Exact solution of Schrödinger equation with q-deformed quantum potentials using Nikiforov–Uvarov method

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In this paper, we present the exact solution of the one-dimensional Schrödinger equation for the q-deformed quantum potentials via the Nikiforov–Uvarov method. The eigenvalues and eigenfunctions of these potentials are obtained via this method. The energy equations and the corresponding wave functions for some special cases of these potentials are briefly discussed. The PT-symmetry and Hermiticity for these potentials are also discussed.

Keywords: Schrödinger equation, q-deformed quantum potential, Woods–Saxon potential, Nikiforov–Uvarov method

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1. Introduction

It is well known that the exact solutions of the Schrödinger equation for some physical potentials play an essential role in non-relativistic quantum mechanics and this is possible only for a few sets of quantum mechanical systems. In recent years, a number of researchers have studied the solution of the Schrödinger equations with various potentials for different applications. These investigations include: Woods–Saxon potential,[1–6] harmonic oscillator potential,[7] Hulthén potential,[8–12] Rosen–Morse potential,[13–19] Manning-Rosen potential,[20–27] Hulthén plus Manning–Rosen,[28] Kratzer potential,[29] pseudoharmonic potential,[30,31] trigonometry Scarf potential,[32] Scarf-type potential,[33] Scarf–Grosche potential,[34] external hyperbolic-tangent potential,[35] and Yukawa potential.[36,37]

Methods, such as the Nikiforov–Uvarov (NU) method,[2–4,6,11,12,14,16,17,19,21,28,32–34,38–43] the supersymetric approach,[44,46–59] the exact quantization rule,[60–65] asymptotic iteration method[66–78] and others have been some of the utility tools in solving the Schrödinger equation, Klein–Gordon and Dirac equation with potential of interest. In this paper, we adopt the Nikiforov–Uvarov method, which has been used to solve several quantum mechanical problems in physics and applied mathematics.

By means of the Nikiforov–Uvarov method, the solutions of the Dirac equation with spin and pseudospin symmetries for the trigonometric Scarf potential in D-dimensions were obtained by Falaye and Oyewumi (2011).[32] In addition, Ikhdair and Sever (2012) (and the references therein) used the NU method to obtain the approximate analytical solutions of the Dirac equations with the reflectionless-type and Rosen–Morse potentials including the spin–orbit centrifugal (pseudo-centrifugal) term.[19]

Very recently, Abdalla *et al.*[79] derived analytical solutions of the Schrödinger wave equation for some q-deformed potentials in terms of Heun functions. In this paper, we solve the one-dimensional Schrödinger equation with the q-deformed quantum potential (i.e., the Woods–Saxon plus Rosen–Morse plus symmetrical double well potential) using the Nikiforov–Uvarov method. In addition, by extension, other interesting related results are obtained.

The rest of the paper is organized as follows: in Section 2, the NU method is reviewed. In Section 3, the bound state solutions are obtained for real, complex, PT-symmetric and non-PT-symmetric cases of this potential. Section 4 contains discussion of some special cases. The conclusion is given in Section 5.

2. NU Method

The NU method is based on solving a second-order linear differential equation by reducing it to a generalized equation of the hypergeometric type.[38,39] By introducing an appropriate coordinate transformation $z = z(r)$, this equation can be re-written in the following form:

$$\Psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\Psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\Psi(z) = 0, \tag{1}$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials of at most the second degree, and $\tilde{\tau}(z)$ is a first-degree polynomial. Taking the following factorization:

$$\Psi(z) = y(z)\phi(z), \tag{2}$$

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equation (1) reduces to the equation of the hypergeometric type

$$\sigma(z)y''(z)+\tau(z)y'(z)+\lambda y(z)=0, \tag{3}$$

where

$$\tau(z)=\tilde{\tau}(z)+2\pi(z). \tag{4}$$

The function π and the parameter λ in this method are defined as follows:

$$\pi(z) = \left(\frac{\sigma'(z)-\tilde{\tau}(z)}{2}\right) \pm\sqrt{\left(\frac{\sigma'(z)-\tilde{\tau}(z)}{2}\right)^2+\tilde{\sigma}(z)+k\sigma(z)}, \tag{5}$$

and

$$\lambda=k+\pi'(z). \tag{6}$$

In order to find the value of k, the expression under the square root must be a square of a polynomial, and this gives the polynomial $\pi(z)$ which is dependent on the transformation function $z(r)$. In addition, the parameter λ defined in Eq. (6) takes the form

$$\lambda=\lambda_n=-n\tau'-\left[\frac{n(n-1)}{2}\sigma''\right]. \tag{7}$$

The polynomial solutions $y_n(z)$ are given by the Rodrigue relation

$$y_n(z)=\frac{N_n}{\rho(z)}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\sigma^n(z)\rho(z)\right], \tag{8}$$

where N_n is the normalization constant and $\rho(z)$ is the weight function satisfying

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\sigma(z)\rho(z)\right]=\tau(z)\rho(z). \tag{9}$$

The second part of the wave function $\phi(z)$ can be obtained from the relation

$$\pi(z)=\sigma(z)\frac{\mathrm{d}}{\mathrm{d}z}[\ln\phi(z)]. \tag{10}$$

This method has been a very useful tool in the fields of physics and applied mathematics since its introduction.[2–4,6,11,12,14,16,17,19,21,28,32–34,38,39]

3. Bound-state solutions

3.1. Exact solutions of Schrödinger equation with q-deformed quantum potentials

The one-dimensional time independent Schrödinger equation ($l=0$) can be written as ($\hbar=2\mu=1$)

$$\frac{\mathrm{d}^2\Psi(x)}{\mathrm{d}x^2}+[E-V(x)]\Psi(x)=0, \tag{11}$$

where E is the energy, $V(x)$ is the potential energy, and $\psi(x)$ is the corresponding wave function. We consider the q-deformed quantum potentials, that is, the q-deformed generalized Woods–Saxon plus Rosen–Morse plus generalized symmetrical double well potential which can be expressed as

$$\begin{aligned}V(x) = & \frac{V_1\mathrm{e}^{-2\alpha x}}{1+q\mathrm{e}^{-2\alpha x}}+\frac{V_2\mathrm{e}^{-4\alpha x}}{(1+q\mathrm{e}^{-2\alpha x})^2}\\ & +V_3\,\mathrm{sech}_q^2(\alpha x)+V_4\tanh_q(\alpha x)\\ & -V_5\tanh_q^2(\alpha x)+V_6\,\mathrm{sech}_q^2(\alpha x).\end{aligned} \tag{12}$$

Here, the following deformed hyperbolic function described by Arai in (1991) has been used:[80]

$$\begin{aligned}&\sinh_q x=\frac{\mathrm{e}^x-q\mathrm{e}^{-x}}{2},\quad \cosh_q x=\frac{\mathrm{e}^x+q\mathrm{e}^{-x}}{2},\\ &\tanh_q x=\frac{\sinh_q x}{\cosh_q x},\quad \mathrm{sech}_q x=\frac{1}{\cosh_q x},\\ &\mathrm{cosech}_q x=\frac{1}{\sinh_q x},\quad \coth_q x=\frac{\cosh_q x}{\sinh_q x},\\ &\cosh_q^2 x-\sinh_q^2 x=q,\quad \frac{\mathrm{d}}{\mathrm{d}x}\cosh_q x=\sinh_q x,\\ &\frac{\mathrm{d}}{\mathrm{d}x}\sinh_q x=\cosh_q x,\quad \frac{\mathrm{d}}{\mathrm{d}x}\tanh_q x=\frac{q}{\cosh_q^2 x},\\ &\frac{\mathrm{d}}{\mathrm{d}x}\coth_q x=-\frac{q}{\sinh_q^2 x}.\end{aligned} \tag{13}$$

Using the transformation

$$z=-\mathrm{e}^{-2\alpha x}, \tag{14}$$

and substituting Eqs. (12) and (14) and appropriate parameters in Eq. (13) into Eq. (11), we have

$$\begin{aligned}&\Psi''(z)+\frac{1-qz}{z(1-qz)}\Psi'(z)+\frac{1}{z^2(1-qz)^2}\\ &\times\{z^2[\beta_1 q-\beta_2-q^2(\beta_4+\beta_5+\xi)]\}\\ &+\frac{1}{z^2(1-qz)^2}\{z[-\beta_1-4(\beta_3+\beta_6)\\ &-2q\beta_5+2q\xi]+(\beta_4-\xi)\}=0,\end{aligned} \tag{15}$$

where

$$\xi=\frac{-E}{4\alpha^2}\ \text{ and }\ \beta_i=\frac{V_i}{4\alpha^2},\ i=1,\,2,\,3,\,4,\,5,\,6. \tag{16}$$

Comparing Eq. (16) with Eq. (1), we have

$$\begin{aligned}\sigma(z) = & z^2\left[\beta_1 q-\beta_2-q^2(\beta_4+\beta_5+\xi)\right]\\ & +z\{-\beta_1-4(\beta_3+\beta_6)-2q\beta_5+2q\xi\}+(\beta_4-\xi),\\ \tilde{\tau}(z) = & 1-qz,\ \sigma(z)=z-qz^2.\end{aligned} \tag{17}$$

Inserting the values of these parameters with $\sigma'(z)=1-2qz$ into Eq. (5), we have

$$\pi(z)=-\frac{qz}{2}\pm\frac{1}{2}b\{[2\delta-\eta H]qz-2\delta\},$$

for

$$k = 2q(\beta_4 - \beta_5) - 4(\beta_3 + \beta_6) - \beta_1 + \eta H q\delta, \tag{18}$$

where

$$\eta = \pm 1, \quad \delta = \sqrt{\xi - \beta_4},$$
$$H = \sqrt{\frac{4\beta_2}{q^2} + 16\frac{(\beta_3 + \beta_6)}{q} + 12\beta_5 + 1}. \tag{19}$$

Selecting an appropriate parameter of k in $\pi(z)$ which satisfies the condition $\tilde{\tau}(z) < 0$, one obtains

$$\pi(z) = -\frac{qz}{2} - \frac{1}{2}\{[\delta - \eta H]qz - 2\delta\}$$

for

$$k = 2q(\beta_4 - \beta_5) - 4(\beta_3 + \beta_6) - \beta_1 + \eta H q\delta. \tag{20}$$

From Eq. (6), parameter λ is obtained as

$$\lambda = 2q(\beta_4 - \beta_5) - 4(\beta_3 + \beta_6) - \beta_1 + \eta H q\delta - \frac{q}{2} - \frac{1}{2}\{\{2\delta - \eta H\}qz\}, \tag{21}$$

and using Eq. (7), we have

$$\lambda = \lambda_n = nq[(n+1) + 2\delta - \eta H]. \tag{22}$$

By comparing Eqs. (21) and (22), the exact energy eigenvalue equation is obtained as

$$E_n = -\frac{\alpha^2}{4}\left[\frac{(2qV_4 + qV_5 - V_1)/(q\alpha^2) + V_2/(q^2\alpha^2)}{2n + 1 - \eta\sqrt{H}} - \left(2n + 1 - \eta\sqrt{H}\right)\right]^2 - V_4; \quad n \geq 0,\ q \geq 1,$$

where we have substituted for β_1, β_2, β_3, β_4, β_5, β_6, and δ. From Eq. (9), it can be shown that the weight function $\rho(z)$ is

$$\rho(z) = z^{2\delta}(1 - qz)^{-\eta H}. \tag{23}$$

Substituting Eq. (23) into the Rodrigue's relation (8) and using the properties of Jacobi polynomial,[81,82] we have

$$y_n(z) = N_n \mathrm{P}_n^{(2\delta, -\eta H)}(1 - 2qz). \tag{24}$$

The other part of the wave function obtained from Eq. (9) as

$$\phi(z) = z^{\delta}(1 - qz)^{(1-\eta H)/2}, \tag{25}$$

and therefore, the wave function $\Psi_n(z) = \phi(z)y_n(z)$ is given as

$$\Psi_n(z) = N_n z^{\delta}(1 - qz)^{(1-\eta H)/2}\mathrm{P}_n^{(2\delta, -\eta H)}(1 - 2qz), \tag{26}$$

where N_n is the normalization factor to be determined from the normalization condition

$$\int_0^1 |\Psi_n(z)|^2 \frac{\mathrm{d}z}{2z\alpha} = 1, \tag{27}$$

that is

$$N_n^2 \int_0^1 z^{2c}(1 - qz)^{(1+2d)}\left[\mathrm{P}_n^{(2c,2d)}(z)\right]^2 \mathrm{d}z = 2\alpha, \tag{28}$$

where $c = \delta - 1/2$ and $d = -\eta H/2$. Using the following forms of Jacobi polynomial:[81,82]

$$\mathrm{P}_n^{(c,d)}(z) = 2^{-n}\sum_{r=0}^{n}(-1)^n \binom{n+c}{p}\binom{n+d}{n-p} \times (1-z)^{n-p}(1-z)^p, \tag{29}$$

and

$$\mathrm{P}_n^{(c,d)}(z) = \frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)} \times \sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r+1)}{r+c+1}\left(\frac{z-1}{2}\right)^r, \tag{30}$$

where

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}.$$

Now $\left[\mathrm{P}_n^{(2c,2d)}(x)\right]^2$ can be obtained from Eqs. (29) and (30) as

$$\left[\mathrm{P}_n^{(2c,2d)}(z)\right]^2 = z^{n+r-p}(1 - qz)^p A_{nq}(p,r). \tag{31}$$

Substituting Eq. (31) into Eq. (28), we have

$$N_n^2 A_{nq}(p,r)\int_0^1 z^{n+r-p+2c}(1 - qz)^{1+2d+p}\mathrm{d}z = 2\alpha. \tag{32}$$

Note that

$$\int_0^1 z^{c-a-1}(1 - qz)^{-b}\mathrm{d}z = {}_2\mathrm{F}_1(a, b; a+1; q)\,\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}, \tag{33}$$

provides $\mathrm{Re}(c) < \mathrm{Re}(a) > 0, |\arg(1-q)| < \pi$, and

$$\int_0^1 z^{a-1}(1 - qz)^{-b}\mathrm{d}z = \frac{1}{a}[{}_2\mathrm{F}_1(a, b; a+1; q)]. \tag{34}$$

Using Eq. (34), we obtain the normalization factor N_n as

$$N_n = \sqrt{\frac{2\alpha B_{nq}(p,r)}{A_{nq}(p,r)}}, \tag{35}$$

where

$$A_{nq}(p,r) = \frac{(-1)^n[\Gamma(n+2c+1)]^2\Gamma(n+2d+1)}{\Gamma(n+2c-p+1)\Gamma(r+2d+1)\Gamma(2c+2d+1)} \times \sum_{p=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)}, \tag{36}$$

$$[B_{nq}(p,r)]^{-1} = \frac{1}{n+r-p+2\delta+1}\,{}_2\mathrm{F}_1(n+r-p+2\delta+1, \eta H - 1 - p; n+r-p+2\delta+1; q). \tag{37}$$

3.2. PT-symmetry and non-Hermiticity

When a Hamiltonian commutes with the parity and time reversal operators PT, it is called PT-symmetric Hamiltonian, i.e., $[H,\ PT]=0$. The PT operator satisfies the following relations:

$$PpP^{-1}=-p=TpT^{-1},\ PxP^{-1}=-x,\ TAT^{-1}=-\mathrm{i}A. \quad (38)$$

In this case, the potential parameters in Eq. (12) are set as V_1, $V_2, V_3, V_4, V_5, V_6, q\in\mathbf{R}$ and $\alpha\in\mathbf{IR}$ $(\alpha\to\mathrm{i}\alpha)$ (the set of purely real numbers and purely complex numbers, respectively), this replacement and some manipulations in Eq. (12) lead to the following potential

$$\begin{aligned}V^{\mathrm{PT}}(x) &= \{-V_1[1+q(\cos 2\alpha x-\mathrm{i}\sin 2\alpha x)]\\ &\quad +V_2[\cos 2\alpha x-\mathrm{i}\sin 2\alpha x]-4(V_3+V_6)\\ &\quad -V_4\left[(1-q^2)\cos 2\alpha x+\mathrm{i}(1+q^2)\sin 2\alpha x\right]\\ &\quad +V_5[(1+q^2)\cos 2\alpha x\\ &\quad +\mathrm{i}(1-q^2)\sin 2\alpha x-2q]\}/[(1+q^2)\cos 2\alpha x\\ &\quad +\left(1-q^2\right)\mathrm{i}\sin 2\alpha x+2q]. \end{aligned} \quad (39)$$

Using Eqs. (11) and (39) and making the corresponding parameter replacements in Eqs. (23) and (27), we obtain the energy eigenvalue equations and eigenfunctions, respectively as

$$\begin{aligned}E_n^{\mathrm{PT}} &= \frac{\alpha^2}{4}\left[\frac{(V_1-2qV_4-qV_5)/q\alpha^2-V_2/q^2\alpha^2}{(2n+1-\eta H^{\mathrm{PT}})}\right.\\ &\quad \left.-\left(2n+1-\eta H^{\mathrm{PT}}\right)\right]-V_4,\ \ n\geq 0, q\geq 1, \end{aligned} \quad (40)$$

with

$$H^{\mathrm{PT}}=\sqrt{1-\frac{V_2}{q^2\alpha^2}-4\frac{(V_3+V_6)}{q\alpha^2}-3\frac{V_5}{\alpha^2}},$$

and

$$\Psi_n^{\mathrm{PT}}(z) = C_n z^{\delta}(1-qz)^{\frac{1}{2}(1-\eta H^{\mathrm{PT}})}\mathrm{P}_n^{\left(2\delta,-\eta H^{\mathrm{PT}}\right)}(1-2qz). \quad (41)$$

Using Eq. (41), we can consequently obtain the normalization constant C_n as

$$C_n=\sqrt{\frac{2\alpha B_{nq}^{\mathrm{PT}}(p,r)}{A_{nq}^{\mathrm{PT}}(p,r)}}, \quad (42)$$

with

$$\begin{aligned}&A_{nq}^{\mathrm{PT}}(p,r)\\ &=\frac{(-1)^n[\Gamma(n+2c+1)]^2\Gamma(n+2d^{\mathrm{PT}}+1)}{\Gamma(n+2c-p+1)\Gamma(r+2d^{\mathrm{PT}}+1)\Gamma(2c+2d^{\mathrm{PT}}+1)}\\ &\times\sum_{p=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d^{\mathrm{PT}}+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d^{\mathrm{PT}}+1)}, \end{aligned} \quad (43)$$

and

$$[B_{nq}^{\mathrm{PT}}(p,r)]^{-1} = \frac{1}{n+r-p+2\delta+1}{}_2\mathrm{F}_1(n+r-p+2\delta+1, \eta H^{\mathrm{PT}}-1-p; n+r-p+2\delta+1; q), \quad (44)$$

where

$$d^{\mathrm{PT}}=-\frac{\eta H^{\mathrm{PT}}}{2},\quad H^{\mathrm{PT}}=\sqrt{1-\frac{V_2}{q^2\alpha^2}-4\frac{V_3+V_6}{q\alpha^2}-3\frac{V_5}{\alpha^2}}.$$

3.3. Non-PT-symmetry and non-Hermiticity

Another form of the potential is obtained by assuming some of the parameters as purely imaginary, i.e. $q\to\mathrm{i}q$, $\alpha\to\mathrm{i}\alpha$, $V_1\to\mathrm{i}V_1$, $V_3\to\mathrm{i}V_3$, $V_6\to\mathrm{i}V_6$. Under this replacement and after some manipulations, the potential takes the form

$$\begin{aligned}V^{\mathrm{NPT}}(x) &= \{V_1[q\cos\alpha x+\mathrm{i}(1+q\cos 2\alpha x)]\\ &\quad +V_2[\cos 2\alpha x-\mathrm{i}\sin 2\alpha x]\\ &\quad -\mathrm{i}(V_3+V_6)-V_4[(1+q^2)\cos 2\alpha x\\ &\quad +\mathrm{i}(1-q^2)\sin 2\alpha x]+V_5[(1+q^2)\cos 2\alpha x\\ &\quad +\mathrm{i}(1-q^2)\sin 2\alpha x-2\mathrm{i}q]\}/[(1-q^2)\cos 2\alpha x\\ &\quad +\mathrm{i}(1+q^2)\sin 2\alpha x+2\mathrm{i}q]. \end{aligned} \quad (45)$$

Similarly, we obtain the energy eigenvalue equation and eigenfunction, respectively as

$$\begin{aligned}E_n^{\mathrm{NPT}} &= \frac{\alpha^2}{4}\left[\frac{(V_1-2qV_4-qV_5)/q\alpha^2-V_2/q^2\alpha^2}{2n+1-\eta H^{\mathrm{NPT}}}\right.\\ &\quad \left.-(2n+1-\eta H^{\mathrm{NPT}})\right]^2-V_4;\quad n\geq 0, q\geq 1, \end{aligned} \quad (46)$$

with

$$H^{\mathrm{NPT}}=\sqrt{1+\frac{V_2}{q^2\alpha^2}-4\frac{(V_3+V_6)}{q\alpha^2}-3\frac{V_5}{\alpha^2}} \quad (47)$$

and

$$\begin{aligned}\Psi_n^{\mathrm{NPT}}(z) &= D_n z^{\delta}(1-qz)^{(1-\eta H^{\mathrm{NPT}})/2}\\ &\quad \times\mathrm{P}_n^{\left(2\delta,-\eta H^{\mathrm{NPT}}\right)}(1-2qz), \end{aligned} \quad (48)$$

where the normalization constant D_n is obtained as

$$D_n=\sqrt{\frac{2\alpha B_{nq}^{\mathrm{NPT}}(p,r)}{A_{nq}^{\mathrm{NPT}}(p,r)}}, \quad (49)$$

and

$$\begin{aligned}&A_{nq}^{\mathrm{NPT}}(p,r)\\ &=\frac{(-1)^n[\Gamma(n+2c+1)]^2\Gamma(n+2d^{\mathrm{NPT}}+1)}{\Gamma(n+2c-p+1)\Gamma(r+2d^{\mathrm{NPT}}+1)\Gamma(2c+2d^{\mathrm{NPT}}+1)}\\ &\times\sum_{p=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d^{\mathrm{NPT}}+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d^{\mathrm{NPT}}+1)}, \end{aligned} \quad (50)$$

$$\left[B_{nq}^{\mathrm{NPT}}(p,r)\right]^{-1} = \frac{1}{n+r-p+2\delta+1}\,{}_2\mathrm{F}_1(n+r-p+2\delta+1, \eta H^{\mathrm{NPT}}-1-p; n+r-p+2\delta+1; q), \quad (51)$$

$$d = -\frac{\eta H^{\mathrm{NPT}}}{2}, \quad H^{\mathrm{NPT}} = \sqrt{1+\frac{V_2}{q^2\alpha^2}-4\frac{(V_3+V_6)}{q\alpha^2}-3\frac{V_5}{\alpha^2}}.$$

4. Special cases

In this section, in the framework of the NU method, the energy eigenvalues and the corresponding wave functions for the Schrödinger equation with the q-deformed Woods–Saxon potential, Rosen–Morse potential and symmetrical double well potential are obtained by setting appropriate parameters in Eq. (12) to zero. The PT-symmetric and the non-PT-symmetric solution of this potential are also considered.

4.1. Woods–Saxon potential

If $V_3 = V_4 = V_5 = V_6 = 0$, the potential in Eq. (12) turns to the Woods–Saxon potential[1–5]

$$V^{\mathrm{WS},1}(x) = \frac{-V_1 \mathrm{e}^{-2\alpha x}}{1+q\mathrm{e}^{-2\alpha x}} + \frac{-V_2 \mathrm{e}^{-4\alpha x}}{1+q\mathrm{e}^{-2\alpha x}}, \quad (52)$$

the energy-eigenvalue equation and wave function are obtained, respectively as

$$E_n^{\mathrm{WS},1} = -\frac{\alpha^2}{4}\left[\frac{V_1/(q\alpha^2)+V_2/(q^2\alpha^2)}{(2n+1-H^{\mathrm{WS},1})} - (2n+1-H^{\mathrm{WS},1})\right]^2, \quad n \geq 0,\ q \geq 1,\ \eta = 1, \quad (53)$$

and

$$\Psi_n^{\mathrm{WS},1}(z) = \mathrm{P}_n z^{\sqrt{\xi}}(1-qz)^{\left(1-\sqrt{H^{\mathrm{WS},1}}\right)/2} \times \mathrm{P}_n^{\left(2\sqrt{\xi},\,-\sqrt{H^{\mathrm{WS},1}}\right)}(1-2qz), \quad (54)$$

where

$$c^{\mathrm{WS}} = \sqrt{\xi}-\frac{1}{2},\ d^{\mathrm{WS}} = -\frac{H^{\mathrm{WS},1}}{2},\ H^{\mathrm{WS},1} = \sqrt{\frac{V_2}{q^2\alpha^2}+1}.$$

These results are consistent with the results of Meyur and Debnath (2010).[89] Using Eq. (54), the normalization constant P_n is obtained as

$$\mathrm{P}_n = \sqrt{\frac{2\alpha B_{nq}^{\mathrm{WS}}(p,r)}{A_{nq}^{\mathrm{WS}}(p,r)}}, \quad (55)$$

where

$$A_{nq}^{\mathrm{WS}}(p,r) = \frac{(-1)^n[\Gamma(n+2c^{\mathrm{WS}}+1)]^2\Gamma(n+2d^{\mathrm{WS}}+1)}{\Gamma(n+2c^{\mathrm{WS}}-p+1)\Gamma(r+2d^{\mathrm{WS}}+1)\Gamma(2c^{\mathrm{WS}}+2d^{\mathrm{WS}}+1)} \times \sum_{p=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c^{\mathrm{WS}}+2d^{\mathrm{WS}}+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d^{\mathrm{WS}}+1)}, \quad (56)$$

$$\left[B_{nq}^{\mathrm{WS}}(p,r)\right]^{-1} = \frac{1}{n+r-p+2\delta+1}\,{}_2\mathrm{F}_1(n+r-p+2\delta+1, \eta H^{\mathrm{WS},1}-1-p; n+r-p+2\delta+1; q). \quad (57)$$

Now, the case in which α is a pure imaginary parameter is considered, i.e., $\alpha \to \mathrm{i}\alpha$, after this substitution and some manipulations, the potential takes the form

$$V^{\mathrm{WS},2}(x) = \frac{-V_1[1+q(\cos 2\alpha x - \mathrm{i}\sin 2\alpha x)]+V_2[\cos 2\alpha x - \mathrm{i}\sin 2\alpha x]}{(1+q^2)\cos 2\alpha x + \mathrm{i}(1-q^2)\sin 2\alpha x + 2q}. \quad (58)$$

The real positive energy eigenvalues are obtained as

$$E_n^{\mathrm{WS},2} = \frac{\alpha^2}{4}\left[\frac{V_1/(q\alpha^2)-V_2/(q^2\alpha^2)}{2n+1-\sqrt{H^{\mathrm{WS},2}}} - \left(2n+1-\sqrt{H^{\mathrm{WS},2}}\right)\right]^2, \quad \eta = 1,\ n \geq 0,\ q \geq 1, \quad (59)$$

where

$$H^{\mathrm{WS},2} = \sqrt{1-\frac{V_2}{q^2\alpha^2}}.$$

The above equation (59) is consistent with the results of Meyur and Debnath (2010)[89] and Ikhdair and Sever (2007)[90] for $\hbar = 2\mu = 1$ and $\alpha = \alpha_1/2$. By assuming some of the parameters as purely imaginary, i.e., $q \to \mathrm{i}q$, $\alpha \to \mathrm{i}\alpha$ and $V_1 \to \mathrm{i}V_1$, then, we have potential of the form

$$V^{\mathrm{WS},3}(x) = \frac{V_1[q\cos\alpha x + \mathrm{i}(1+q\cos 2\alpha x)]+V_2[\cos 2\alpha x - \mathrm{i}\sin 2\alpha x]}{(1-q^2)\cos 2\alpha x + \mathrm{i}(1+q^2)\sin 2\alpha x + 2\mathrm{i}q}. \quad (60)$$

and the corresponding energy eigenvalues are obtained as

$$E_n^{\mathrm{WS},3} = \frac{\alpha^2}{4}\left[\frac{V_1/(q\alpha^2)-V_2/(q^2\alpha^2)}{2n+1-\sqrt{H^{\mathrm{WS},1}}} - \left(2n+1-\sqrt{H^{\mathrm{WS},1}}\right)\right]^2, \quad n \geq 0,\ q \geq 1. \quad (61)$$

4.2. Rosen–Morse Potential

If $V_1 = V_2 = V_5 = V_6 = 0$, then the potential in Eq. (12) reduces to the Rosen–Morse potential[18,83–87]

$$V^{\mathrm{RM},1}(x) = -V_3\,\mathrm{sech}_q^2(\alpha x) - V_4\tanh_q(\alpha x). \quad (62)$$

The energy eigenvalue equation and corresponding wave function are obtained respectively as

$$E_n^{\mathrm{RM},1} = -\frac{\alpha^2}{4}\left(2n+1+H^{\mathrm{RM},1}\right)^2 - \frac{V_4^2}{\alpha^2}\left(2n+1+H^{\mathrm{RM},1}\right)^{-2}, \quad n \geq 0,\ q \geq 1 \quad (63)$$

and

$$\Psi_n(z) = R_n z^{\delta}(1-qz)^{(1+H^{\mathrm{RM},1})/2} \times \mathrm{P}_n^{(2\delta, H^{\mathrm{RM},1})}(1-2qz),\ \eta=-1. \quad (64)$$

These results are consistent with the results of Akbarieh and Motavalli (2008),[14] Ikhdair (2010)[17] and Oyewumi (2012)[18] for the relativistic limits. Using the above equation (64), we obtained the normalization constant R_n as

$$R_n^{\mathrm{RM},1} = \sqrt{\frac{2\alpha B_{nq}^{\mathrm{RM},1}(p,r)}{A_{nq}^{\mathrm{RM},1}(p,r)}}, \quad (65)$$

where

$$A_{nq}^{\mathrm{RM},1}(p,r) = \frac{(-1)^n[\Gamma(n+2c+1)]^2\Gamma(n+2d^{\mathrm{RM},1}+1)}{\Gamma(n+2c-p+1)\Gamma(r+2d^{\mathrm{RM},1}+1)\Gamma(2c+2d^{\mathrm{RM},1}+1)} \sum_{p=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d^{\mathrm{RM},1}+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d^{\mathrm{RM},1}+1)}, \quad (66)$$

$$\left[B_{nq}^{\mathrm{RM},1}(p,r)\right]^{-1} = \frac{1}{n+r-p+2\delta+1}\,{}_2\mathrm{F}_1(n+r-p+2\delta+1, \eta H-1-p; n+r-p+2\delta+1; q), \quad (67)$$

$$d^{\mathrm{RM},1} = \frac{H^{\mathrm{RM},1}}{2},\quad H^{\mathrm{RM},1} = \sqrt{\frac{V_3}{q\alpha^2}+1}.$$

Again, a case in which α is a purely imaginary parameter is considered, i.e., $\alpha \to \mathrm{i}\alpha$, then equation (62) takes the form

$$V^{\mathrm{RM},2}(x) = \frac{-4V_3 - V_4\left[(1-q^2)\cos 2\alpha x + \mathrm{i}(1+q^2)\sin 2\alpha x\right]}{(1+q^2)\cos 2\alpha x + (1-q^2)\mathrm{i}\sin 2\alpha x + 2q}. \quad (68)$$

The energy eigenvalues are given as

$$E_n^{\mathrm{RM},2} = \frac{\alpha^2}{4}\left[\frac{2V_4/\alpha^2}{(2n+1-H^{\mathrm{RM},2})} - \left(2n+1-H^{\mathrm{RM},2}\right)\right]^2 - V_4,$$
$$n \geq 0,\ q \geq 1$$
$$\text{with } H^{\mathrm{RM},2} = \sqrt{1-4\frac{V_3}{q\alpha^2}}. \quad (69)$$

Now, by setting some of the parameters as purely imaginary, i.e., $q \to \mathrm{i}q$, $\alpha \to \mathrm{i}\alpha$, $V_3 \to \mathrm{i}V_3$, and $V_4 \to \mathrm{i}V_4$, then equation (62) takes the form

$$V^{\mathrm{RM},3}(x) = \frac{-\mathrm{i}V_3 - V_4\left[(1+q^2)\cos 2\alpha x + \mathrm{i}(1-q^2)\sin 2\alpha x\right]}{(1-q^2)\cos 2\alpha x + \mathrm{i}(1+q^2)\sin 2\alpha x + 2\mathrm{i}q}. \quad (70)$$

The energy eigenvalue equation is also found to be the same as Eq. (69).

4.3. Symmetrical double-well potential

Choosing $V_1 = V_2 = V_3 = V_4 = 0$ in Eq. (12), we have

$$V^{\mathrm{SDW},1}(x) = V_5\tanh_q^2(\alpha x) - V_6\,\mathrm{sech}_q^2(\alpha x). \quad (71)$$

Then, energy eigenvalues and the corresponding wave functions are obtained respectively as

$$E_n^{\mathrm{SDW},1} = -\frac{\alpha^2}{4}\left[\frac{V_5/\alpha^2}{2n+1-\eta H^{\mathrm{SDW},1}}\left(2n+1-\eta H^{\mathrm{SDW},1}\right)\right]^2,$$
$$n \geq 0,\ q \geq 1 \quad (72)$$

and

$$\Psi_n(z) = W_n z^{\sqrt{\xi}}(1-qz)^{(1-\eta H^{\mathrm{SDW},1})/2} \times \mathrm{P}_n^{\left(2\sqrt{\xi},\ -\eta H^{\mathrm{SDW},1}\right)}(1-2qz),\ \eta=\pm 1. \quad (73)$$

Using the above equation, we obtained the normalization constant W_n as

$$W_n = \sqrt{\frac{2\alpha B_{nq}^{\mathrm{SDW},1}(p,r)}{A_{nq}^{\mathrm{SDW},1}(p,r)}}, \quad (74)$$

where

$$A_{nq}^{\mathrm{SDW},1}(p,r) = \frac{(-1)^n[\Gamma(n+2c^{\mathrm{WS}}+1)]^2\Gamma(n+2d^{\mathrm{SDW},1}+1)}{\Gamma(n+2c^{\mathrm{WS}}-p+1)\Gamma(r+2d^{\mathrm{SDW},1}+1)\Gamma(2c^{\mathrm{WS}}+2d^{\mathrm{SDW},1}+1)} \sum_{p=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c^{\mathrm{WS}}+2d^{\mathrm{SDW},1}+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)}, \quad (75)$$

$$\left[B_{nq}^{\mathrm{SDW},1}(p,r)\right]^{-1} = \frac{1}{n+r-p+2\delta+1}\,{}_2\mathrm{F}_1(n+r-p+2\delta+1, \eta H^{\mathrm{SDW},1}-1-p; n+r-p+2\delta+1; q), \quad (76)$$

$$d^{\mathrm{SDW},1} = -\frac{\eta H^{\mathrm{SDW},1}}{2},\ H^{\mathrm{SDW},1} = \sqrt{4\frac{V_6}{q\alpha^2}+\frac{3V_5}{\alpha^2}+1}.$$

Let us consider the case where at least one of the potential parameters is complex. If α is a purely imaginary parameter, i.e., $\alpha \to \mathrm{i}\alpha$, the symmetrical double well potential can be written as

$$V^{\mathrm{SDW},2}(x) = \frac{-4V_6+V_5[(1+q^2)\cos 2\alpha x+\mathrm{i}(1-q^2)\sin 2\alpha x-2q]}{(1+q^2)\cos 2\alpha x+(1-q^2)\mathrm{i}\sin 2\alpha x+2q}. \quad (77)$$

By using Eqs. (11) and (76), and making the corresponding parameter substitutions in Eq. (23), we obtained the real positive energy eigenvalue equation as

$$E_n^{\mathrm{SDW},2} = \frac{\alpha^2}{4}\left[\frac{V_5/\alpha^2}{2n+1-\eta\sqrt{1-4V_6/(q\alpha^2)-3V_5/\alpha^2}} - \left(2n+1-\eta\sqrt{1-4\frac{V_6}{q\alpha^2}-3\frac{V_5}{\alpha^2}}\right)\right]^2, \quad n\geq 0,\ q\geq 1. \quad (78)$$

When $\alpha\to\mathrm{i}\alpha$, $V_6\to\mathrm{i}V_6$, and $q\in\mathbf{R}$, then, $V(x)$ takes the form

$$V^{\mathrm{SDW},3}(x) = \frac{-\mathrm{i}V_6+V_5[(1+q^2)\cos 2\alpha x+\mathrm{i}(1-q^2)\sin 2\alpha x-2\mathrm{i}q]}{(1-q^2)\cos 2\alpha x+\mathrm{i}(1+q^2)\sin 2\alpha x+2\mathrm{i}q}. \quad (79)$$

The positive energy eigenvalues are also found to be the same as the one obtained in Eq. (78).

5. Conclusion

In this paper we solve the Schrödinger equation for Woods–Saxon potential plus Rosen–Morse potential plus symmetrical double well potential via NU method. The results include the energy spectrum of Wood-Saxon potential, Rosen–Morse potential and Symmetrical double well potential obtained by setting some parameters to zero. The PT-symmetric and non-PT-symmetric of Wood-Saxon potential, Rosen–Morse potential and the symmetrical double well potential are also discussed.

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