Stochastic responses of Duffing–Van der Pol vibro-impact system under additive colored noise excitation*

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A response analysis procedure is developed for a vibro-impact system excited by colored noise. The non-smooth transformation is used to convert the vibro-impact system into a new system without impact term. With the help of the modified quasi-conservative averaging, the total energy of the new system can be approximated as a Markov process, and the stationary probability density function (PDF) of the total energy is derived. The response PDFs of the original system are obtained using the analytical solution of the stationary PDF of the total energy. The validity of the theoretical results is tested through comparison with the corresponding simulation results. Moreover, stochastic bifurcations are also explored.

Keywords: vibro-impact system, colored noise, non-smooth transformation, stochastic bifurcation

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1. Introduction

The stochastic averaging method was proposed first by Stratonovich[1] to deal with the noise-excited dynamic systems and justified rigorously by Khasminskii.[2] The basic idea is to approximate the system response by a Markov vector process on condition that the random excitation can be replaced by white noise. The response probability density is then governed by a second-order partial differential equation called the Fokker–Planck–Kolmogorov (FPK) equation. There are two main versions of the stochastic averaging method: the standard stochastic averaging (amplitude) and the stochastic averaging of energy envelope (total energy).[3,4] Reviews of the method and its applications have been given by Roberts and Spanos[5] and Zhu and Cai.[6] One of the new developments of the stochastic averaging method is the extension of the stochastic averaging of energy envelope to the cases of non-white noise excitation.[7,8]

Recent work of the stochastic averaging can be found in Refs. [9] and [10]. The work mentioned above gives the applications of the method in smooth systems. However many non-smooth factors[11] exist in engineering applications, such as impacts, collisions, and dry frictions. Due to the existence of the non-smooth factors, the theories of smooth systems cannot be applied to the non-smooth systems. Some achievements of non-smooth systems, which have attracted many scholars, have been gained[12] in the past few decades. As an important type of non-smooth system, the vibro-impact system has been studied widely. Some interesting phenomena, such as grazing bifurcation,[13,14] torus bifurcation,[15] chatter and sticking motions,[16–19] have been observed in this type of system. With the help of the non-smooth transformation, the stochastic averaging method has been extended to study the stochastic response of the vibro-impact systems in Refs. [20]–[24]. The methods used in those papers can only be applied to the nonlinear systems subjected to white noise. However, the white noise idealization may not be entirely satisfactory in some cases. A modified version of quasi-conservative averaging was proposed by Roberts[7,25] to deal with the stochastic systems involving a non-white additive stationary random excitation. The modified version was extended by Cai[26] to the case with both additive and multiplicative excitations.

In this paper, we explore the stationary probability density functions (PDF) of a Duffing–Van der Pol vibro-impact system excited by additive colored noise.[27–32] The non-smooth transformation of state variables and the modified quasi-conservative averaging are used to obtain an approximate Markov process of total energy, and the stationary PDF can be obtained by solving the corresponding FPK equation. Then the joint stationary PDF of the original displacement and velocity can be formulated by using the result of the stationary PDF of the total energy. Further, the corresponding PDFs of displacement and velocity can be derived. Using the stationary PDFs, we investigate the effects of different system parameters and observe P-bifurcations.

The remainder of this paper is organized as follows. The non-smooth transformation is introduced and used to transform the vibro-impact system into a new system without impact term in Section 2. Approximate stationary PDFs are derived by using the modified quasi-conservative averaging method in Section 3. Analytical results are verified by nu-

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merical simulations in Section 4. Furthermore, the stochastic bifurcations are also explored. Conclusions are presented in Section 5.

2. System description and non-smooth transformation

2.1. System description

Considering a Duffing–Van der Pol vibro-impact system with Gaussian colored noise, the corresponding differential equation of motion is

$$\ddot{x}+(c_2x^2-c_1)\dot{x}+ax+bx^3=\xi(t),\quad x>0, \tag{1a}$$

$$\dot{x}_+=-r\dot{x}_-,x=0, \tag{1b}$$

where a, b, c_1, and c_2 are positive parameters, and $\xi(t)$ is Gaussian colored noise satisfying the following conditions:

$$\langle\xi(t)\rangle=0,\quad \langle\xi(t)\xi(s)\rangle=\frac{D}{\tau}\exp\left(-\frac{|t-s|}{\tau}\right). \tag{2}$$

Here τ and D denote the correlation time and intensity of the colored noise $\xi(t)$, respectively. The dot, as usual, indicates the differentiation with respect to time. In Eq. (1b), r is the restitution coefficient, $\dot{x}_+$ and $\dot{x}_-$ are the velocities of the system just after and before the impacts, respectively. So, Eq. (1b) gives the change of velocity $\dot{x}$ at the time instant of impact t_*, and r denotes the impact loss ($0<r\leq 1$). Obviously, the impact loss of the system becomes very small when r approaches to 1. Under the condition of $r=1$, the impact loss is zero in the system.

2.2. Non-smooth transformation

Following Zhuravlev,[33,34] the non-smooth transformations of the state variables are introduced as follows:

$$x=x_1=|y|,\quad \dot{x}=x_2=\dot{y}\mathrm{sgn}(y),\quad \ddot{x}=\ddot{y}\mathrm{sgn}(y), \tag{3}$$

where $\mathrm{sgn}(\cdot)$ represents the signum function. Obviously, the transformation maps the domain of the original phase plane $(x,\dot{x})$ onto the whole phase plane $(y,\dot{y})$, and makes the transformed velocity $\dot{y}$ continuous at the impact instants in the special case of elastic impact (i.e., $r=1$). When Eq. (3) is introduced into Eq. (1), the equations of new variables can be written as

$$\ddot{y}+(c_2y^2-c_1)\dot{y}+ay+by^3=\mathrm{sgn}(y)\xi(t),\quad t\neq t_*, \tag{4a}$$

$$\dot{y}_+=r\dot{y}_-,\quad t=t_*. \tag{4b}$$

After this transformation, the jump of transformed velocity $\dot{y}$ becomes proportional to $1-r$ instead of $1+r$ for the jump of the original velocity $\dot{x}$. Considering $y(t_*)=0$ at the instant of impact t_* and an equation of Dirac delta function (i.e., $\delta(t-t_*)=|\dot{y}|\,\delta(y)$), we can write the impulsive term as

$$(\dot{y}_--\dot{y}_+)\delta(t-t_*)=(1-r)\dot{y}\,|\dot{y}|\,\delta(y). \tag{5}$$

By substituting Eq. (5) into Eq. (4), the transformed equation of motion can be written as

$$\begin{aligned}&\ddot{y}+(c_2y^2-c_1)\dot{y}+(1-r)\dot{y}\,|\dot{y}|\,\delta(y)+ay+by^3\\ &=\mathrm{sgn}(y)\xi(t).\end{aligned} \tag{6}$$

Letting $h(y,\dot{y})=(c_2y^2-c_1)\dot{y}+(1-r)\dot{y}\,|\dot{y}|\,\delta(y)$, $u(y)=ay+by^3$, Eq. (6) becomes the following simple equation:

$$\ddot{y}+h(y,\dot{y})+u(y)=\mathrm{sgn}(y)\xi(t). \tag{7}$$

After the previous procedure, the original impact system (1) is reduced to one without impact term expressed by Eq. (6). The term $(1-r)\dot{y}\,|\dot{y}|\,\delta(y)$ in Eq. (6) describes the impact loss of the system, which can be regarded as an impulsive damping term.

3. Modified quasi-conservative averaging

The corresponding undamped free motion of Eq. (6) can be described as follows:

$$\ddot{y}+ay+by^3=0. \tag{8}$$

The total energy and the potential energy of the system are

$$E=\frac{1}{2}\dot{y}^2+U(y), \tag{9}$$

$$U(y)=\int_0^y(az+bz^3)\mathrm{d}z=\frac{a}{2}y^2+\frac{b}{4}y^4. \tag{10}$$

The energy relation (9) can be written in the form of

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\sqrt{2E-2U(y)}. \tag{11}$$

Then the period of the free vibration can be written as

$$T(E)=\int_0^T\mathrm{d}t=2\int_{y_{\min}}^{y_{\max}}\frac{\mathrm{d}y}{\sqrt{2E-2U(y)}}, \tag{12}$$

where $y_{\max}$ and $y_{\min}$ denote the maximum and the minimum displacements at energy E, respectively. Assuming that a is the positive root of $U(y)=E$, the period can be obtained as

$$T=4T_{1/4}=4\int_0^a\frac{\mathrm{d}y}{\sqrt{2E-2U(y)}}. \tag{13}$$

It is clear that T depends on energy level E.

On the condition that coefficients c_1, c_2, $1-r$, and D are all small and proportional to a small parameter, Eq. (6) is a quasi-conservative system. The T given in Eq. (12) may be treated as the quasi-period of the system. Letting

$$\mathrm{sgn}(y)\sqrt{U(y)}=\sqrt{E}\cos\phi, \tag{14a}$$

$$\dot{y}=-\sqrt{2E}\sin\phi,\quad 0\leq\phi<2\pi, \tag{14b}$$

Eq. (6) is equivalent to a pair of first-order differential equations of energy process and phase process

$$\dot{E}=\sqrt{2E}h(E,\phi)\sin\phi-\sqrt{2E}\sin\phi\,\xi(t)\mathrm{sgn}(y(E,\phi)), \tag{15a}$$

$$\dot{\phi} = \frac{1}{\sqrt{2E}}\left(h(E,\phi)\cos\phi + \frac{u(E,\phi)}{\cos\phi}\right) - \frac{\cos\phi}{\sqrt{2E}}\xi(t)\mathrm{sgn}(y(E,\phi)), \tag{15b}$$

where $h(E,\phi)$, $u(E,\phi)$, and $y(E,\phi)$ are obtained from $h(y,\dot{y})$, $u(y,\dot{y})$, and y, respectively, by replacing y and $\dot{y}$ with E and ϕ according to transformation (14). Under the previous assumption, Eq. (15) indicates that the energy process $E(t)$ varies slowly, and the phase process $\phi(t)$ does not.

By assuming that the correlation time of the excitation is much shorter than the relaxation time of the energy process, $E(t)$ can be approximated as a Markov process governed by the mean Itô–type stochastic differential equation

$$\mathrm{d}E = m(E)\mathrm{d}t + \sigma(E)\mathrm{d}W(t), \tag{16}$$

where $W(t)$ denotes a unit Wiener process, $m(E)$ and $\sigma(E)$ are the mean drift coefficient and the mean diffusion coefficient, respectively, which can be obtained as follows:

$$m(E) = \langle\sqrt{2E}\sin\phi h(E,\phi)\rangle_t + \left\langle\int_{-\infty}^{0}(\sin\phi(t+\mu)\sin\phi(t) + \cos\phi(t+\mu)\cos\phi(t))R(\mu)\mathrm{d}\mu\right\rangle_t, \tag{17a}$$

$$\sigma^2(E) = 2E\left\langle\int_{-\infty}^{\infty}\sin\phi(t+\mu)\sin\phi(t)R(\mu)\mathrm{d}\mu\right\rangle_t. \tag{17b}$$

In Eqs. (17a) and (17b), $\langle\cdot\rangle_t$ denotes the time averaging which can be carried out over one quasi-period T as follows:

$$\langle[\cdot]\rangle_t = \frac{1}{T}\int_0^T[\cdot]\,\mathrm{d}t, \tag{18}$$

and $R(\mu)$ is the autocorrelation function of $\xi(t)$. For a given E, y and $\dot{y}$ are treated as periodic functions with period T, which can be obtained by solving Eq. (8). So $\sin\phi(t)$ and $\cos\phi(t)$, being functions of y and $\dot{y}$, can be expanded in Fourier series as follows:[7]

$$\sin\phi = \sum_{n=1}^{\infty}a_n\sin[(2n-1)w_Tt], \tag{19a}$$

$$\cos\phi = \sum_{n=1}^{\infty}b_n\cos[(2n-1)w_Tt], \tag{19b}$$

where $w_T = 2\pi/T$, and the coefficients a_n and b_n are

$$a_n = \frac{2}{T}\int_0^T\sin\phi(t)\sin[(2n-1)w_Tt]\,\mathrm{d}t = -\frac{2}{\sqrt{2E}T_{1/4}}\int_0^{T_{1/4}}\dot{y}\sin[(2n-1)w_Tt]\,\mathrm{d}t, \tag{20a}$$

$$b_n = \frac{2}{T}\int_0^T\cos\phi(t)\cos[(2n-1)w_Tt]\,\mathrm{d}t = \frac{1}{\sqrt{E}T_{1/4}}\int_0^{T_{1/4}}y\sqrt{2a+by^2}\cos[(2n-1)w_Tt]\,\mathrm{d}t. \tag{20b}$$

By substituting Eqs. (19a) and (19b) into Eqs. (17a) and (17b), the mean drift coefficient and the mean diffusion coefficient can be written as

$$m(E) = \frac{1}{T}\int_0^T(-\dot{y}h(y,\dot{y}))\,\mathrm{d}t + \frac{\pi}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\Phi_{11}^{(1)}[(2n-1)w_T] = -\frac{1}{T_{1/4}}\left[\int_0^{T_{1/4}}(c_2y^2-c_1)\dot{y}^2\,\mathrm{d}t + (1-r)E\right] + \frac{\pi}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\Phi_{11}^{(1)}[(2n-1)w_T], \tag{21a}$$

$$\sigma^2(E) = 2\pi E\sum_{n=1}^{\infty}a_n^2\Phi_{11}^{(2)}[(2n-1)w_T], \tag{21b}$$

where $\Phi_{11}^{(1)}(w)$ and $\Phi_{11}^{(2)}(w)$ are defined as

$$\Phi_{11}^{(1)}(w) = \frac{1}{\pi}\int_{-\infty}^{0}\cos(w\mu)R(\mu)\,\mathrm{d}\mu, \tag{22a}$$

$$\Phi_{11}^{(2)}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(w\mu)R(\mu)\,\mathrm{d}\mu. \tag{22b}$$

In this paper, $\Phi_{11}^{(1)}(w)$ is equal to $\Phi_{11}^{(2)}(w)$, which is the spectral density of $\xi(t)$.

Based on the knowledge of the drift and diffusion coefficients, the FPK equation [35] for $E(t)$ is

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial E}[m(E)p(E,t|E_0,t_0)] - \frac{1}{2}\frac{\partial^2}{\partial E^2}[\sigma^2(E)p(E,t|E_0,t_0)] = 0, \tag{23}$$

where $p(E,t|E_0,t_0)$ represents the transition probability density for $E(t)$. The stationary probability density $p(E)$ can be solved from the reduced FPK equation

$$\frac{\partial p}{\partial E}[m(E)p(E)] - \frac{1}{2}\frac{\partial^2}{\partial E^2}[\sigma^2(E)p(E)] = 0. \tag{24}$$

The solution of Eq. (24) is[4,36]

$$p(E) = \frac{c}{\sigma^2(E)}\exp\left(2\int_0^E\frac{m(e)}{\sigma^2(e)}\mathrm{d}e\right), \tag{25}$$

where c is a normalization constant. The joint stationary PDF of the transformed variables y and $\dot{y}$ can be obtained through Eq. (14) as [37]

$$\widetilde{p}_{Y_1,Y_2}(y,\dot{y}) = \frac{p(E)}{T(E)}\Big|_{E=\frac{1}{2}\dot{y}^2+\frac{a}{2}y^2+\frac{b}{4}y^4}. \tag{26}$$

Then the joint stationary PDF of the original displacement x_1 and velocity x_2 is obtained by using the inverse transformation of Eq. (3) as

$$p(x_1,x_2) = 2\widetilde{p}_{Y_1,Y_2}(x_1,x_2), \quad x_1 \geq 0. \tag{27}$$

The corresponding stationary PDFs of displacement x_1 and velocity x_2, which mean the marginal distribution of Eq. (27), can be derived as

$$p_{X_1}(x_1) = \int_{-\infty}^{+\infty} p(x_1, s)\,\mathrm{d}s, \tag{28}$$

$$p_{X_2}(x_2) = \int_{-\infty}^{+\infty} p(s, x_2)\,\mathrm{d}s. \tag{29}$$

4. Analysis of stationary PDFs

To illustrate the validity of the aforementioned method, the analytical stationary PDFs and the numerical simulation results will be shown in this section. Before continuing this section, it is worth explaining how to obtain the following results. First, we instruct how to calculate drift and diffusion coefficients. For a given E, we first calculate y and $\dot{y}$ in $0 \le t \le T_{1/4}$ from Eq. (8), then use the results to obtain the Fourier coefficients from Eq. (20), and finally obtain the mean drift coefficient and the mean diffusion coefficient from Eq. (21). The Fourier coefficients a_n and b_n ($n = 1, 2, 3$) are shown in Fig. 1. The remaining Fourier coefficients are found to be close to zero. This means that the first two terms are the main part of the Fourier series. Thus three terms may be retained in the Fourier expansions to calculate the drift and diffusion coefficients. Second, we illustrate the methods used to obtain the numerical solutions. The numerical solutions of the vibro-impact system (1) can be obtained by using the second-order Runge–Kutta algorithm.[38] Two different step sizes are selected to improve the computation precision in the integration process. An appropriately larger time step Δt is chosen until the condition $x \le 0$ is satisfied. Once $x \le 0$, we move the system state by a larger time step backward, and repeat the integration by reducing the step size to a pre-assumed precision $\widetilde{\Delta t} = t/N$. In this case, we believe that the time is instant impact when the system state is less than zero. Then the larger time step is recovered after time interval Δt. The response PDFs can be obtained by the Monte Carlo simulation method with simulation data length 6×10^8.

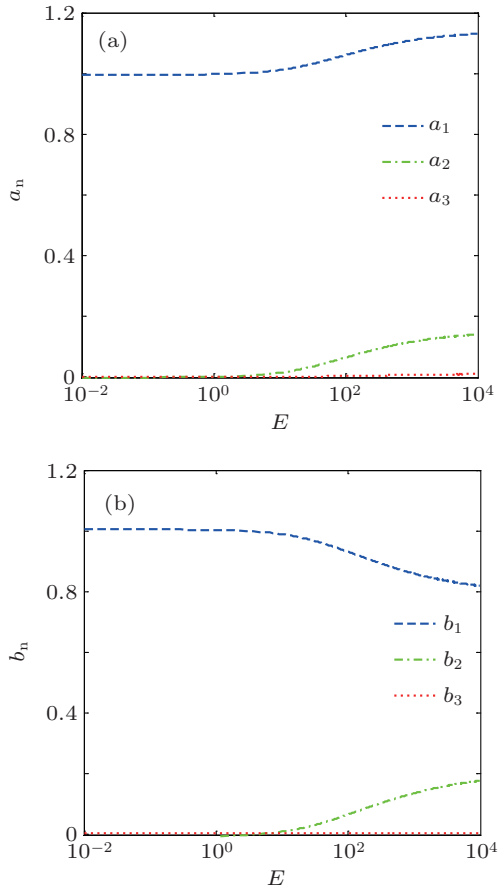


Fig. 1. (color online) Fourier coefficients versus energy level E for the case of $a = 1.0$, $b = 0.01$, $c_1 = 0.01$, $c_2 = 0.01$, $r = 0.98$, $D = 0.01$, and $\tau = 0.01$: (a) a_n, (b) b_n, $(n = 1, 2, 3)$.

4.1. Effectiveness of the method

In this subsection, the validity of the present method is tested through comparing the analytical results with the numerical results. When the parameters are fixed at $a = 1.0$, $b = 0.01$, $c_1 = 0.01$, and $r = 0.98$, the effects of other parameters are considered. Figure 2 shows the stationary PDFs of total energy E, displacement x_1, and velocity x_2 for two different values of coefficient c_2. Figure 3 indicates the PDFs with different noise intensities. Figure 4 displays the effect of the correlation time. These figures demonstrate the good agreement between the numerical and the analytical results. According to Fig. 2, it is clear that the peaks of all stationary PDFs appear at the constrained positions, and the increase of c_2 can lead to higher peaks of stationary PDFs of total energy E, displacement x_1, and velocity x_2. The higher peak of the stationary PDF means the larger probability that the system will stay close to the equilibrium state. However, with the increase of noise intensity, Fig. 3 shows lower peaks of stationary PDFs. Figures 2 and 3 indicate that the influences of c_2 and D on the stationary PDF are different. As can be seen from Fig. 4, the PDF peaks first decrease slightly and then increase with correlation time increasing. Additionally, it is interesting to note that the peaks of displacement PDFs are almost twice as high as the peaks of velocity PDFs and the velocity PDFs are symmetric.

4.2. Stochastic bifurcations

Figures 2–4 show the agreement between the analytical results and the simulation results, however no bifurcations occur when changing parameters c_2, D, and τ. In this subsection, we focus on stochastic bifurcations under the effects of different parameters. For this purpose, the influences of c_1 and r will be considered.

In order to investigate the stochastic bifurcation, parameters $a = 1.0$, $b = 0.01$, $c_1 = 0.01$, $c_2 = 0.01$, $r = 0.98$, $D = 0.01$, and $\tau = 0.01$ are chosen as the original values. The corresponding analytical and numerical results of joint PDFs

are shown in Fig. 5. First, the damping coefficient c_1 is investigated with $r = 0.98$. Figure 6 provides analytical and numerical results of the joint PDFs when $c_1 = 0.02$. Figure 5 has a singular peak at the equilibrium position, however the PDF in Fig. 6 presents the shape of a crater. These figures demonstrate the occurrence of the stochastic P-bifurcation according to the concept of stochastic bifurcation[39,40] when parameter c_1 increases from 0.01 to 0.02. Second, the effect of the restitute coefficient is explored when $c_1 = 0.01$. Similar to the above analysis, Fig. 7 reveals the variation of joint PDFs when $r = 0.99$. Figure 7 illustrates the same shape as Fig. 6. Therefore, the stochastic P-bifurcation exists when r changes from 0.98 to 0.99.

To illustrate the procedure of the stochastic P-bifurcation, Fig. 8 shows the sections of analytical results of joint PDFs on the surface $x_1 = 0.2$ about different bifurcation parameters. Figures 8(a) and 8(b) clearly show the evolutions of the curves from one peak to two peaks with the bifurcation parameters increasing. The similarity between Figs. 6 and 7 indicates that the influences of r and c_1 on the stochastic P-bifurcation are similar.

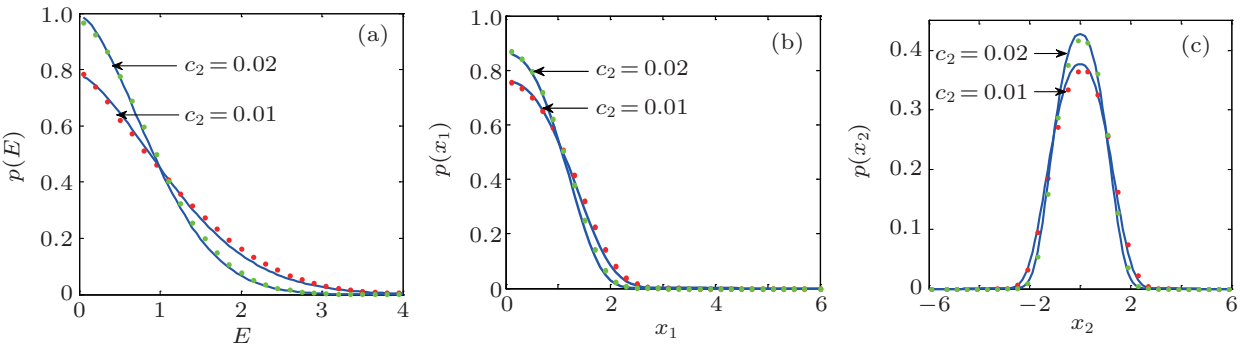


Fig. 2. (color online) Stationary PDFs obtained analytically and numerically for different c_2 in the case of $a = 1.0$, $b = 0.01$, $c_1 = 0.01$, $r = 0.98$, $D = 0.01$, and $\tau = 0.01$: (a) PDFs of total energy E; (b) PDFs of displacement x_1; (c) PDFs of velocity x_2. Here lines are analytical results; dots are Monte–Carlo results.

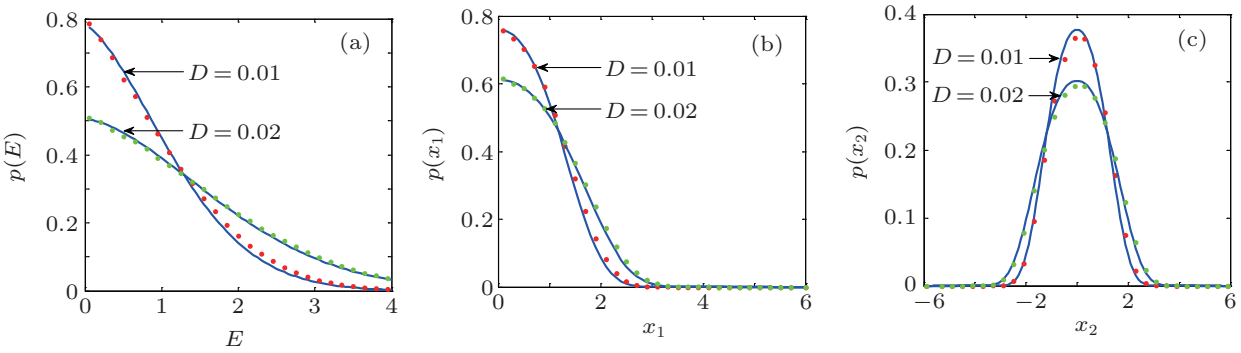


Fig. 3. (color online) Stationary PDFs obtained analytically and numerically for different noise intensity D in the case of $a = 1.0$, $b = 0.01$, $c_1 = 0.01$, $c_2 = 0.01$, $r = 0.98$, and $\tau = 0.01$: (a) PDFs of total energy E; (b) PDFs of displacement x_1; (c) PDFs of velocity x_2. Here lines are analytical results; dots are Monte–Carlo results.

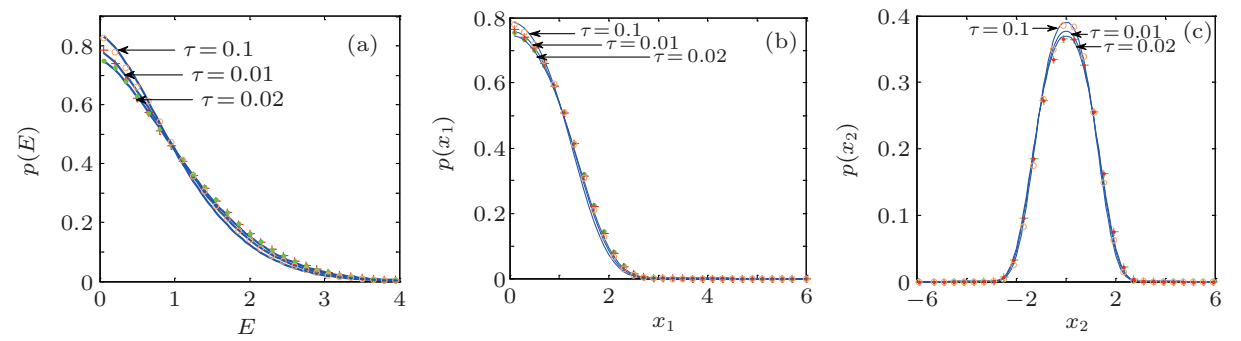


Fig. 4. (color online) Stationary PDFs obtained analytically and numerically for different correlation time τ in the case of $a = 1.0$, $b = 0.01$, $c_1 = 0.01$, $c_2 = 0.01$, $r = 0.98$, and $D = 0.01$: (a) PDFs of total energy E; (b) PDFs of displacement x_1; (c) PDFs of velocity x_2. Here lines are analytical results; symbols are Monte–Carlo results.

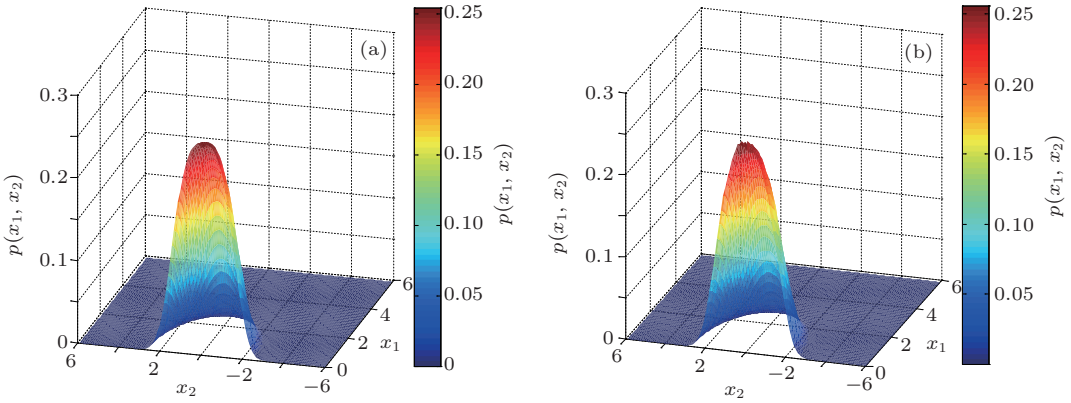


Fig. 5. (color online) Joint PDFs of displacement x_1 and velocity x_2: (a) analytical result; (b) Monte-Carlo result.

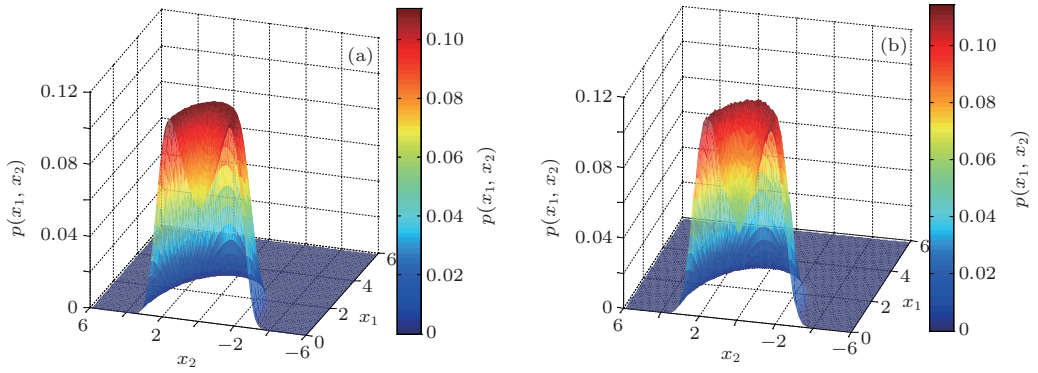


Fig. 6. (color online) Joint PDFs of displacement x_1 and velocity x_2 for $c_1 = 0.02$: (a) analytical result; (b) Monte-Carlo result.

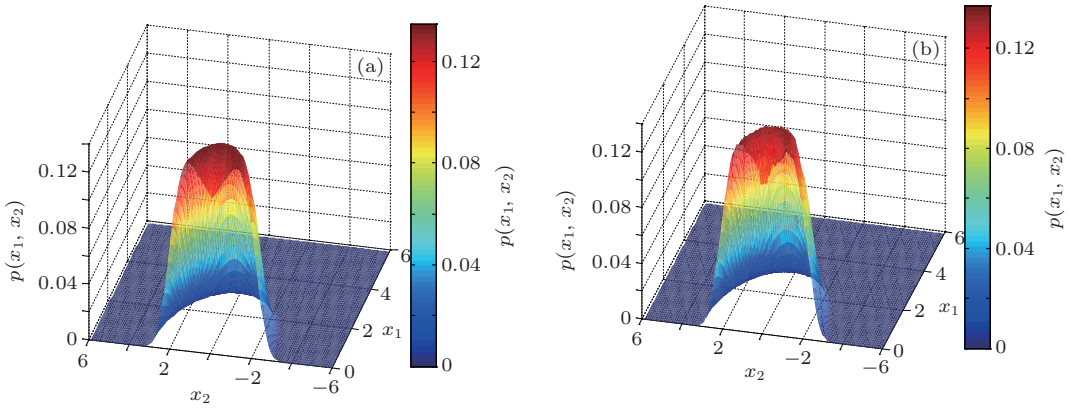


Fig. 7. (color online) Joint PDFs of displacement x_1 and velocity x_2 for $r = 0.99$: (a) analytical result; (b) Monte-Carlo result.

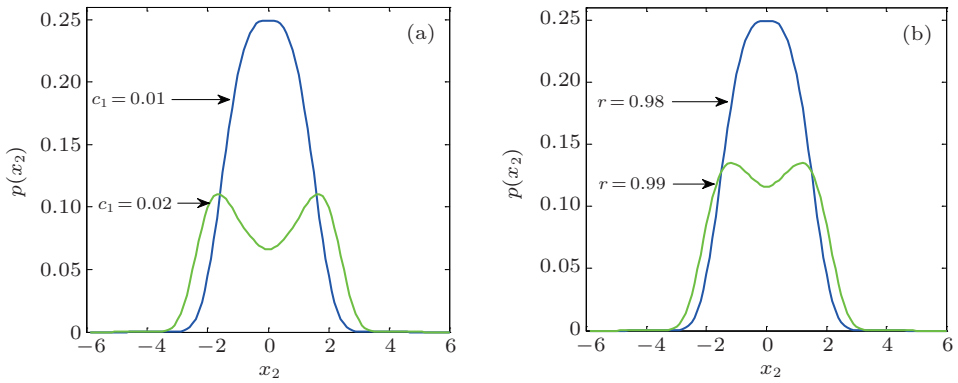


Fig. 8. (color online) Section graphs of joint PDFs on the surface $x_1 = 0.2$ for different bifurcation parameters at (a) different c_1 and (b) different r.

5. Conclusion

An effective approach is presented for the stationary PDFs of a single degree of freedom vibro-impact system under colored noise excitation. The non-smooth transformation and the modified quasi-conservative stochastic averaging are used to obtain the approximate analytical results of the response PDFs. The validity of the method is verified by the numerical simulations. The stochastic bifurcations of the vibro-impact system are also explored.

References

[1] Stratonovitch R L 1963 *Topics in the Theory of Random Noise* (Vol. **1**) (New York: Gordon and Breach)
[2] Khasminskii R Z 1966 *Theory of Probability & Its Applications* **11** 390
[3] Zhu W Q 1992 *Random Vibration* (Beijing: Science Press) (in Chinese)
[4] Lin Y K and Cai G Q 1995 *Probabilistic Structural Dynamics* (New York: McGraw-Hill)
[5] Roberts J B and Spanos P D 1986 *Int. J. Non-Linear Mech.* **21** 111
[6] Zhu W Q and Cai G Q 2002 *Acta Mech. Sin.* **18** 551
[7] Roberts J B 1983 *IUTAM Symposium on Random Vibrations and Reliability* October 31–November 6, 1982 Frankfurt, Germany, p. 285
[8] Dimentberg M, Cai G Q and Lin Y K 1995 *Int. J. Non-Linear Mech.* **30** 677
[9] Zeng Y and Zhu W Q 2010 *Int. J. Non-Linear Mech.* **45** 572
[10] Xu Y, Gu R C, Zhang H Q, Xu W and Duan J Q 2011 *Phys. Rev. E* **83** 056215
[11] Brogliatoa B 1999 *Nonsmooth Mechanics: Models, Dynamics and Control* (London: Springer)
[12] Bernardo M di, Budd C J, Champneys A R and Kowalczyk P 2007 *Piecewise-smooth Dynamical Systems: Theory and Applications* (London: Springer)
[13] Chin W, Ott E, Nusse H E and Grebogi C 1994 *Phys. Rev. E* **50** 4427
[14] Weger J D, Binks D and Molenaar J 1996 *Phys. Rev. Lett.* **76** 3951
[15] Luo G W, Chu Y D, Zhang Y L and Zhang J G 2006 *J. Sound Vib.* **298** 154
[16] Budd C and Dux F 1994 *Phil. Trans. R. Soc. Lond. A* **347** 365
[17] Wagg D J and Bishop S R 2001 *Int. J. Bifurcation Chaos* **11** 57
[18] Wagg D J 2004 *Chaos Soliton. Fract.* **22** 541
[19] Feng J Q, Xu W and Niu Y J 2010 *Acta Phys. Sin.* **59** 157 (in Chinese)
[20] Huang Z L, Liu Z H and Zhu W Q 2004 *J. Sound Vib.* **275** 223
[21] Feng J Q, Xu W and Wang R 2008 *J. Sound Vib.* **309** 730
[22] Feng J Q, Xu W, Rong H W and Wang R 2009 *Int. J. Non-Linear Mech.* **44** 51
[23] Rong H W, Wang X D, Xu W and Fang T 2009 *J. Sound Vib.* **327** 173
[24] Su M B and Rong H W 2011 *Chin. Phys. B* **20** 060501
[25] Roberts J B 1978 *J. Sound Vib.* **60** 177
[26] Cai G Q 1995 *J. Eng. Mech.* **121** 633
[27] Xie W X, Xu W and Cai L 2006 *Acta Phys. Sin.* **55** 1639 (in Chinese)
[28] Gan Z N, Ma J, Zhang G Y and Chen Y 2008 *Chin. Phys. B* **17** 4047
[29] Wang B, Wu X Q and Shao J H 2009 *Acta Phys. Sin.* **58** 1391 (in Chinese)
[30] Yang J H and Liu X B 2010 *Chin. Phys. B* **19** 050504
[31] Zhang X Y, Xu W and Zhou B C 2011 *Acta Phys. Sin.* **60** 060514 (in Chinese)
[32] Xiao R, Wang C J and Zhang L 2012 *Chin. Phys. B* **21** 110504
[33] Zhuravlev V F 1976 *Mech. Solids* **11** 23
[34] Dimentberg M F and Iourtchenko D V 2004 *Nonlinear Dyn.* **36** 229
[35] Gardiner C W 2004 *Handbook of Stochastic Methods* (New York: Springer)
[36] Zhu W Q 2003 *Nonlinear Stochastic Dynamics and Control: Framework of Hamiltonian Theory* (Beijing: Science Press) (in Chinese)
[37] Krenk S and Roberts J B 1999 *J. Appl. Mech.* **66** 225
[38] Zhang H Q, Xu W and Xu Y 2009 *Physica A* **388** 781
[39] Ludwig Arnold 1998 *Random Dynamical Systems* (New York: Springer)
[40] Xu W, He Q, Rong H W and Fang T 2003 *Proceedings of the Fifth International Conference on stochastic Structural Dynamics-SSD03* May 26–28, 2003 Hangzhou, China, p. 509