Invariant expression for tunneling rate under canonical transformations*

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The original formula to calculate the tunneling rate through event horizons is apparently dependent on the type of coordinates used. In this paper, we propose an invariant expression under canonical transformations to study the tunneling effect. Moreover, the problem of factor 2 is solved naturally. As an application of this expression, we obtain the same tunneling rate both in the Schwarzschild and the Painlevé coordinates. It is shown that once the suitable formula to calculate tunneling rate is correctly identified, the tunneling method is manifestly covariant.

Keywords: Hawking temperature, tunneling method, invariant expression of tunneling rate

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1. Introduction

The discovery of thermal radiation of the black holes was a milestone in the development of black hole physics.[1] It, however, gave rise to a famous paradox, the information loss paradox of black hole physics at the same time.[2] The seriousness of this paradox is that the information loss contradicts one of the basic principles of quantum mechanics–unitary evolution. An initially pure quantum state, by collapsing to a black hole and then evaporating completely, has evolved into a mixed state. However, according to unitary principle, a pure state should always stay pure. Hawking once said: "It seemed the only way the information could come out would be if the radiation was not exactly thermal but had subtle correlations."

A method to describe Hawking radiation as a tunneling process where a particle moves into dynamic geometry was developed by Kraus and Wilczek[3] and elaborated upon by Parikh and Wilczek.[4–6] They took the self-gravitation and energy conservation into account and gave a leading correction to traditional emission rate, satisfying information conservation. After initially dealing with massless particles tunneling out of the Schwarzschild black holes, this method has been applied successfully to a wide variety of space-times.[7–21] Then the massless particle tunneling was extended to massive particle tunneling[22,23] and charged particle tunneling.[24,25] Recently, the fermion and gravitino tunnelings were calculated by Hamilton–Jacobi equations.[26–29]

Although this semi-classical method of modeling Hawking radiation as a tunneling effect has gained a lot of interest and has been greatly developed, the commonly used expression[4–6] to compute the tunneling probability

$$\Gamma \sim \exp(-2\mathrm{Im}S) = \exp\left(-2\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r \mathrm{d}r\right) \tag{1}$$

is apparently dependent on the type of coordinates used (e.g., Schwarzschild, Painlevé, isotropic),[30] where S is the action for the trajectory, p_r is canonical momentum conjugate to r, $r_{\mathrm{in}} = 2m$, which is the initial radius of the black hole, and $r_{\mathrm{out}} = 2(m-E)$ is the final radius, where E is the energy of the tunneling particle.

Chowdhury[31] once proposed a transmission coefficient of tunneling that did not depend on the canonical frame chosen and is expressed as

$$\begin{aligned}\Gamma &\sim \exp\left(-\mathrm{Im}\oint p_r \mathrm{d}r\right)\\ &= \exp\left[-\mathrm{Im}\left\{\int_{\mathrm{in}}^{\mathrm{out}} (p_r)_{\mathrm{out}}\mathrm{d}r + \int_{\mathrm{out}}^{\mathrm{in}} (p_r)_{\mathrm{in}}\mathrm{d}r\right\}\right],\end{aligned} \tag{2}$$

where $(p_r)_{\mathrm{out}}$ and $(p_r)_{\mathrm{in}}$ are the radial momenta of the outgoing and incoming particles, respectively. Some authors studied the tunneling effect by adopting Eq. (2).[30–33] However, the invariant expression, $\mathrm{Im}\oint p_r \mathrm{d}r$, gives half the correct result, which is called the problem of factor 2. The temporal contribution to the imaginary action coming from the time part is considered in order to solve this problem in Refs. [34]–[36], but this method is unnatural. In our opinion, the cause of the trouble is that the formula of the action in Eq. (1), $S = \int p_r \mathrm{d}r$, is not invariant itself.

Our aim is to solve the above problems. We propose a suitable formula which is invariant under canonical transformations to calculate tunneling rate. The results are in agreement with the standard results. Throughout the paper, the units $G = c = \hbar = 1$ are used.

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2. Invariant expression for tunneling rate

The line element of the Schwarzschild black hole is

$$\mathrm{d}s^2 = -\Big(1-\frac{2m}{r}\Big)\mathrm{d}s^2 + \Big(1-\frac{2m}{r}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \quad (3)$$

In the Schwarzschild coordinate, the radial null geodesic satisfies $\mathrm{d}t/\mathrm{d}r = \pm r/(r-2m)$. The momentum p_r in Eq. (1) can be eliminated by using Hamilton's equation

$$\frac{\mathrm{d}H}{\mathrm{d}p_r} = \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}. \quad (4)$$

The imaginary part of the action is therefore

$$\mathrm{Im}S = \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int \frac{\mathrm{d}H}{\mathrm{d}r/\mathrm{d}t}\mathrm{d}r. \quad (5)$$

In Ref. [4], the Hamiltonian H is interpreted as Arnowitt–Deser–Misner (ADM) mass. The total mass is fixed and the hole mass is allowed to fluctuate, $\mathrm{d}H = \mathrm{d}(m-E) = -\mathrm{d}E$. In addition, we should replace m with $m-E$ in the geodesic equation and in the line element equation, when the particle of energy E travels on the geodesics. Therefore, equation (5) becomes

$$\begin{aligned}\mathrm{Im}S &= -\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r\mathrm{d}E' \\ &= -\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{r\mathrm{d}r\mathrm{d}E'}{r-2(m-E')}. \end{aligned} \quad (6)$$

There is a pole at the horizon in the above equation. Using the Feynman prescription to displace the energy from E' to $E'-\mathrm{i}\varepsilon$, we obtain

$$\mathrm{Im}S = -\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{r\mathrm{d}r\mathrm{d}E'}{r-2(m-E'+\mathrm{i}\varepsilon)}. \quad (7)$$

We compute this integral and have

$$\mathrm{Im}S = 2\pi mE\Big(1-\frac{E}{2m}\Big). \quad (8)$$

This result is half of the correct one calculated in the Painlevé coordinates.

This difference, however, is not due to the bad behavior at the event horizon in the standard Schwarzschild coordinates. We can choose the isotropic coordinates where the line element is not singular at the horizon. The result is $\mathrm{Im}\int p_r\mathrm{d}r = 2\pi mE[1-(E/2m)]$, which is also half of the correct result.[30]

It is noted that the formula of the action in Eq. (1) itself, $S = \int p_r\mathrm{d}r$, is not invariant. Stodolsky[37] proposed that the invariant phase of a quantum mechanical particle in curved space-time traveling from A to B is

$$S = \int_A^B p_\mu\,\mathrm{d}x^\mu, \quad (9)$$

where p_μ is the four momenta of the particle. Equation (9) is commonly used for calculating the interference phase of neutrino oscillation in curved space-time.[38–45] In fact, S in the expression, $\Gamma \sim \exp(-2\mathrm{Im}S)$, is also the quantum mechanical phase of a particle. Therefore, equation (9) can be used to study the tunneling effect. In the Schwarzschild space-time, when the radial motion of a particle with energy E is considered only, the imaginary part of the action can be written as

$$\begin{aligned}\mathrm{Im}S &= \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_\mu\,\mathrm{d}x^\mu = \mathrm{Im}\Big(-\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} E\,\mathrm{d}t + \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r\mathrm{d}r\Big) \\ &= \mathrm{Im}\Big(-\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} E\frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r + \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int \mathrm{d}p_r'\mathrm{d}r\Big). \end{aligned} \quad (10)$$

Using Hamilton's equation (4), we have

$$\begin{aligned}\mathrm{Im}S &= \mathrm{Im}\Big[-\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r\mathrm{d}E' - \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r\mathrm{d}E'\Big] \\ &= -2\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r\mathrm{d}E' = 4\pi mE\Big(1-\frac{E}{2m}\Big). \end{aligned} \quad (11)$$

Therefore, we obtain the correct tunneling rate

$$\begin{aligned}\Gamma \sim \exp(-2\mathrm{Im}S) &= \exp\Big[-8\pi mE\Big(1-\frac{E}{2m}\Big)\Big] \\ &= \exp(\Delta S_{\mathrm{BH}}). \end{aligned} \quad (12)$$

According to the coordinate transformation

$$t_{\mathrm{p}} = t + 2\sqrt{2mr} + 2m\ln\frac{\sqrt{r}-\sqrt{2m}}{\sqrt{r}+\sqrt{2m}}, \quad (13)$$

the line element in the Painlevé coordinates can be obtained as

$$\mathrm{d}s^2 = -\Big(1-\frac{2m}{r}\Big)\mathrm{d}t_{\mathrm{p}}^2 + \mathrm{d}r^2 + 2\sqrt{\frac{2m}{r}}\mathrm{d}t_{\mathrm{p}}\mathrm{d}r + r^2\mathrm{d}\Omega^2. \quad (14)$$

From $\mathrm{d}s^2 = 0$, $\mathrm{d}r/\mathrm{d}t_{\mathrm{p}}$ along the radial geodesics is $\mathrm{d}r/\mathrm{d}t_{\mathrm{p}} = \pm 1 - \sqrt{(2m/r)}$. The imaginary part of the action in Painlevé coordinates is

$$\begin{aligned}\mathrm{Im}S &= \mathrm{Im}\Big[-\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} E\frac{\mathrm{d}t_{\mathrm{p}}}{\mathrm{d}r}\mathrm{d}r + \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\frac{\mathrm{d}H}{\dot{r}}\mathrm{d}r\Big]_{\mathrm{out}} \\ &= \mathrm{Im}\Big(-2\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{\mathrm{d}t_{\mathrm{p}}}{\mathrm{d}r}\mathrm{d}r\mathrm{d}E'\Big) = 8\pi E\Big(m-\frac{E}{2}\Big). \end{aligned} \quad (15)$$

By substituting Eq. (15) into the formula of tunneling rate, $\Gamma \sim \exp(-2\mathrm{Im}S)$, the wrong result is obtained.

According to the path integral, the probability amplitude of a particle traveling a path from a to b is

$$\phi_{ab} \sim \mathrm{e}^{\mathrm{i}S_{ab}}, \quad (16)$$

where $S_{ab} = \int_a^b L\mathrm{d}t$ is the classical action and L is the Lagrangian of a particle. The probability is given by

$$\Gamma \sim \phi_{ab}\phi_{ab}^*. \quad (17)$$

From the quantum mechanics in flat space-time, $\phi_{ab} = \langle b|a\rangle$ and $\phi_{ba} = \langle a|b\rangle$, we have $\phi_{ab}^* = \phi_{ba}$. For the process of tunneling the barrier, S_{ab} can be divided into real part and imaginary part, $S_{ab} = \mathrm{Re}S_{ab} + \mathrm{i}\mathrm{Im}S_{ab}$, and equation (17) can be written as

$$\Gamma \sim \phi_{ab}\phi_{ba} \sim \mathrm{e}^{\mathrm{i}S_{ab}}\mathrm{e}^{\mathrm{i}S_{ba}} = \mathrm{e}^{\mathrm{i}\mathrm{Re}S_{ab}-\mathrm{Im}S_{ab}}\mathrm{e}^{\mathrm{i}\mathrm{Re}S_{ba}-\mathrm{Im}S_{ba}}$$

$$= \mathrm{e}^{\mathrm{i}(\mathrm{Re}S_{ab}+\mathrm{Re}S_{ba})}\,\mathrm{e}^{-\mathrm{Im}(S_{ab}+S_{ba})}. \quad (18)$$

Since the tunneling probability must be real, $\mathrm{Re}S_{ab}+\mathrm{Re}S_{ba}=0$, and tunneling rate is

$$\Gamma \sim \mathrm{e}^{-\mathrm{Im}(S_{ab}+S_{ba})}. \quad (19)$$

From $\phi^*_{ab}=\phi_{ba}$, we can obtain $\mathrm{Im}S_{ab}=\mathrm{Im}S_{ba}$. Hence, the familiar formula, $\Gamma \sim \mathrm{e}^{-2\mathrm{Im}S_{ab}}$, is obtained. Thus, $\Gamma \sim \mathrm{e}^{-\mathrm{Im}(S_{ab}+S_{ba})}$ is equivalent to $\Gamma \sim \mathrm{e}^{-2\mathrm{Im}S_{ab}}$, from elementary quantum mechanics in flat space-time. However, the two formulae are very different when tunneling effect is studied in black hole physics.

Next, let a be r_{in} and b be r_{out}. The imaginary part of the action from r_{in} to r_{out} in Schwarzschild space-time is Eq. (11), $\mathrm{Im}S_{\mathrm{out}}=4\pi mE[1-E/(2m)]$. The imaginary part of the action from r_{out} to r_{in} is

$$\begin{aligned}\mathrm{Im}S_{\mathrm{in}} &= \mathrm{Im}\int_{\mathrm{out}}^{\mathrm{in}}(p_\mu)_{\mathrm{in}}\,\mathrm{d}x^\mu = -\mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}}(p_\mu)_{\mathrm{in}}\,\mathrm{d}x^\mu \\ &= \mathrm{Im}\left(2\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r\,\mathrm{d}E'\right)_{\mathrm{in}}, \end{aligned} \quad (20)$$

where $(p_\mu)_{\mathrm{in}}$ is the four momenta of the incoming particle. Substituting $\mathrm{d}r/\mathrm{d}t=-[1-2(m-E)/r]$ for incoming particles into Eq. (20), we can obtain

$$\mathrm{Im}S_{\mathrm{in}} = 4\pi mE\left(1-\frac{E}{2m}\right). \quad (21)$$

Therefore, $\mathrm{Im}S_{\mathrm{in}}$ is equal to $\mathrm{Im}S_{\mathrm{out}}$ in Schwarzschild coordinates. From Eq. (19), we can obtain the tunneling rate Eq. (12) which is in agreement with the standard result.

We substitute $\mathrm{d}r/\mathrm{d}t_{\mathrm{p}}=-1-\sqrt{2m/r}$ into Eq. (20) to calculate $\mathrm{Im}S_{\mathrm{in}}$ in the Painlevé coordinates

$$\mathrm{Im}S_{\mathrm{in}} = 2\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^E \frac{\mathrm{d}r\,\mathrm{d}E'}{-1-\sqrt{2(m-E')/r}}. \quad (22)$$

There is not a pole at the horizon. Henceforth, S_{in} is real and its imaginary part vanishes. Substituting $\mathrm{Im}S_{\mathrm{out}}$ (Eq. (15)) and $\mathrm{Im}S_{\mathrm{in}}$ (Eq. (22)) into Eq. (19), the correct tunneling rate in the Painlevé coordinates is given.

From the calculation of the tunneling rate in the Schwarzschild and the Painlevé coordinates, we can conclude that the quantity, $\mathrm{Im}S_{\mathrm{out}}+\mathrm{Im}S_{\mathrm{in}}$, does not depend on the type of coordinates used. Therefore

$$\begin{aligned}&\Gamma \sim \exp[-(\mathrm{Im}S_{\mathrm{out}}+\mathrm{Im}S_{\mathrm{in}})] \\ &= \exp\left[-\left(\mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}}(p_\mu)_{\mathrm{out}}\,\mathrm{d}x^\mu + \mathrm{Im}\int_{\mathrm{out}}^{\mathrm{in}}(p_\mu)_{\mathrm{in}}\,\mathrm{d}x^\mu\right)\right] \end{aligned} \quad (23)$$

can be regarded as an invariant expression for the tunneling rate under canonical transformations. Equation (23) can be rewritten as

$$\Gamma \sim \exp\left(-\mathrm{Im}\oint p_\mu\,\mathrm{d}x^\mu\right). \quad (24)$$

3. Discussion and conclusion

Tunneling is the system penetrating a barrier quantum mechanically, which is impossible classically. When $\mathrm{Im}S_{\mathrm{out}}$ is equal to $\mathrm{Im}S_{\mathrm{in}}$, the two expressions of tunneling rate, $\Gamma \sim \exp(-2\mathrm{Im}S_{\mathrm{out}})$ and $\Gamma \sim \exp[-(\mathrm{Im}S_{\mathrm{out}}+\mathrm{Im}S_{\mathrm{in}})]$, are equivalent. However, they are different for the particles tunneling out of black holes. Only the latter can be extended to the study of tunneling effect from the black holes. The reason is that the ingoing particles face no barrier at all. In the Painlevé coordinates the line element is regular at the horizon and there is not a pole at the horizon. Therefore, $\mathrm{Im}S_{\mathrm{in}}$ is equal to zero. In the Schwarzschild coordinates, the coordinate singularity exists at the event horizon. The coordinate singularity just leads to a pole in the calculation of $\mathrm{Im}S_{\mathrm{in}}$ in Eq. (20) and $\mathrm{Im}S_{\mathrm{out}}=\mathrm{Im}S_{\mathrm{in}}$. By adopting Eq. (23), we obtain the same tunneling rate in both the Schwarzschild coordinates and the Painlevé coordinates. Therefore, the expression (23) is invariant for studying tunneling effect of black holes.

If we disregard the quadratic term in Eq. (12), the tunneling rate reduces to $\Gamma \sim \exp(-8\pi mE)$. The Hawking temperature is therefore $T=1/8\pi m$. We show that the temperature of the black holes originally calculated by Hawking is undoubtedly right and the incorrect results (the problem of factor of 2) obtained previously were attributed to the fact that the suitable formula had not been used to calculate tunneling rate. The value of the black hole radiation temperature is very important for quantum gravity calculations which try to reproduce the black hole entropy by counting microscopic states.

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