}\boldsymbol{\eta}^{\mathrm{T}}(s)\mathcal{G}\boldsymbol{\eta}(s)\,\mathrm{d}s$, which cause the conservatism of the synchronization criterion. In the next section, it will be shown that Theorem 1 can provide larger delay bounds than the methods of Ref. [24] through one numerical example .

Theorem 1 provides the synchronization criterion for system (5) in the framework of LMIs when the disturbance $\boldsymbol{w}(t) = 0_{2m\cdot 1}$. Based on the results of Theorem 1, an $\mathcal{H}_\infty$ stabilization controller design method for system (5) will be introduced as Theorem 2.

Before introducing Theorem 2, the following notations of several matrices are defined for simplicity:

$$
\begin{aligned}
\tilde{\boldsymbol{\zeta}}(t) &= [\boldsymbol{\zeta}^{\mathrm{T}}(t), \boldsymbol{w}^{\mathrm{T}}(t)]^{\mathrm{T}},\\
\tilde{\boldsymbol{\Pi}}_1 &= [\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_3, \tilde{\boldsymbol{e}}_6 + \tilde{\boldsymbol{e}}_7, \tilde{\boldsymbol{e}}_{11} + \tilde{\boldsymbol{e}}_{12}],\\
\tilde{\boldsymbol{\Pi}}_2 &= [\tilde{\boldsymbol{e}}_4, \tilde{\boldsymbol{e}}_5, \tilde{\boldsymbol{e}}_1 - \tilde{\boldsymbol{e}}_3, \tilde{\boldsymbol{e}}_8 - \tilde{\boldsymbol{e}}_{10}],\\
\tilde{\boldsymbol{\Pi}}_3 &= [\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_4, \tilde{\boldsymbol{e}}_8], \quad \tilde{\boldsymbol{\Pi}}_4 = [\tilde{\boldsymbol{e}}_3, \tilde{\boldsymbol{e}}_5, \tilde{\boldsymbol{e}}_{10}],\\
\tilde{\boldsymbol{\Pi}}_5 &= [\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3],\\
\tilde{\boldsymbol{\Pi}}_6 &= [\tilde{\boldsymbol{e}}_6, \tilde{\boldsymbol{e}}_1 - \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_{11}, \tilde{\boldsymbol{e}}_7, \tilde{\boldsymbol{e}}_2 - \tilde{\boldsymbol{e}}_3, \tilde{\boldsymbol{e}}_{12}],\\
\tilde{\boldsymbol{\Xi}}_1 &= \mathrm{sym}\{\tilde{\boldsymbol{\Pi}}_1\mathcal{R}\tilde{\boldsymbol{\Pi}}_2^{\mathrm{T}}\},\\
\tilde{\boldsymbol{\Xi}}_2 &= \tilde{\boldsymbol{\Pi}}_3\mathcal{N}\tilde{\boldsymbol{\Pi}}_3^{\mathrm{T}} - \tilde{\boldsymbol{\Pi}}_4\mathcal{N}\tilde{\boldsymbol{\Pi}}_4^{\mathrm{T}}\\
&\quad + \mathrm{sym}\left\{[\tilde{\boldsymbol{e}}_8, \tilde{\boldsymbol{e}}_1]\begin{bmatrix}\boldsymbol{\Lambda} - \Delta\\ K_{\mathrm{p}}\Delta\end{bmatrix}\tilde{\boldsymbol{e}}_4^{\mathrm{T}}\right\},\\
\tilde{\boldsymbol{\Xi}}_3 &= h_{\mathrm{U}}^2\tilde{\boldsymbol{\Pi}}_3\mathcal{G}\tilde{\boldsymbol{\Pi}}_3^{\mathrm{T}} + h_{\mathrm{U}}\tilde{\boldsymbol{\Pi}}_5\,\mathrm{diag}\{\boldsymbol{P}_1, -\boldsymbol{P}_1 + \boldsymbol{P}_2, -\boldsymbol{P}_2\}\tilde{\boldsymbol{\Pi}}_5^{\mathrm{T}}\\
&\quad - \tilde{\boldsymbol{\Pi}}_6\boldsymbol{\Omega}_3\tilde{\boldsymbol{\Pi}}_6^{\mathrm{T}},\\
\tilde{\boldsymbol{\Xi}}_4 &= (h_{\mathrm{U}}^2/2)^2\tilde{\boldsymbol{e}}_4\boldsymbol{Q}\tilde{\boldsymbol{e}}_4^{\mathrm{T}} - (h_{\mathrm{U}}\tilde{\boldsymbol{e}}_1 - \tilde{\boldsymbol{e}}_6 - \tilde{\boldsymbol{e}}_7)\\
&\quad \times \boldsymbol{Q}(h_{\mathrm{U}}\tilde{\boldsymbol{e}}_1 - \tilde{\boldsymbol{e}}_6 - \tilde{\boldsymbol{e}}_7)^{\mathrm{T}},\\
\tilde{\boldsymbol{\Theta}} &= \sum_{i=1}^{3}\mathrm{sym}\left\{\tilde{\boldsymbol{e}}_i K_{\mathrm{p}} H_i\tilde{\boldsymbol{e}}_{i+7}^{\mathrm{T}} - \tilde{\boldsymbol{e}}_{i+7}H_i\tilde{\boldsymbol{e}}_{i+7}^{\mathrm{T}}\right\},\\
\tilde{\boldsymbol{\Psi}} &= \tilde{\boldsymbol{e}}_1 + \tilde{\boldsymbol{e}}_4,\\
\tilde{\boldsymbol{\Gamma}} &= X(-A\tilde{\boldsymbol{e}}_1^{\mathrm{T}} - \tilde{\boldsymbol{e}}_4^{\mathrm{T}} + \boldsymbol{W}_0\tilde{\boldsymbol{e}}_8^{\mathrm{T}} + \boldsymbol{W}_1\tilde{\boldsymbol{e}}_9^{\mathrm{T}} + D\tilde{\boldsymbol{e}}_{13}^{\mathrm{T}}) - \boldsymbol{Y}\tilde{\boldsymbol{e}}_1^{\mathrm{T}},\\
\tilde{\boldsymbol{\Phi}} &= \sum_{i=1}^{4}\tilde{\boldsymbol{\Xi}}_i + \tilde{\boldsymbol{\Theta}} + \mathrm{sym}\{\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Gamma}}\},\\
\tilde{\boldsymbol{\Sigma}} &= \tilde{\boldsymbol{e}}_1 F^{\mathrm{T}} F\tilde{\boldsymbol{e}}_1^{\mathrm{T}} - \gamma^2\tilde{\boldsymbol{e}}_{13}\tilde{\boldsymbol{e}}_{13}^{\mathrm{T}},
\end{aligned}
\tag{19}
$$

where $\tilde{\boldsymbol{e}}_i \in \mathbb{R}^{(12n+2m)\times n}$ are defined as block entry matrices; e.g., $\tilde{\boldsymbol{e}}_3 = [0_{n\cdot 2n}, \boldsymbol{I}_n, 0_{n\cdot 9n}, 0_{n\cdot 2m}]^{\mathrm{T}}$ and $\tilde{\boldsymbol{e}}_{13} = [0_{2m\cdot 12n}, \boldsymbol{I}_{2m}]^{\mathrm{T}}$, and $\boldsymbol{\zeta}(t)$ and $\boldsymbol{\Omega}_i$ $(i = 1,2,3)$ are defined by Eq. (6).

Now, we have the following theorem.

Theorem 2 For given positive scalars h_{U} and γ, the master system (1) and the slave system (2) are synchronized for $0 \le h(t) \le h_{\mathrm{U}}$ with $\mathcal{H}_\infty$ disturbance attenuation level γ, if there exist positive definite matrices $\mathcal{R} \in \mathbb{R}^{4n\times 4n}$, $\mathcal{N} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G} \in \mathbb{R}^{3n\times 3n}$, $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, positive diagonal matrices $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$, $\boldsymbol{\Delta} \in \mathbb{R}^{n\times n}$, $H_i \in \mathbb{R}^{3n\times 3n}(i = 1,2,3)$, any symmetric matrices $\boldsymbol{P}_i \in \mathbb{R}^{n\times n}$ $(i = 1,2)$ and any matrices $\mathcal{S} \in \mathbb{R}^{3n\times 3n}$, $\boldsymbol{X} \in \mathbb{R}^{n\times n}$, $\boldsymbol{Y} \in \mathbb{R}^{n\times n}$ satisfying the following LMIs:

$$\tilde{\boldsymbol{\Sigma}} + \tilde{\boldsymbol{\Phi}} < 0, \tag{20}$$

$$\boldsymbol{\Omega}_3 \ge 0, \tag{21}$$

where $\tilde{\boldsymbol{\Sigma}}$ and $\tilde{\boldsymbol{\Phi}}$ are defined in (19), and $\boldsymbol{\Omega}_3$ in (6). Then, the $\mathcal{H}_\infty$ synchronization gain can be calculated as $K = X^{-1}Y$ and, by use of this gain, $\mathcal{H}_\infty$ control is performed.

Proof Based on the Lyapunov–Krasovskii functional (9), the deriving process of $\tilde{\boldsymbol{\Phi}}$ is very similar to the proof of Theorem 1, so it is omitted. At this time, the augmented vector $\tilde{\boldsymbol{\zeta}}(t)$ can be utilized instead of the $\boldsymbol{\zeta}(t)$ used in Theorem 1.

From Definition 1, with the zero initial condition, J can be represented as follows:

$$
\begin{aligned}
\boldsymbol{J} &= \int_0^\infty \left\{z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{\boldsymbol{V}}(\boldsymbol{\eta}(t))\right\}\mathrm{d}t\\
&\quad + V(\boldsymbol{\eta}(t))|_{t=0} - V(\boldsymbol{\eta}(t))|_{t=\infty}.
\end{aligned}
\tag{22}
$$

Considering $\boldsymbol{V}(\boldsymbol{\eta}(t))|_{t=0} = 0$ and $\boldsymbol{V}(\boldsymbol{\eta}(t))|_{t=\infty} \to 0$, $\boldsymbol{J}$ is bounded as

$$
\begin{aligned}
\boldsymbol{J} &\le \int_0^\infty \left\{\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{w}^{\mathrm{T}}(t)\boldsymbol{w}(t) + \dot{\boldsymbol{V}}(\boldsymbol{\eta}(t))\right\}\mathrm{d}t\\
&= \int_0^\infty \left\{\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{\eta}(t) - \gamma^2\boldsymbol{w}^{\mathrm{T}}(t)\boldsymbol{w}(t) + \dot{\boldsymbol{V}}(\boldsymbol{\eta}(t))\right\}\mathrm{d}t\\
&= \int_0^\infty \tilde{\boldsymbol{\zeta}}^{\mathrm{T}}(t)\left(\tilde{\boldsymbol{\Sigma}} + \tilde{\boldsymbol{\Phi}}\right)\tilde{\boldsymbol{\zeta}}(t)\,\mathrm{d}t\\
&= \boldsymbol{J}^*.
\end{aligned}
\tag{23}
$$

Thus, $\tilde{\boldsymbol{\Sigma}} + \tilde{\boldsymbol{\Phi}} < 0$ is equivalent to $\boldsymbol{J}^* < 0$, i.e., $\boldsymbol{J} \le \boldsymbol{J}^* < 0$. Therefore, if $\tilde{\boldsymbol{\Sigma}} + \tilde{\boldsymbol{\Phi}} < 0$, then system (5) with the controller gain $\boldsymbol{K} = \boldsymbol{X}^{-1}\boldsymbol{Y}$ is asymptotically synchronous and $\mathcal{H}_\infty$ control is performed. ■

Remark 2 In the previous works,[29–34] $\mathcal{H}_\infty$ synchronization problems between master and slave systems with disturbances in only slave parts were investigated. As mentioned in the Introduction, it is desirable to study the $\mathcal{H}_\infty$ synchronization problem for nonlinear neural networks with the disturbances in both master and slave parts since disturbances can also exist in the master part. Thus, disturbances are considered in both master and slave parts and its $\mathcal{H}_\infty$ synchronization criterion is first proposed in Theorem 2.

5. Numerical examples

In this section, we provide two numerical examples to illustrate the effectiveness of the proposed synchronization criteria proposed in Theorems 1 and 2.

Example 1 Consider the following neural networks:

(i) Master system:

$$\dot{\boldsymbol{x}}_{\rm m}(t) = -\boldsymbol{A}\boldsymbol{x}_{\rm m}(t) + \boldsymbol{W}_0\boldsymbol{f}(\boldsymbol{x}_{\rm m}(t)) + \boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}_{\rm m}(t-h(t)));$$

(ii) slave system:

$$\begin{aligned}\dot{\boldsymbol{x}}_{\rm s}(t) &= -A\boldsymbol{x}_{\rm s}(t) + \boldsymbol{W}_0\boldsymbol{f}(\boldsymbol{x}_{\rm s}(t)) \\ &\quad + \boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}_{\rm s}(t-h(t))) + \boldsymbol{u}(t),\end{aligned} \tag{24}$$

with $\boldsymbol{f}(\boldsymbol{x}) = \tanh(\boldsymbol{x})$, which means $\boldsymbol{K}_{\rm p} = \boldsymbol{I}_n$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{W}_0 = \begin{bmatrix} 2 & -0.1 \\ -5 & 2 \end{bmatrix},$$
$$\boldsymbol{W}_1 = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -1.5 \end{bmatrix}.$$

It should be noted that the system (24) is borrowed from the works of Ref. [24] which investigated the problem of synchronization for chaotic nonlinear continuous neural networks with time-varying delay. Since the error dynamics of systems presented in Example 1 also include the term of $-Ae(t)$, Theorem 1 can be applicable to the system (24) to investigate maximum delay bounds for guaranteeing synchronization between master and slave system. For the master–slave networks above, the results of upper bounds on time-delay and synchronization gain K by application of Theorem 1 are listed in Table 1. From Table 1, it can be confirmed that the obtained results are much larger than those of Ref. [24] which utilized

Table 1. Upper bounds of time-delay $h_{\rm U}$ and gain K (Example 1).

Methods	$h_{\rm U}$	K
Ref. [24] ($N=6$)	1.9686	$\begin{bmatrix} 12.7158 & -6.3187 \\ -0.4436 & 12.2130 \end{bmatrix}$
Theorem 1	290.3107	$\begin{bmatrix} 50.9460 & 0.2381 \\ 0.4078 & 1.4856 \end{bmatrix} \times 10^5$

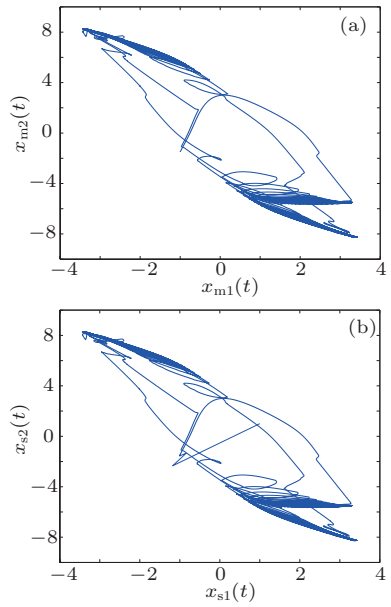


Fig. 1. (color online) x_1–x_2 plane of the networks (24) with S1: (a) master part and (b) slave part (Example 1).

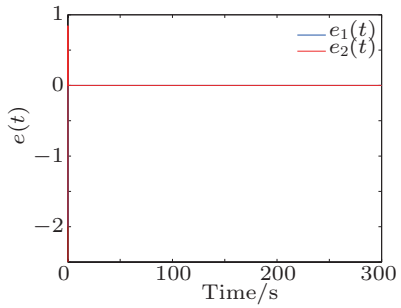


Fig. 2. (color online) Synchronization error trajectories of the networks (24) with S1 (Example 1).

the delay decomposition approach. In order to confirm the obtained results with the time-varying delay conditions, let us assume $h(t) = 100|\sin(t)|$ which means $h_{\rm U} = 100$ and activation functions as $\boldsymbol{f}(\cdot) = \tanh(\cdot)$. The simulation results for the state responses of master and slave parts and the synchronization error trajectories, $\boldsymbol{\eta}(t)$, of the networks (24) are shown in Figs. 1 and 2. At this time, the gain is

$$\boldsymbol{K} = \begin{bmatrix} 1.2027 & 0.3636 \\ 0.3602 & 1.7505 \end{bmatrix} \times 10^4. \tag{25}$$

These figures show that the networks with the errors converge to zero for given initial values of the state by $\boldsymbol{x}_{\rm m}(\sigma) = [-1,-1.5]^{\rm T}(\sigma \in [-h_{\rm U}, 0])$ and $\boldsymbol{x}_{\rm s}(\sigma) = [1,1]^{\rm T}(\sigma \in [-h_{\rm U}, 0])$.

Example 2 Consider the following neural networks with the disturbances.

(i) Master system:

$$\begin{aligned}\dot{\boldsymbol{x}}_{\rm m}(t) &= -\boldsymbol{\alpha}(\boldsymbol{x}_{\rm m}(t)) + \boldsymbol{W}_0\boldsymbol{f}(\boldsymbol{x}_{\rm m}(t)) \\ &\quad + \boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}_{\rm m}(t-h(t))) + \boldsymbol{b} + \boldsymbol{D}_{\rm m}\boldsymbol{w}_{\rm m}(t), \\ z_{\rm m}(t) &= F\boldsymbol{x}_{\rm m}(t);\end{aligned}$$

(ii) Slave system:

$$\begin{aligned}\dot{\boldsymbol{x}}_{\rm s}(t) &= -\boldsymbol{\alpha}(\boldsymbol{x}_{\rm s}(t)) + \boldsymbol{W}_0\boldsymbol{f}(\boldsymbol{x}_{\rm s}(t)) \\ &\quad + \boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}_{\rm s}(t-h(t))) + \boldsymbol{b} + \boldsymbol{D}_s\boldsymbol{w}_{\rm s}(t) + u(t), \\ z_{\rm s}(t) &= \boldsymbol{F}\boldsymbol{x}_{\rm s}(t),\end{aligned} \tag{26}$$

with $\boldsymbol{b} = 0_{2\cdot 1}$ and $\boldsymbol{f}(\boldsymbol{x}) = \tanh(x)$, which means $\boldsymbol{K}_{\rm p} = \boldsymbol{I}_n$, where

$$\boldsymbol{\alpha}(x) = \begin{bmatrix} 2x_1 - \sin(x_1) \\ 2x_2 + \sin(x_2) \end{bmatrix},$$
$$\boldsymbol{W}_0 = \begin{bmatrix} 2.1 & -0.12 \\ -5.1 & 3.2 \end{bmatrix}, \quad \boldsymbol{W}_1 = \begin{bmatrix} -1.6 & -0.1 \\ -0.2 & -2.4 \end{bmatrix},$$
$$\boldsymbol{D}_{\rm m} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{D}_{\rm s} = \boldsymbol{D}_{\rm m}, \quad \boldsymbol{F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\rm T}.$$

Table 2. Gain $\boldsymbol{K}$ with fixed $h_{\rm U} = 5$ and $\gamma = 0.5$ (Example 2).

Methods	γ	$h_{\rm U}$	K
Theorem 2	0.5	5	$\begin{bmatrix} 151.9586 & 113.6306 \\ 127.6281 & 416.6092 \end{bmatrix}$

The results of synchronization gain K with the disturbances and upper bound on time-delay $h_U = 5$ and the $\mathcal{H}_\infty$ performance $\gamma = 0.5$ by Theorem 2 are listed in Table 2. To confirm the obtained results, simulation with the condition listed in Table 3 is conducted in Figs. 3–6. These figures show for the state responses of master and slave parts and the synchronization error trajectories, $\eta(t)$, of network (26) with the conditions of time-delay given in Table 3 when initial values of the state are chosen as the same values in Example 1. Also, the utilized activation function in simulation is $f(\cdot) = \tanh(\cdot)$. For comparison, in the same condition, the gain K is obtained for the network (26) without disturbances. From Figs. 3–6, it can be confirmed that the synchronization error converges to zero.

Table 3. Simulation conditions (Example 2).

No.	h_U	$h(t)$	K	$w(t)$
C2-1	5	$5\lvert\sin(t)\rvert$	$\begin{bmatrix} 48.2758 & 15.3196 \\ 17.5774 & 274.4456 \end{bmatrix}$	$0_{2\cdot 1}$
C2-2	5	$5\lvert\sin(t)\rvert$	$\begin{bmatrix} 151.9586 & 113.6306 \\ 127.6281 & 416.6092 \end{bmatrix}$	$\begin{bmatrix} 10\sin(\pi t) \text{ at } 20\text{–}50 \text{ s} \\ 20\sin(\pi t) \text{ at } 30\text{–}70 \text{ s} \end{bmatrix}$

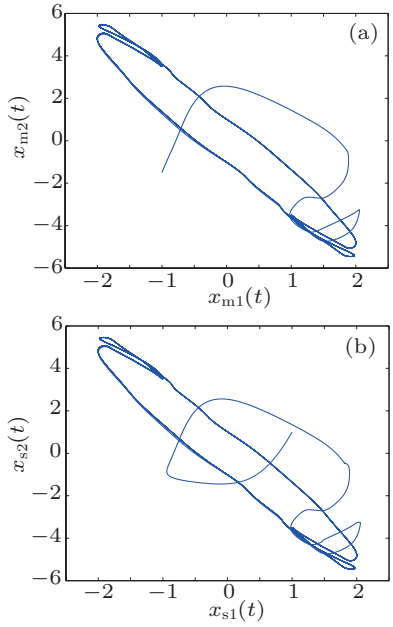


Fig. 3. (color online) x_1–x_2 plane of the networks (26) with C2-1: (a) master part and (b) slave part (Example 2).

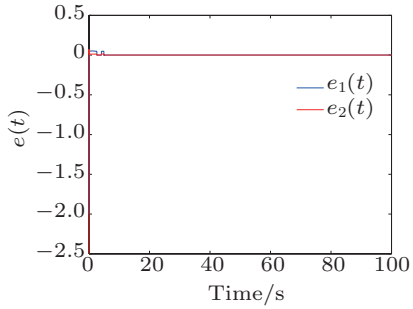


Fig. 4. (color online) Synchronization error trajectories of the networks (26) with C2-1 (Example 2).

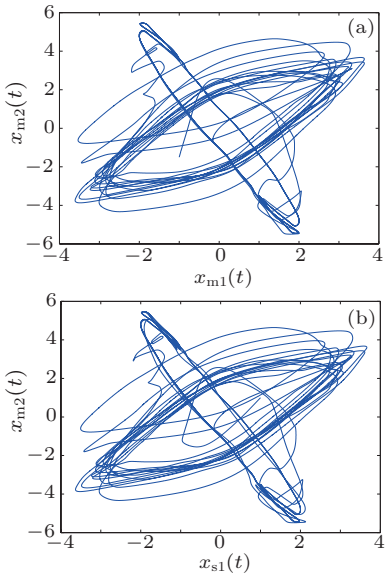


Fig. 5. (color online) x_1–x_2 plane of the networks (26) with C2-2: (a) master part and (b) slave part (Example 2).

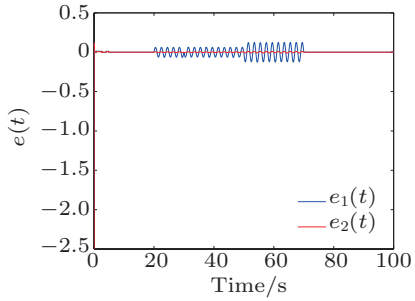


Fig. 6. (color online) Synchronization error trajectories of the networks (26) with C2-2 (Example 2).

6. Conclusion

In this paper, the new synchronization criteria for chaotic neural networks with time varying delays and disturbances were proposed. In Theorem 1, an improved synchronization criterion without disturbances was proposed and it was shown that the feasible region of synchronization criterion can be enhanced by comparing maximum delay bound with the results of Ref. [24]. In Theorem 2, an $\mathcal{H}_\infty$ synchronization criterion for chaotic neural networks with time varying delays and disturbances in both master and slave parts was introduced and its effectiveness was illustrated via one numerical example.

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