Minimum detection efficiency for the loophole-free confirmation of quantum contextuality*

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Klyachko–Can–Binicioğlu–Shumovsky (KCBS) inequality is a Bell-like inequality, the violation of which can be used to confirm the existence of quantum contextuality. However, the imperfection of detection efficiency may cause the so-called loophole in actual KCBS's experiments. We derive an alternative KCBS inequality to deal with the loophole in actual KCBS's experiments. We prove that if the experimental data violate this KCBS inequality, the loophole-free violation of the original KCBS inequality will occur. We show that the minimum detection efficiency needed for a loophole-free violation of the KCBS inequality is about 0.9738.

Keywords: Klyachko–Can–Binicioğlu–Shumovsky (KCBS) inequality, quantum contextuality

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1. Introduction

The classical physics[1,2] states that physical systems can be described by local objective properties (physical reality) which are independent of observation. The classical correlations rely on two basic mechanisms. Either the classical correlated events share a common cause or one event influences the other. However, the quantum theory opens a new world of non-classical correlations. Quantum nonlocality is a property that the local measurements are dependent on other spacelike separated measurements; quantum contextuality is a property that the measurements are dependent on other compatible measurements. Quantum contextuality can be seen as a natural generalization of quantum nonlocality since it does not privilege spacelike separated measurements. Recently, there have been theoretical results about quantum contextuality, such as quantum contextuality of relativistic effects,[3] quantum contextuality in continuous-variable states,[4] and the state-independent quantum contextuality.[5,6]

Bell inequalities[7–9] and noncontextuality inequalities[1,10–13] are the basic tools for characterizing quantum nonlocality and quantum contextuality, respectively. Since then, the experimental confirmation of quantum nonlocality and quantum contextuality has become one of the fundamental challenges of modern physics.[14–25] So far, however, the experiments that have been performed to verify quantum nonlocality and quantum contextuality are not completely conclusive. Take quantum nonlocality for example. Any incontrovertible experimental confirmations must simultaneously satisfy the following three conditions: (i) observers' measurement choices are random, i.e., they are not correlated with each other or hidden variable;[26–28] (2) measurements of different observers are spacelike separated; (3) the detected data are representative of all those emitted from the source. Any dissatisfaction of the above three conditions means that there are possibly loopholes in an experimental confirmation of quantum nonlocality. For the experimental confirmation of quantum contextuality, the third condition is especially important. Since almost all experiment tests of quantum contextuality have nothing to do with the first two conditions, the loopholes of these experiment tests can only be caused by imperfection of the detector's efficiency. Thus, if the third condition is unsatisfactory and at the same time the detection efficiency is too low, the so-called detection loophole arises. The present work focuses on the issue of how to achieve a loophole-free confirmation of quantum contextuality, and at the same time offer the minimum detection efficiency for the loophole-free confirmation.

The detection efficiency is defined as the ratio between the numbers of detected particles and actually emitted particles by source. An interesting question is what the minimum detection efficiency is for a loophole-free confirmation of quantum contextuality. In fact, the detection loophole exists in many quantum contextuality experiments. One usually adopts photon experiments for characterizing quantum contextuality. However, the optical detection efficiencies are still too low to close the detection loophole. Since the detection efficiency is the product of the transmission efficiency and the detector efficiency, the loss of photons in the transmission from

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the source to the observer's location can drastically reduce the detection efficiency. Thus, it is very meaningful to find a theoretical protocol to close the detection loophole and calculate the minimum detection efficiency needed for a loophole-free confirmation.

Klyachko–Can–Binicioğlu–Shumovsky (KCBS) inequality[1] is the simplest noncontextual inequality, which can be violated by a qutrit, namely, the simplest physical system violating a noncontextual inequality. Furthermore, KCBS inequality is fundamentally important in quantum theory, since the quantum violation of other noncontextual inequalities can be traced back to the quantum violation of it.[29] We derive an alternative KCBS inequality to deal with the loophole in actual KCBS's experiments. We prove that if the experimental data can violate this KCBS inequality, it must result in the loophole-free violation of the original KCBS inequality. Then, we show that the minimum detection efficiency needed for a loophole-free violation of KCBS inequality is about 0.9738.

2. Minimum detection efficiency for the loophole-free violation of KCBS inequality

Noncontextual hidden variable theory states that the measurement results are irrelevant to other compatible measurements. Noncontextual inequalities are restrictions that must be satisfied by any noncontextual hidden variable theory. The KCBS inequality is the simplest and the most important noncontextual inequality. Its quantum violation has been recently reported in a test with photons.[20] Consider measurements denoted as A_i with outcomes $a_i = 0, 1$ $(i = 0, 1, 2, 3, 4)$. All measurements A_i are compatible with A_{i+1} (the sum modulo is 5), and their measurement results are exclusive, i.e., there is always $a_i a_{i+1} = 0$. The measurements A_i and A_{i+1} are just analogue to mutual orthogonal projection operators in quantum theory. Then, the KCBS inequality can be expressed as

$$\sum_{i=0}^{4} P(A_i = 0, A_{i+1} = 1) \leq 2, \tag{1}$$

where $P(A_i = 0, A_{i+1} = 1)$ represents the probability of obtaining the result 0 in the measurement A_i, and the result 1 in the compatible measurement A_{i+1}. Since these two measurements are compatible, the order of performing them has no effect on results. The events $(A_i = 0, A_{i+1} = 1)$ and $(A_{i+1} = 0, A_{i+2} = 1)$ are exclusive (the sum modulo is 5), i.e., they cannot be true at the same time. In consequence, according to noncontextual hidden variable theory, at most two events of $(A_i = 0, A_{i+1} = 1)$ can be simultaneously true. This is the reason why the bound 2 arises.

The KCBS inequality of Eq. (1) applies to the ideal case in which all measurements A_i would give results, and only in the ideal case one can assert that the violation of Eq. (1) confirms the existence of quantum contextuality. Therefore, in order to obtain a loophole-free confirmation of quantum contextuality, we essentially should obtain a violation of the following inequality:

$$\sum_{i=0}^{4} P(A_i = 0, A_{i+1} = 1 | \Lambda_0) \leq 2, \tag{2}$$

where Λ_0 represents the ensemble where all measurements successfully give results, and $P(A_i = 0, A_{i+1} = 1 | \Lambda_0)$ represent the probabilities of the measurement result event $(A_i = 0, A_{i+1} = 1)$ in the ensemble Λ_0. The violation of Eq. (2) is a (detection) loophole-free confirmation of existence of quantum contextuality.

However, $P(A_i = 0, A_{i+1} = 1 | \Lambda_0)$ are inaccessible in actual experiments. The usual approach is that those "not detected" events should be disregarded and only the coincident events contribute to the estimation of $P(A_i = 0, A_{i+1} = 1)$. A coincident event means that the measurements A_i and A_{i+1} both successfully give a result in one trial. Thus, in actual experiments, one essentially calculates the conditional probabilities $P(A_i = 0, A_{i+1} = 1 | \Lambda_{A_i A_{i+1}})$, where $\Lambda_{A_i A_{i+1}}$ represents the ensemble where all measurements A_i and A_{i+1} successfully give results 0 or 1.

Here, for convenience we define all notations used in the following text. Λ_{A_i} denotes the ensemble where measurement A_i successfully gives result 0 or 1; $\Lambda_{A_i A_j}$ denotes the ensemble where both measurement A_i and A_j successfully give results 0 or 1, similar to $\Lambda_{A_i A_j A_k}$, etc. According to the above notations, the ensemble Λ_0 essentially can be expressed as $\Lambda_{A_0 A_1 A_2 A_3 A_4}$. $P(\Lambda_{A_i})$ denotes the probability that the measurement A_i successfully gives a result, and $P(\Lambda_{A_i A_j})$ denotes the possibility that the measurements A_i and A_j both successfully give a result, similar to $P(\Lambda_{A_i A_j A_k})$, etc. $P(\Lambda_{A_i} | \Lambda_{A_j})$ denotes the conditional probability that A_i successfully gives a result, given that A_j has successfully given a result.

Lemma 1 If we define

$$\delta = \min_i P(\Lambda_0 | \Lambda_{A_i A_{i+1}}), \tag{3}$$

where $P(\Lambda_0 | \Lambda_{A_i A_{i+1}})$ is the conditional probability that all measurements successfully give results under the condition that measurements A_i and A_{i+1} both successfully give results, and $\min_i$ is taken over all five measurement settings. From Eqs. (2) and (3), we can obtain

$$\sum_{i=0}^{4} P(A_i = 0, A_{i+1} = 1 | \Lambda_{A_i A_{i+1}}) \leq 2 + 3(1 - \delta). \tag{4}$$

Proof It is obvious that $\Lambda_0 \subset \Lambda_{A_i A_{i+1}}$, the ensemble $\Lambda_{A_i A_{i+1}}$ can be split into two disjointed sub-ensembles Λ_0 and its complement $\Lambda_0^{\mathrm{c}} = \Lambda_{A_i A_{i+1}} \backslash \Lambda_0$. We can obtain

$$\begin{aligned} &P(A_i = 0, A_{i+1} = 1 | \Lambda_{A_i A_{i+1}}) - \delta P(A_i = 0, A_{i+1} = 1 | \Lambda_0) \\ &= P(\Lambda_0^{\mathrm{c}} | \Lambda_{A_i A_{i+1}}) P(A_i = 0, A_{i+1} = 1 | \Lambda_0^{\mathrm{c}}) \end{aligned}$$

$$
\begin{aligned}
&+P(\Lambda_0|\Lambda_{A_iA_{i+1}})P(A_i=0,A_{i+1}=1|\Lambda_0)\\
&-\delta P(A_i=0,A_{i+1}=1|\Lambda_0)\\
\le\ & P(\Lambda_0^{\rm c}|\Lambda_{A_iA_{i+1}})+\left[P(\Lambda_0|\Lambda_{A_iA_{i+1}})-\delta\right]\\
=\ & 1-\delta. \qquad (5)
\end{aligned}
$$

Combining Eqs. (2) and (5), we obtain

$$
\begin{aligned}
&\sum_{i=0}^{4}P(A_i=0,A_{i+1}=1|\Lambda_{A_iA_{i+1}})\\
\le\ & \delta\sum_{i=0}^{4}P(A_i=0,A_{i+1}=1|\Lambda_0)+5(1-\delta)\\
\le\ & 2\delta+5(1-\delta)\\
=\ & 2+3(1-\delta). \qquad (6)
\end{aligned}
$$

In order to obtain the minimum detection efficiency required for loophole-free violation of KCBS inequality, we should first get the relation between δ and the detection efficiency. For simplicity, we assume that the detection efficiencies of all five measurements A_i are equal and independent of each other. This means that $P(\Lambda_{A_i})=P(\Lambda_{A_j})=P(\Lambda_{A_i}|\Lambda_{A_j})=\eta$ for $i\neq j$.

Lemma 2 $\delta\ge 3\eta-2$.

Proof As mentioned above, the ensemble Λ_0 essentially can be expressed as $\Lambda_{A_0A_1A_2A_3A_4}$, so $P(\Lambda_0|\Lambda_{A_iA_{i+1}})=P(\Lambda_{A_0A_1A_2A_3A_4}|\Lambda_{A_iA_{i+1}})$. Now we calculate $P(\Lambda_{A_0A_1A_2A_3A_4}|\Lambda_{A_0A_1})$, the calculations of other $P(\Lambda_{A_0A_1A_2A_3A_4}|\Lambda_{A_iA_{i+1}})$ are the same.

$$
\begin{aligned}
&P(\Lambda_{A_0A_1A_2A_3A_4}|\Lambda_{A_0A_1})\\
=\ & P(\Lambda_{A_0A_1}\cap\Lambda_{A_2A_3A_4}|\Lambda_{A_0A_1})\\
=\ & P(\Lambda_{A_0A_1}|\Lambda_{A_0A_1})+P(\Lambda_{A_2A_3A_4}|\Lambda_{A_0A_1})\\
&-P(\Lambda_{A_0A_1}\cup\Lambda_{A_2A_3A_4}|\Lambda_{A_0A_1})\\
\ge\ & P(\Lambda_{A_2A_3A_4}|\Lambda_{A_0A_1})\\
=\ & P(\Lambda_{A_2}\cap\Lambda_{A_3A_4}|\Lambda_{A_0A_1})\\
=\ & P(\Lambda_{A_2}|\Lambda_{A_0A_1})+P(\Lambda_{A_3A_4}|\Lambda_{A_0A_1})-P(\Lambda_{A_2}\cup\Lambda_{A_3A_4}|\Lambda_{A_0A_1})\\
\ge\ & P(\Lambda_{A_3A_4}|\Lambda_{A_0A_1})+(\eta-1)\\
=\ & \left[P(\Lambda_{A_3}|\Lambda_{A_0A_1})+P(\Lambda_{A_4}|\Lambda_{A_0A_1})-P(\Lambda_{A_3}\cup\Lambda_{A_4}|\Lambda_{A_0A_1})\right]\\
&+(\eta-1)\\
\ge\ & (2\eta-1)+(\eta-1)\\
=\ & 3\eta-2. \qquad (7)
\end{aligned}
$$

Combining Eqs. (3) and (7), we prove the Lemma 2.

From Lemma 1 and Lemma 2, we finally obtain the following theorem.

Theorem 1 In actual KCBS experiments, if the detector's efficiency is η and the experimental data violate the following inequality:

$$
\sum_{i=0}^{4}P(A_i=0,A_{i+1}=1|\Lambda_{A_iA_{i+1}})\le 2+9(1-\eta), \qquad (8)
$$

then we can assert that the experimental data must result in violation of Eq. (2) and it is a loophole-free confirmation of quantum contextuality.

The maximum value of $\sum_{i=0}^{4}P(A_i=0,A_{i+1}=1|\Lambda_{A_iA_{i+1}})$ in quantum theory is $\sqrt{5}$.[1] It occurs for the state $|\psi\rangle=(0,0,1)^{\rm T}$ and the measurements $A_i=|v_i\rangle\langle v_i|$, where

$$
\begin{aligned}
|v_0\rangle &= N_0\left(1,0,\sqrt{\cos(\pi/5)}\right)^{\rm T},\\
|v_{1,4}\rangle &= N_1\left(\cos(4\pi/5),\pm\sin(4\pi/5),\sqrt{\cos(\pi/5)}\right)^{\rm T},\\
|v_{2,3}\rangle &= N_2\left(\cos(2\pi/5),\mp\sin(2\pi/5),\sqrt{\cos(\pi/5)}\right)^{\rm T},
\end{aligned}
$$

all N_i are normalization factors. Hence, in order to obtain the violation of the inequality of the Eq. (8), the detection efficiency must satisfy $\eta\ge(11-\sqrt{5})/9\approx 0.9738$.

3. Conclusion

The imperfection of detection efficiency may cause the so-called loophole in actual KCBS's experiments. An alternative KCBS inequality has been derived and it can be used to deal with the loophole in actual KCBS's experiments. If the experimental data can violate this alternative KCBS inequality, it must result in the loophole-free violation of the original KCBS inequality. We show that the minimum detection efficiency needed for a loophole-free violation of KCBS inequality is about 0.9738.

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