Existence of heteroclinic orbits in a novel three-order dynamical system*

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In this paper, we design a novel three-order autonomous system. Numerical simulations reveal the complex chaotic behaviors of the system. By applying the undetermined coefficient method, we find a heteroclinic orbit in the system. As a result, the Ši'lnikov criterion along with some other given conditions guarantees that the system has both Smale horseshoes and chaos of horseshoe type.

Keywords: novel chaotic system, heteroclinic orbit, Ši'lnikov criterion, undetermined coefficient method

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1. Introduction

The Lorenz system[1] has become a paradigm for the investigation and applications of chaos theory. Many researchers have constructed various chaotic systems based on it. In 1999, Chen *et al*. developed a Chen system[2] by the feedback control method on the Lorenz System. In 2002, Lü *et al*. proposed a Lü system[3] which serves as the bridge that connects the Lorenz system and the Chen system, and the term "Lorenz system family" has been put forward subsequently.[4] Liu *et al*. have established the Liu system[5] which contains quadratic terms. In recent years, researchers around the world have constructed a large number of new chaotic systems such as the smooth multi-wing system,[6] hyper-chaotic system,[7,8] invariable Lyapunov exponent system,[9] (2+1)-dimensional generalized Breor–Kaup system,[10] chaotic equation in the imaginary axis,[11] and so on.[12–20] This work demonstrates that constructing new chaotic systems is vital to enrich the contents of chaos.

However, it is not easy to prove the existence of chaotic attractors by theoretical analysis. The Ši'lnikov criterion,[21,22] which guarantees that a system has both Smale horseshoes and chaos of horseshoe type, has played a key role in ensuring that the system is chaotic.

A pivotal method, also known as the undetermined coefficient method, was proposed by Zhou, Tang, and Chen to show the existence of Chen's attractor.[23] The existence problem of Ši'lnikov-type orbits has therefore become an active research field with much progress made, such as the discoveries of orbits in the Lorenz family system,[24] the generalized Lorenz canonical form of dynamics system,[25] the Liu system,[26] the Lü system, Zhou's system,[27] the coupled Duffing's systems,[28] a modified Lorenz system,[29] and other chaotic systems[30–36] among several others based on the Ši'lnikov criterion.

The rest of the paper is organized as follows. In Section 2, we design a simple three-order autonomous system and study some properties like symmetry and characteristics of trajectory under parameter shifting. In Section 3, we review some basic concepts and terminologies related to homoclinic and heteroclinic orbits. We devote Section 4 to investigating the heteroclinic orbit of the system by using the undetermined coefficient method, and derive the algebraic expression of heteroclinic orbits. In Section 5, we prove the uniform convergence of its series expansion. A summary is given in Section 6.

2. Structure and some dynamical properties of the novel system

2.1. Structure of the novel system

The novel system is described by the following differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -ay + yz, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x - by, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = -cz + x^2 + dy^2, \end{cases} \tag{1}$$

where $a, c, d \in \mathbb{R}$, and $-b \in \mathbb{R}^+$. This system may exhibit a chaotic attractor as shown in Fig. 1 when $a = 11$, $b = -0.5$, $c = 8$, and $d = 4$.

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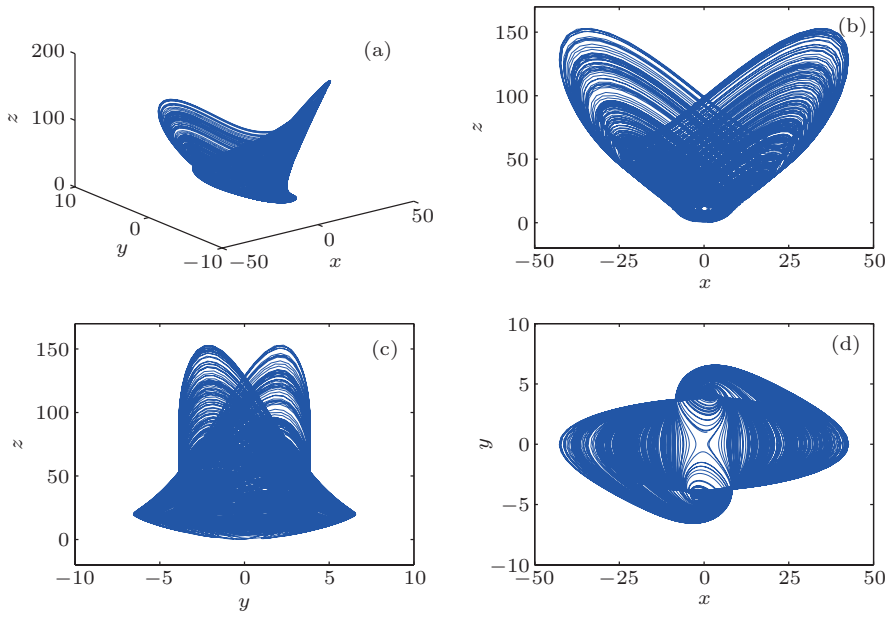


Fig. 1. (color online) Phase portraits of the system in (a) $x(t)$–$y(t)$–$z(t)$ space, (b) $x(t)$–$z(t)$ plane, (c) $y(t)$–$z(t)$ plane, and (d) $x(t)$–$y(t)$ plane. Initial condition: $(x_0, y_0, z_0) = (1, 1, 1)$.

2.2. Some dynamical properties of the novel system

2.2.1. Symmetry

System (1) is symmetric with respect to the z axis, that is, the system will not change under the transformation $(x, y, z) \to (-x, -y, z)$.

2.2.2. Dissipativity

System (1) satisfies

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -b - c = -7.5, \tag{2}$$

where V denotes the phase volume. Since $\nabla V = -7.5 < 0$, the system is dissipative, and its exponential dissipative rate is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \mathrm{e}^{-7.5}. \tag{3}$$

2.2.3. Equilibrium and stability

The characteristic equation of system (1) at equilibrium points (x, y, z) is

$$\begin{aligned} &\lambda^3 + (b+c)\lambda^2 + (-2xy + z - a + bc)\lambda \\ &-2bxy + 2dy^2 + cz - ac = 0. \end{aligned} \tag{4}$$

The three equilibria of system (1) are

$$E_1 = (0,0,0), \quad E_{2,3} = \left(\mp b\sqrt{\frac{ac}{b^2+d}}, \pm\sqrt{\frac{ac}{b^2+d}}, a \right). \tag{5}$$

The characteristic equation of system (1) at E_1 is

$$(\lambda + c)(\lambda^2 + b\lambda - a) = 0. \tag{6}$$

Obviously, equation (6) has two negative real roots and one positive real root since $-a < 0$, so E_1 is a saddle node. The characteristic equation at the equilibrium points E_2 and E_3 is

$$\lambda^3 + (b+c)\lambda^2 + bc\left(1 + \frac{2a}{b^2+d}\right)\lambda + 2ac = 0. \tag{7}$$

Setting $\lambda = \mu - (b+c)/3$ yields

$$\mu^3 + p\mu + q = 0, \tag{8}$$

where

$$p = bc\left(1 + \frac{2a}{b^2+d}\right) - \frac{(b+c)^2}{3}, \tag{9}$$

$$q = \frac{2(b+c)^2}{27} - \frac{bc(b+c)}{3}\left(1 + \frac{2a}{b^2+d}\right) + 2ac. \tag{10}$$

Furthermore, denote

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3, \tag{11}$$

when $\Delta > 0$. By the Cardan formula, the algebraic equation (8) has a unique negative real root, γ, and a conjugate pair of complex roots, $a \pm \mathrm{i}b$, with

$$\gamma = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}, \tag{12}$$

$$\alpha = -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}\right), \tag{13}$$

$$\beta = \frac{\sqrt{3}}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}\right). \tag{14}$$

Therefore, when $\Delta > 0$, three roots of the characteristic polynomial (7) are

$$\lambda_1 = -\frac{b+c}{3} + \gamma, \tag{15}$$

$$\lambda_{2,3} = -\frac{b+c}{3} + \alpha \pm \mathrm{i}\beta, \tag{16}$$

respectively, where $\lambda_1 < 0$ and $\mathrm{i} = \sqrt{-1}$, to ensure that the real part of the complex conjugate roots is positive. It is further required that

$$\mu^3 + p\mu + q = 0, \tag{17}$$

$$-\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} > \frac{2(b+c)}{3} > 0. \tag{18}$$

Thus, the equilibrium points E_2 and E_3 are saddle foci at the same time.

For $a = 11$, $b = -0.5$, $c = 8$, and $d = 4$, the equilibrium and their eigenvalues are given by

$$\begin{cases} E_1 = (0,0,0), \\ (\lambda_1, \lambda_2, \lambda_3) = (-8, -3.076033, 3.576033); \\ E_2 = (2.2752, 4.3386, 11), \\ (\lambda_1, \lambda_2, \lambda_3) = (-11.1372, 1.8186 \pm 3.5349\mathrm{i}); \\ E_3 = (-2.2752, -4.3386, 11), \\ (\lambda_1, \lambda_2, \lambda_3) = (-11.1372, 1.8186 \pm 3.5349\mathrm{i}). \end{cases} \tag{19}$$

Then, when $\Delta > 0$, one can easily obtain that the two points E_2 and E_3 are of hyperbolic saddle foci type.

2.2.4. Poincaré section and Lyapunov exponents

A powerful method introduced by Poincaré for examining the motion of dynamical systems is that of a Poincaré section. Let S be an $(n-1)$-dimensional surface transverse to the trajectories of a dynamical system. Consider a point $\boldsymbol{x}_0$ on S at time $t = 0$. As the trajectory starts at $\boldsymbol{x}_0$, it will eventually return to S from $\boldsymbol{x}_1$ after a certain period. If we consider all initial points on S, we can define a map P from S to itself such that

$$\boldsymbol{x}_1 = P(\boldsymbol{x}_0), \tag{20}$$

and in general after $k+1$ intersections,

$$\boldsymbol{x}_{k+1} = P(\boldsymbol{x}_k). \tag{21}$$

The mapping P is called the return map or Poincaré map of the dynamical system.

We can use this method to investigate the behavior of the autonomous system in the following way. Choose a Poincaré section (for example, the x–y plane). Choose a strobe frequency ω_s which is hopefully the period of a potential periodic attractor. Only plot points in the x–y plane corresponding to a time interval $2\pi\omega_s$. Then, we ignore transients.

For a periodic attractor and the correct strobe frequency, we will only observe one plotted point in the x–y plane.

If the natural frequency of the system is $\omega_0 = \frac{p}{q}\omega_s$, then we will obtain q points.

If the natural frequency is an irrational multiple of the strobe frequency, we will obtain a closed curve on the Poincaré section.

If we do not in fact have a periodic attractor but a chaotic attractor, the Poincaré section will be fractal in nature.

For system (1), its Lyapunov exponents are $\mathrm{LE}_1 = 0.64263$, $\mathrm{LE}_2 = 0.00051662$, and $\mathrm{LE}_3 = -8.1432$. The Poincaré section diagram in the y–z plane with transversing plane $x = 0$ is drawn in Fig. 2. We observe that the Poincaré section shows some characteristics of a fractal.

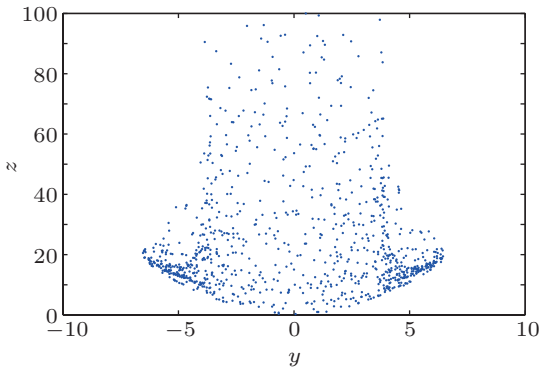


Fig. 2. (color online) The Poincaré section diagram in the y–z plane with transversing plane $x = 0$.

2.2.5. A brief description of the formation of the system

In system (1), one can adjust the value of a after setting $b = -0.5$, $c = 8$, and $d = 4$. Some values of a that lead the system to exhibit various dynamical behaviors are selected and the corresponding phase portraits of the systems are shown in Figs. 3 and 4, and their characteristics are shown in Table 1.

Table 1. Characteristics of the phase portraits under parameter shifting.

a	LE_1	LE_2	LE_3	Characteristics
−21	0.001449	−0.54157	−6.9599	the trajectory will converge to a limit cycle
9	0.004077	−0.051691	−7.453	converge to a limit cycle in another shape
30	0.59895	0.0010753	−8.1	the system can also be chaotic

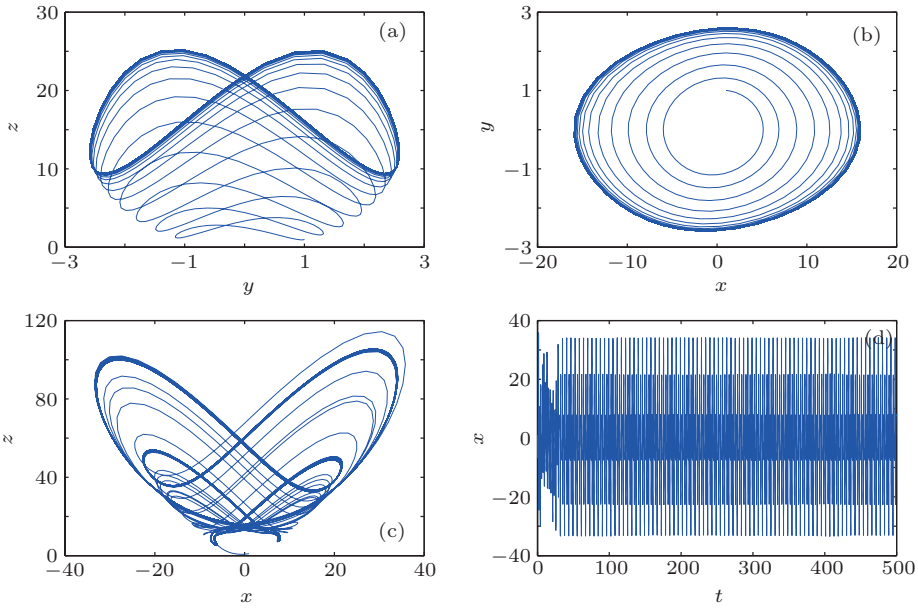


Fig. 3. (color online) The trajectory will converge to a limit cycle when $a = -21$ in (a) $y(t)$–$z(t)$ plane, (b) $x(t)$–$y(t)$ plane. The trajectory will also converge to a limit cycle when $a = 9$ in (c) $x(t)$–$z(t)$ plane. (d) Variation of x with respect to t. Initial condition: $(x_0, y_0, z_0) = (1, 1, 1)$.

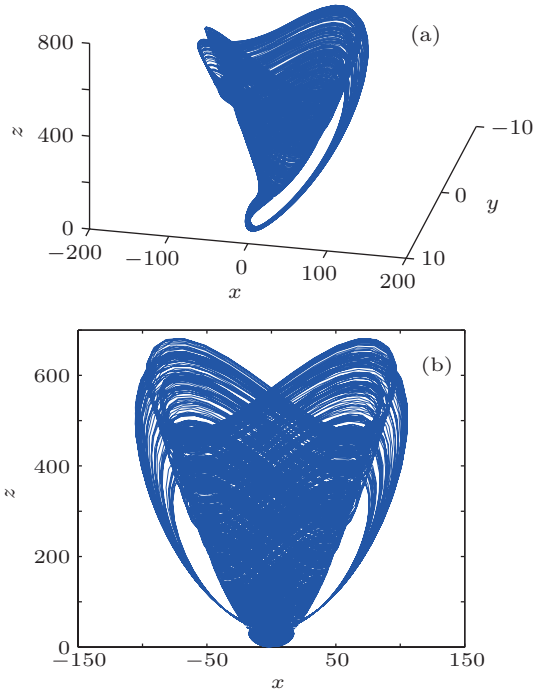


Fig. 4. (color online) The system is also chaotic when $a = 30$ in (a) $x(t)$–$y(t)$–$z(t)$ space, (b) $x(t)$–$z(t)$ plane. Initial condition: $(x_0, y_0, z_0) = (1, 1, 1)$.

3. Homoclinic and heteroclinic orbit

Consider the third-order autonomous system in the form of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3, \tag{22}$$

where the vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$ belongs to class $\mathrm{C}^r (r \geq 2)$.

Let $x \in \mathbb{R}^3$ be the state variable of system (22), and $t \in \mathbb{R}$ be the time. Suppose that f has at least an equilibrium point E. The point E is called a hyperbolic saddle focus for system (22), if the eigenvalues of the Jacobian $A = Df(E)$ are α and $\rho \pm \omega$, where $\rho\alpha < 0$ and $\rho \neq 0$.

A homoclinic orbit $\gamma(t)$ refers to a bounded trajectory of system (22) that is doubly asymptotic to an equilibrium point E of the system, i.e., $\lim\limits_{t\to+\infty} \gamma(t) = \lim\limits_{t\to-\infty} \gamma(t) = E$.

A heteroclinic orbit $\delta(t)$ is similarly defined except that there are two distinct saddle foci, E_1 and E_2, being connected by the orbit. One corresponds to the forward asymptotic time, and the other to the reverse asymptotic time limit. Namely, $\lim\limits_{t\to+\infty} \gamma(t) = E_1$, and $\lim\limits_{t\to-\infty} \gamma(t) = E_2$.

The heteroclinic and the homoclinic Ši'lnikov criteria for the existence of chaos, are summarized as the following two theorems.

Theorem 1[21,22] Suppose that two distinct equilibrium points, denoted by χ_{e}^1 and χ_{e}^2, of system (22) are saddle foci, whose characteristic values γ_k and $\rho_k \pm \mathrm{i}\omega_k$ $(k = 1, 2)$ satisfy the following Ši'lnikov inequalities:

$$|\gamma_k| > |\rho_k| > 0, \quad \omega \neq 0, \tag{23}$$

under constraint

$$\rho_1\rho_2 > 0 \ \text{ or } \ \gamma_1\gamma_2 > 0. \tag{24}$$

Supposing that there exists a heteroclinic orbit joining χ_{e}^1 and χ_{e}^2, we draw the following conclusions.

(i) The Ši'lnikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics.

(ii) For any sufficiently small C^1 perturbation g of f, the perturbed system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x), \quad x \in \mathbb{R}^3, \tag{25}$$

has at least a finite number of Smale horseshoes in the discrete dynamics of the Ši'lnikov map defined near the heteroclinic orbit.

(iii) Both the original system (22) and the perturbed system (25) have horseshoe-type chaos.

Theorem 2 (Homoclinic Ši'lnikov theorem[21,22]) Suppose that one equilibrium point of system (22), denoted by χ_e, is saddle focus, whose eigenvalues γ and $\rho \pm i\omega$ satisfy the following Ši'lnikov condition:

$$\gamma\rho < 0, \quad |\gamma| > |\rho| > 0, \quad \omega \neq 0. \tag{26}$$

Supposing that there exists a homoclinic orbit connecting χ_e, we draw the following conclusions.

(i) The Ši'lnikov map, defined in a neighborhood of the homoclinic orbit of the system, possesses a countable number of Smale horseshoes in its discrete dynamics.

(ii) For any sufficiently small C^1 perturbation g of f, the perturbed system

$$\frac{dx}{dt} = g(x), \quad x \in \mathbb{R}^3, \tag{27}$$

has at least a finite number of Smale horseshoes in the discrete dynamics of the Ši'lnikov map defined near the homoclinic orbit.

(iii) Both the original system (22) and the perturbed system (27) exhibit horseshoe-type chaos.

4. Existence of heteroclinic orbits in the novel system

This section will prove the existence of the heteroclinic orbit of system (1) that links equilibria E_2 and E_3.

First, it follows from (1) that

$$x = -\dot{y} - by, \tag{28}$$

$$\dot{x} = -\ddot{y} - b\dot{y}, \tag{29}$$

$$z = \frac{-\ddot{y} - b\dot{y} + ay}{y}, \tag{30}$$

$$\dot{z} = \frac{-y^{(3)}y - b\ddot{y}y + \ddot{y}\dot{y} + b\dot{y}^2}{y^2}. \tag{31}$$

Substituting these into the third equation of system (1) gives

$$\begin{aligned} &-y^{(3)}y - (b+c)\ddot{y}y + \ddot{y}\dot{y} + b\dot{y}^2 - bc\dot{y}y - \dot{y}^2y^2 \\ &+ acy^2 - 2b\dot{y}y^3 - (b^2+d)y^4 = 0. \end{aligned} \tag{32}$$

If $y(t)$ is found, then $x(t)$ and $z(t)$ will also be determined. Therefore, finding the heteroclinic orbit of system (1) is now changed to seek a function $\psi(t)$ such that $y(t) = \psi(t)$ satisfies (32) and

$$\psi(t) \to -\sqrt{\frac{ac}{b^2+d}}, \quad \text{as } t \to +\infty, \tag{33}$$

$$\psi(t) \to \sqrt{\frac{ac}{b^2+d}}, \quad \text{as } t \to -\infty, \tag{34}$$

or

$$\psi(t) \to \sqrt{\frac{ac}{b^2+d}}, \quad \text{as } t \to +\infty, \tag{35}$$

$$\psi(t) \to -\sqrt{\frac{ac}{b^2+d}}, \quad \text{as } t \to -\infty. \tag{36}$$

Without loss of generality, one may stipulate a definite direction as follows: from E_2 to E_3 corresponds to $t \to +\infty$, while from E_3 to E_2 to $t \to -\infty$.

Next, suppose that, for $t > 0$,

$$\begin{aligned} y(t) = \psi(t) &= -\delta + \sum_{k=1}^{\infty} b_k e^{k\beta t}, \\ \delta &= \sqrt{\frac{ac}{b^2+d}}, \quad t > 0, \end{aligned} \tag{37}$$

where $\beta < 0$ is an undetermined constant, and $\{b_k\}$ $(k \geq 1)$ are undetermined coefficients.

Substituting Eq. (37) into Eq. (32) yields

$$\sum_{k=1}^{\infty} \delta G(k\beta) b_k e^{k\beta t} = H_2 + H_3 + H_4, \tag{38}$$

where

$$\begin{aligned} H_2 = &\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}\{[l^3 - l^2(k-l)]\beta^3 \\ &+ [(b+c)l^2 + (\delta^2 - b)l(k-l)]\beta^2 \\ &+ b(c+6\delta^2)l\beta + 5ac\} b_l b_{k-l} e^{k\beta t}, \end{aligned} \tag{39}$$

$$\begin{aligned} H_3 = &\sum_{k=3}^{\infty}\sum_{l=2}^{k-1}\sum_{m=1}^{l-1}\{-\delta[(2(l-m)\beta + 6b)m\beta \\ &+ 4(b^2+d)]\} b_m b_{l-m} b_{k-l} e^{k\beta t}, \end{aligned} \tag{40}$$

$$\begin{aligned} H_4 = &\sum_{k=4}^{\infty}\sum_{l=3}^{k-1}\sum_{m=2}^{l-1}\sum_{n=1}^{m-1}[n(m-n)\beta^2 + 2bn\beta \\ &+ (b^2+d)] b_n b_{m-n} b_{l-m} b_{k-l} e^{k\beta t}, \end{aligned} \tag{41}$$

and

$$G(k\beta) = (k\beta)^3 + (b+c)(k\beta)^2 + b(c+2\delta^2)(k\beta) + 2ac. \tag{42}$$

Assume that $b_1 \neq 0$, otherwise one can inductively have $b_k = 0$ for all $k > 1$.

Consequently, b_k is completely determined by β, a, b, c, d, and b_1. Thus,

$$b_k = \psi_k b_1^k, \quad k > 1, \tag{43}$$

where ψ_k $(k > 1)$ are some known functions depending on β, a, b, c, and d.

Now, comparing the coefficients of $e^{k\beta t}(k \geq 1)$ of the same power terms, one can obtain

$$\beta^3 + (b+c)\beta^2 + b(c+2\delta^2)\beta + 2ac = 0, \text{ as } k = 1, \tag{44}$$

which is just the characteristic polynomial of the Jacobian of the linearized equation of system (1) evaluated at the equilibrium point E_2 or E_3. Since Eq. (8) has the unique negative root for given parameters, there exists $\beta < 0$ such that $G(\beta) = 0$. For $k > 1$,

$$G(k\beta) = (k\beta)^3 + (b+c)(k\beta)^2 + b(c+2\delta^2)(k\beta) + 2ac \neq 0. \quad (45)$$

For $k = 2$,

$$b_2 = \frac{H_2^{\mathrm{II}} b_1^2}{\delta G(2\beta)}, \quad (46)$$

where

$$H_2^{\mathrm{II}} = (c+\delta^2)\beta^2 + b(c+6\delta^2)\beta + 5ac. \quad (47)$$

For $k = 3$,

$$b_3 = \frac{H_2^{\mathrm{III}} + H_3^{\mathrm{III}}}{\delta G(3\beta)}, \quad (48)$$

where

$$H_2^{\mathrm{III}} = \sum_{l=1}^{2} \{[l^3 - l^2(3-l)]\beta^3 + [(b+c)l^2 + (\delta^2 - b)l(3-l)]\beta^2 + b(c+6\delta^2)l\beta + 5ac\} b_l b_{3-l}, \quad (49)$$

$$H_3^{\mathrm{III}} = -\delta[(2\beta + 6b)\beta - 4(b^2+d)]. \quad (50)$$

Finally, for $k \geq 4$,

$$b_k = \frac{H_2^{\mathrm{K}} + H_3^{\mathrm{K}} + H_4^{\mathrm{K}}}{\delta G(k\beta)}, \quad (51)$$

where

$$H_2^{\mathrm{K}} = \sum_{l=1}^{k-1} \{[l^3 - l^2(k-l)]\beta^3 + [(b+c)l^2 + (\delta^2 - b)l(k-l)]\beta^2 + b(c+6\delta^2)l\beta + 5ac\} b_l b_{k-l}, \quad (52)$$

$$H_3^{\mathrm{K}} = \sum_{l=2}^{k-1} \sum_{m=1}^{l-1} -\delta[(2(l-m)\beta + 6b)m\beta + 4(b^2+d)] b_m b_{l-m} b_{k-l}, \quad (53)$$

$$H_4^{\mathrm{K}} = \sum_{l=3}^{k-1} \sum_{m=2}^{l-1} \sum_{n=1}^{m-1} [n(m-n)\beta^2 + 2bn\beta + (b^2+d)] b_n b_{m-n} b_{l-m} b_{k-l}. \quad (54)$$

As a result, β is completely determined by a, b, c, and d, and β_k $(k \geq 2)$ is completely determined by a, b, c, d, and β.

In fact, the first part of the heteroclinic orbit corresponding to $t > 0$ has been determined. Next, the second part corresponding to $t < 0$ will be constructed. Due to the symmetry of system (1) around the z axis and $((x(t), y(t), z(t))$ is a solution, then $(-x(t), -y(t), z(t))$ is also a solution for system (1). Thus,

$$x(t) = \delta - \sum_{k=1}^{\infty} b_k \mathrm{e}^{-k\beta t}, \quad \delta = \sqrt{\frac{ac}{b^2+d}}, \quad t < 0. \quad (55)$$

Then, it follows that system (1) has a heteroclinic orbit which connects the equilibrium points E_3 and E_2, and takes the following form:

$$y(t) = \psi(t) = \begin{cases} -\delta + \sum\limits_{k=1}^{\infty} b_k \mathrm{e}^{k\beta t}, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \\ \delta - \sum\limits_{k=1}^{\infty} b_k \mathrm{e}^{-k\beta t}, & \text{for } t < 0. \end{cases} \quad (56)$$

From the continuity of the solution, one has

$$\sum_{k=1}^{\infty} b_k = \delta, \quad (57)$$

which will determine the value of b_1.

5. Uniform convergence of series expansion of heteroclinic orbits

This section investigates the uniform convergence of the series expansion of the heteroclinic orbit. For simplicity, one may only consider the case in which system (1) has the special parameter set that generates attractors. For other parameter sets, the proof is similar if the heteroclinic orbit exists.

For system (1) with parameter $a = 11$, $b = -0.5$, $c = 8$, $d = 4$, and $\delta = \sqrt{(ac)/(b^2+d)} = 4.55037167195617$, the values of β and b_k can be determined by Eqs. (44), (46), (48), and (51) as $|b_2| = 0.15763|b_1^2|$, $|b_3| = 0.0252|b_1^3|$, and $|b_4| = 0.0034325|b_1^4|$. One can inductively prove that $|b_k| < 4^{-k}|b_1^k|$ numerically $(k \geq 4)$. Thus, b_k is bounded, that is, there exists $M > 0$ such that $|b_k| \leq M$ $(k = 1, 2, \ldots)$.

Consequently, $\sum\limits_{k=1}^{\infty} |b_k \mathrm{e}^{k\beta t}| \leq M \sum\limits_{k=1}^{\infty} \mathrm{e}^{k\beta t}$ is convergent in $(0, +\infty)$. Thus, $-\delta + \sum\limits_{k=1}^{\infty} b_k \mathrm{e}^{k\beta t}$ is convergent in $(0, +\infty)$. Similarly, the convergence of $\delta - \sum\limits_{k=1}^{\infty} b_k \mathrm{e}^{k\beta t}$ in $(-\infty, 0)$ can also be proved.

Actually, b_1 can be derived from $\sum\limits_{k=1}^{\infty} b_k \mathrm{e}^{k\beta t} = \delta = 4.55037167195617$. Numerical simulation shows that a "stable" b_1 indeed exists near 4.662193368038 with relative error no greater than 1%.

In summary, one has the following theorem.

Theorem 3 System (1) has one Šilinikov heteroclinic orbit with one component in the form of (56), and the corresponding chaos is of horseshoe type. Given that for the typical parameter set: $a = 11$, $b = -0.5$, $c = 8$, $d = 4$, the conditions

$(k\beta)^3+(b+c)(k\beta)^2+b(c+2\delta^2)(k\beta)+2ac\neq 0$ are always satisfied, so there are heteroclinic orbits of Ši'lnikov type, and as a result, a countable number of Smale horseshoes exist. Therefore, there exists an invariant set constituting the complex attractor. That is the essence of the geometric structure of the attractor.

6. Summary

This paper describes a novel three-order autonomous differential equation system and investigates the existence of Ši'lnikov chaos of the system. By using the undetermined coefficient method, we have found the heteroclinic orbit of the system, with an explicit and uniformly convergent algebraic expression. According to the Ši'lnikov criterion, it has Smale horseshoes and horseshoe-type chaos. Our results suggest that combining numerical calculations and theoretical analysis will enable a better understanding of nonlinear dynamic systems. Finally, it is worth mentioning that the undetermined coefficient method is a powerful tool to confirm the existence of Ši'lnikov chaos.

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