Finite-time consensus for leader-following multi-agent systems over switching network topologies*

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Finite-time consensus problem of the leader-following multi-agent system under switching network topologies is studied in this paper. Based on the graph theory, matrix theory, homogeneity with dilation, and LaSalle's invariance principle, the control protocol of each agent using local information is designed, and the detailed analysis of the leader-following finite-time consensus is provided. Some examples and simulation results are given to illustrate the effectiveness of the obtained theoretical results.

Keywords: finite-time consensus, leader-following multi-agent system

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1. Introduction

The consensus problem of multi-agent systems has attracted great attention in many fields, such as biology, physics, robotics, and control engineering. It also has broad applications, such as aggregation behavior of animals[1] and formation control.[2,3] As a type of collective behavior, the consensus of multiple dynamic agents means that the states of agents reach agreement on a common quantity by implementing an appropriate consensus protocol. The consensus problem has received considerable research interest.[4–19] Olfati-Saber and Murray[7] presented a systematic framework to analyze the consensus algorithms. They proposed that the consensus problem could be solved if the diagraph is strongly connected. Ren and Beard[15] further proved that the consensus could be achieved if the union of the dynamically changing interaction graphs has a directed spanning tree frequently enough as the system evolves. Sun *et al.*[16] discussed the average consensus problem of dynamic agents with multiple time-varying communication delays. Lu *et al.*[19] studied the consensus problem over directed networks with arbitrary finite communication delays and nonlinear couplings.

The consensus state and the convergence rate are crucial factors in the consensus problems. The two topics deal with where the multiple dynamic agents reach and how quickly they reach a consensus. A particularly interesting topic is the consensus problem of a group of agents with a leader, where the leader is a special agent whose motion is independent of the others and followed by the others. Such a problem is generally called leader-following consensus problem.[20–23] When an agent is a particle moving under the control of Newton's law, the consensus of leader-following multi-agent systems is considered in Refs. [22] and [23]. Hong *et al.*[22] proposed a distributed control law using local information. Cheng *et al.*[23] provided a rigorous proof for the consensus problem by the extension of LaSalle's invariance principle.

The convergence speed is another active topic of the consensus problem, and it even can reflect the performance of the proposed consensus protocol. Much attention has been focused on how to enhance the convergence speed of the multi-agent systems.[7,24,25] Olfati-Saber[24] studied the ultrafast consensus problem in small-world networks. They found that the second smallest eigenvalue of a regular network can be greatly increased via changing the inter-agent information flow without increasing the total number of network links. Zhou *et al.*[25] characterized the convergence speed for the distributed discrete-time consensus algorithm over a variety of random networks with arbitrary weights. They proposed the asymptotic and per-step convergence factors as measures of the convergence speed and derived the exact value for the per-step convergence factor. Though most of those results were to choose proper interaction graphs, they have not found available protocols with high performance. In other words, although by maximizing the algebraic connectivity of the interaction graph one can increase convergence speed with respect to the linear protocol proposed in Ref. [7], the consensus state can never be reached in finite time. However, it is often required that the consensus can be achieved in finite time in many situations, for example, when the control accuracy is crucial.

There are some results related to finite-time consensus.[26–29] Qing *et al.*[26] developed a framework to design semi-consensus and finite-time semi-consensus proto-

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cols for dynamical networks with switching topologies. Jiang *et al.*[28] established the explicit expression of the consensus state for the entire group analytically. They proved that, in the scenarios with fixed and switching undirected topologies, the agents of the group under a particular type of nonlinear interaction can reach the consensus state in a finite time. Using the theory of finite-time stability, Wang *et al.*[29] proved that, if the sum of time intervals, in which the interaction topology is connected, is sufficiently large, the proposed protocols could solve the finite-time consensus problems. In a network of mobile agents, some of the existing communication links may fail due to the existence of an obstacle between two agents. Because of those problems, this paper studies the finite-time consensus for the leader-following multi-agent system over balanced and switching networks, which is both theoretically interesting and practically significant.

The rest of this paper is organized as follows. The necessary graph notions, preliminary results, and the problem formation are described in Section 2. Finite-time consensus protocols for multi-agent systems over balanced and switching networks are proposed in Section 3. Some examples and simulation results are given in Section 4 to illustrate the effectiveness of the obtained theoretical results, and the conclusions are drawn in Section 5.

2. Preliminaries

Throughout this paper, $\boldsymbol{R}^n$ denotes the set of n-dimensional real column vectors, $\boldsymbol{I}_n$ denotes the n-dimensional identity matrix, $\mathbf{1} = (1,1,\ldots,1)^{\mathrm{T}}$ is an n-dimensional column vector, and $\mathbf{0}$ denotes a zero vector or zero matrix with an appropriate dimension.

Let $\boldsymbol{G} = \{\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{A}\}$ be a weighted graph with a node set $\boldsymbol{V} = \{v_1, v_2, \ldots, v_n\}$, where node $i \in I$ represents the i-th agent, and the finite index set $\boldsymbol{I} = \{1,2,\ldots,n\}$ is the node indexes of $\boldsymbol{G}$. $\boldsymbol{E} \subseteq \boldsymbol{V} \times \boldsymbol{V}$ is the set of edges, whose elements denote the directed communication links between the agents. An edge in $\boldsymbol{G}$ is denoted by an ordered pair (v_i, v_j). $(v_i, v_j) \in \boldsymbol{E}$ if and only if the j-th agent can send information to the i-th agent directly. In this case, v_j is called the parent of v_i, and v_i is called the child of v_j. The neighborhood of the i-th agent is denoted by $N_i = \{v_j \in V | (v_i, v_j) \in \boldsymbol{E}\}$, which is the set of all the parents of v_i. The node v_i is called a source if it has no parent but only children. $\boldsymbol{A} = [a_{ij}]$ is the weighted adjacency matrix of a weighted graph with the nonnegative adjacency elements $a_{ij} \geq 0$, $\forall\, i, j \in I$. Moreover, $a_{ij} > 0$ if $(v_j, v_i) \in \boldsymbol{E}$; $a_{ij} = 0$ if $(v_j, v_i) \notin \boldsymbol{E}$, and $a_{ii} = 0$ for all $i \in \boldsymbol{I}$. A path between two distinct nodes v_i and v_j means a sequence of distinct edges of the form $(v_i, v_{k_1}), (v_{k_1}, v_{k_2}), \ldots, (v_{k_l}, v_j)$. A graph is said to be connected, if there is a path between any two distinct nodes of the graph. In addition, a graph is called balanced if and only if $\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij}$, $\forall\, i, j \in I$. Obviously, an undirected graph is certainly balanced. For the leader-following multi-agent systems, we consider another graph $\bar{\boldsymbol{G}} = \{\bar{\boldsymbol{V}}, \bar{\boldsymbol{E}}, \bar{\boldsymbol{A}}\}$ associated with the system consisting of n agents (called followers) and one leader denoted by $\mathbf{0}$, where $\bar{\boldsymbol{V}} = \{\boldsymbol{V}, v_0\}$. Suppose that vector $\boldsymbol{P} = (p_1, p_2, \ldots, p_n)$ with $p_i > 0$ if agent i is a neighbor of the leader and otherwise $p_i = 0$ $(i \in I)$. Assume that $p_i = 1$ once the i-th agent is a neighbor of the leader. If there is a path in $\bar{\boldsymbol{G}}$ from every other node to node 0, the node 0 is said to be globally reachable in $\bar{\boldsymbol{G}}$, which is much weaker than strong connection.

Before giving the main results, some important lemmas are given first for the further analysis.

Lemma 1 (LaSalle's Invariance Principle[30]) Let $\boldsymbol{x}(t)$ be a solution of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x}(0) = x_0 \in \boldsymbol{R}^n$, where $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{R}^n$ is continuous on the open subset $\boldsymbol{U} \subseteq \boldsymbol{R}^n$, and $\boldsymbol{V} : \boldsymbol{U} \to \boldsymbol{R}$ is a local Lipschitz function such that $\boldsymbol{D}^+\boldsymbol{V}(\boldsymbol{x}(t)) \leq 0$, where $\boldsymbol{D}^+$ denotes the upper Dini derivative. Suppose that $\boldsymbol{S} = \{\boldsymbol{x} \in \boldsymbol{U} : \boldsymbol{D}^+\boldsymbol{V}(\boldsymbol{x}) = 0\}$, and $\boldsymbol{\Theta}^+(x_0)$ is the positive limit set. Then $\boldsymbol{\Theta}^+(x_0) \cap \boldsymbol{U}$ is contained in the union of all solutions that remain in $\boldsymbol{S}$.

Next, the homogeneity with dilation (see Ref. [31] for details) is given for the finite-time convergence analysis.

A function $\boldsymbol{V}(x)$ is homogeneous of degree $\kappa \in \boldsymbol{R}$ with dilation $(r_1, r_2, \ldots, r_n)$ $(r_i > 0,\ i = 1,2,\ldots,n)$ if, for any $\varepsilon > 0$,

$$\boldsymbol{V}(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^{\kappa}\boldsymbol{V}(x).$$

For the n-dimensional system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(x), \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \boldsymbol{R}^n, \tag{1}$$

a continuous vector field $\boldsymbol{f}(x) = (f_1(x), f_2(x), \ldots, f_n(x))^{\mathrm{T}}$ is homogeneous of degree $\kappa \in \boldsymbol{R}$ with dilation $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$, if, for any $\varepsilon > 0$,

$$\boldsymbol{f}_i(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^{\kappa + r_i}\boldsymbol{f}_i(x),\ i = 1,2,\ldots,n.$$

System (1) is called homogeneous if its vector field is homogeneous. Furthermore, the following system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(x) + \tilde{\boldsymbol{f}}(x), \quad \tilde{\boldsymbol{f}}(0) = \mathbf{0}, \quad \boldsymbol{x} \in \boldsymbol{R}^n, \tag{2}$$

is said to be locally homogeneous of degree $\kappa \in \boldsymbol{R}$ with respect to the dilation $(r_1, r_2, \ldots, r_n)$, if $\boldsymbol{f}(x)$ is homogeneous of degree $\kappa \in R$ with respect to the dilation $(r_1, r_2, \ldots, r_n)$ and $\tilde{\boldsymbol{f}}$ is a continuous vector field satisfying

$$\lim_{\varepsilon \to 0} \frac{\tilde{f}_i(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \ldots, \varepsilon^{r_n}x_n)}{\varepsilon^{\kappa + r_i}} = 0,\ \forall\ x \neq 0,\ \ i \in I. \tag{3}$$

Let $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in \boldsymbol{R}^n$ and $\mathrm{sig}(\boldsymbol{x})^{\alpha} = |x|^{\alpha}\mathrm{sign(x)}$, where $\mathrm{sign}(\cdot)$ denotes the sign function and $|x|$ denotes the absolute value of the real number x.

Lemma 2[27] Suppose that system (1) is homogeneous of degree $\kappa \in \boldsymbol{R}$ with dilation $(r_1, r_2, \ldots, r_n)$, the function $\boldsymbol{f}(x)$

is continuous and $x=0$ is its asymptotically stable equilibrium. If homogeneity degree $\kappa<0$, the equilibrium of system (1) is finite-time stable. Moreover, if Eq. (3) holds, then the equilibrium of system (2) is locally finite-time stable.

Consider the following continuous-time multi-agent system:

$$\dot{x}_i(t)=u_i(t),\quad i\in I, \tag{4}$$

where $x_i(t)\in\boldsymbol{R}$ is the state and $u_i(t)\in\boldsymbol{R}$ is the control input of the i-th agent. The state of the virtual leader of system (4) is $x_0(t)\in\boldsymbol{R}$. Assume that the state of the leader is invariant. That is, $x_0(t)=x_0(0)$ for all $t\geq 0$.

Definition 1 The leader-following multi-agent system (4) is said to achieve the finite-time consensus, if for any initial states, there is a setting time $T_0\in[0,+\infty)$, such that the solution of system (4) satisfies

$$\lim_{t\to T_0}\|x_i(t)-x_0(t)\|=0,\quad x_i(t)=x_0(t),\ \forall\ t\geq T_0,\ \ i\in I.$$

3. Main Results

In this section, the consensus protocols are used to solve the finite-time consensus problems for the leader-following multi-agent systems over balanced and switching network topologies.

3.1. Finite-time consensus over balanced network topology

For system (4) with balanced network topologies, we design the following protocol:

$$\begin{aligned}u_i(t)\ =\ &\sum_{v_j\in N_i(t)}a_{ij}(t)\varphi_1(\mathrm{sig}(x_j(t)-x_i(t))^\alpha)\\&-p_i(t)\varphi_2(\mathrm{sig}(x_i(t)-x_0(t))^\alpha),\ \ i\in I,\end{aligned} \tag{5}$$

where $a_{ij}(t)$ is the adjacency weight between agent i and agent j, and $p_i(t)$ is the adjacency weight between the leader and the i-th agent at time instant t; φ_k is a continuous odd function satisfying $x\varphi_k(x)>0$ $(\forall\ x\neq 0)$ and $\varphi_k(x)=c_kx+o(x)$ around $x=0$ for some $c_k>0$, $k=1,2$.

Theorem 1 Suppose that the network graph of the multi-agent system (4) is balanced and the leader is globally reachable. Then, under protocol (5) with $0<\alpha<1$, φ_k being a continuous odd function satisfying $x\varphi_k(x)>0$ $(\forall\ x\neq 0)$, and $\varphi_k(x)=c_kx+o(x)$ around $x=0$ for some constant $c_k>0$ $(k=1,2)$, the leader-following finite-time consensus is achieved.

Proof Let $\delta_i(t)=x_i(t)-x_0(t)$. Then, under protocol (1), system (4) can be written as

$$\begin{aligned}\dot{\delta}_i(t)\ =\ u_i(t)=&\sum_{v_j\in N_i}a_{ij}\varphi_1(\mathrm{sig}(\delta_j(t)-\delta_i(t))^\alpha)\\&-p_i\varphi_2(\mathrm{sig}(\delta_i(t))^\alpha),\quad i\in I.\end{aligned} \tag{6}$$

Choose Lyapunov function $V=\frac{1}{2}\sum\limits_{i=1}^{n}\delta_i^2$. Then along the trajectory of system (6),

$$\begin{aligned}\dot{V}\ =&\sum_{i=1}^{n}\delta_i\dot{\delta}_i=\sum_{i=1}^{n}\delta_i\Big[\sum_{v_j\in N_i}a_{ij}\varphi_1(\mathrm{sig}(\delta_j-\delta_i)^\alpha)\\&-p_i\varphi_2(\mathrm{sig}(\delta_i)^\alpha)\Big]\\=&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\delta_i-\delta_j)\varphi_1(\mathrm{sig}(\delta_j-\delta_i)^\alpha)\\&-\sum_{i=1}^{n}p_i\delta_i\varphi_2(\mathrm{sig}(\delta_i)^\alpha)\\\leq&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\delta_i-\delta_j)\varphi_1(\mathrm{sig}(\delta_j-\delta_i)^\alpha)\\\leq&\ 0,\quad i\in I.\end{aligned}$$

The first inequality comes from the fact that node 0 is globally reachable, i.e., there is at least one $p_i>0$. Note that $\dot{V}=0$ if and only if $\delta_i=\delta_j$, i.e., $x_i=x_j$ for all $i\neq j$. By Lemma 1, $x_1=x_2=\cdots=x_n=x_0$. That is, $\dot{V}=0$ if and only if $\delta_i=\delta_j=0$. Therefore, the origin of system (6) is globally asymptotically stable.

According to the assumptions given to the odd function $\varphi_k(x)=c_kx+o(x)$, $k=1,2$, system (2) can be written as

$$\begin{aligned}\dot{\delta}_i(t)\ =&\ c_1\sum_{v_j\in N_i}a_{ij}\mathrm{sig}(\delta_j-\delta_i)^\alpha+\sum_{v_j\in N_i}a_{ij}o(\mathrm{sig}(\delta_j-\delta_i)^\alpha)\\&-p_i[c_2\mathrm{sig}(\delta_i)^\alpha+o(\mathrm{sig}(\delta_i)^\alpha)]\\=&\ \tilde{\zeta}_i+\bar{\zeta}_i,\end{aligned} \tag{7}$$

where

$$\begin{aligned}\tilde{\zeta}_i&=c_1\sum_{v_j\in N_i}a_{ij}\mathrm{sig}(\delta_j-\delta_i)^\alpha-c_2p_i\mathrm{sig}(\delta_i)^\alpha,\\\bar{\zeta}_i&=\sum_{v_j\in N_i}a_{ij}o(\mathrm{sig}(\delta_j-\delta_i)^\alpha)-p_io(\mathrm{sig}(\delta_i)^\alpha).\end{aligned}$$

Note that when $\bar{\zeta}_i\equiv 0$, system (7) with variables $(\delta_1,\delta_2,\ldots,\delta_n)$ is homogeneous of the degree $\kappa=\alpha-1<0$. Thus, it is locally homogeneous with the degree $k<0$. Consider the derivative of the function V along the trajectory of system (6). Next from the similar arguments as above, one can find the Lyapunov stability of the invariant manifold $\{x_i=x_0,\ \forall\ i\in I\}$ for system (4), which implies the Lyapunov stability of system (6). Hence, the origin of system (6) is globally asymptotically stable.

From the discussion above, system (6) is globally asymptotically stable and locally homogeneous with the degree $k<0$. According to Lemma 2, the origin is a finite-time stable equilibrium of system (6). Note that if the equilibrium of a nonlinear system is globally asymptotically stable and locally finite-time convergent, then it is globally finite-time stable. The reason is that globally asymptotical stability implies finite-time convergence to any given bounded neighborhood of

the equilibrium. Based on this theory, the origin is a globally finite-time stable equilibrium of system (6), which implies that $x_i - x_0 \to 0$ $(\forall\, i \in I)$ in a finite time. This completes the proof.

3.2. Finite-time consensus over switching network topology

In a network of mobile agents, some of the existing communication links may fail due to the existence of an obstacle between two agents. On the other hand, some new links between the nearby agents might be created because the agents come to an effective range of the detection with respect to each other. That is, the topology of such network might be switched or time-variant. Therefore, it is necessary and significant to study the finite-time consensus of the system with switching network topology. For simplicity, we investigate the network over switching topology and time-invariant weights of all the interaction links. That is, at any time when the agent j is the neighbor of the agent i, the connection weight $a_{ij} > 0$ is a constant.

Let $\boldsymbol{G}_\sigma = \{\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{A}_{\sigma(t)}\}$ be a collection of balanced graph with n vertices. Note that $\boldsymbol{G}_\sigma$ is a finite set if all the weights are chosen from a finite set. The finite set $\Gamma = \{0, 1, 2, \ldots, N\}$ is the index set of graph $\boldsymbol{G}_\sigma$. We introduce a switching signal $\sigma(t) : R^+ \to \Gamma$ and a switching time sequence $t_0 = 0, t_1, \ldots, t_s, \ldots$, at which the interaction topology changes. For any $t \in [t_s, t_{s+1})$, the topology graph $\boldsymbol{G}_{\sigma(t)} = \boldsymbol{G}_s$ is active. Correspondingly, the agents of network (4) employ the protocol u_i $(\boldsymbol{G}_{\sigma(t)})$ in the form of (5). Under these circumstances, the dynamics of the leader-following multi-agent system becomes

$$\begin{aligned} \dot{x}_i(t) \;=\; & \sum_{v_j \in N_i(t)} a_{ij}(t)\varphi_1(\mathrm{sig}(x_j(t) - x_i(t))^\alpha) \\ & - p_i(t)\varphi_2(\mathrm{sig}(x_i(t) - x_0(t))^\alpha), \\ & t \in [t_s, t_{s+1}),\ s = 0, 1, \ldots,\ i \in I. \end{aligned} \tag{8}$$

The following lemma is given for the analysis of the finite-time consensus over switching systems.

Lemma 3[27] Consider a family of systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_\sigma(x),\ \ \boldsymbol{f}_\sigma(0) = 0,\ \ \ \boldsymbol{x} \in \boldsymbol{R}^n. \tag{9}$$

Let Γ denote the finite switching index set; $\sigma(t) : [0, \infty) \to \Gamma$ is a piecewise constant function of time; f_k is continuous with respect to x for fixed $k \in \Gamma$; and τ is the dwell time. If the switching system (9) is asymptotically stable, and $\dot{x} = f_k(x)$ for any fixed $k \in \Gamma$ is finite-time stable, then system (9) is finite-time stable.

Theorem 2 Suppose that the network graph of system (4) is balanced and switched, and the leader is globally reachable. Then for any initial state and arbitrary switching signal $\sigma(t) : R^+ \to \Gamma$, under protocol (5) with $0 < \alpha < 1$, φ_k being a continuous odd function satisfying $x\varphi_k(x) > 0$ $(\forall\, x \neq 0)$, and $\varphi_k(x) = c_k x + o(x)$ around $x = 0$ for some constant $c_k > 0$, $k = 1, 2$, the leader-following finite-time consensus is achieved.

Proof Let $\delta_i(t) = x_i(t) - x_0(t)$. Then, equation (8) can be written as

$$\begin{aligned} \dot{\delta}_i(t) \;=\; u_i = & \sum_{v_j \in N_i(t)} a_{ij}(t)\varphi_1(\mathrm{sig}(\delta_j(t) - \delta_i(t))^\alpha) \\ & - p_i(t)\varphi_2(\mathrm{sig}(\delta_i(t))^\alpha), \\ & \forall\ t \in [t_s, t_{s+1}),\ s = 0, 1, 2, \ldots,\ i \in I. \end{aligned} \tag{10}$$

For any $t \in [t_s, t_{s+1})$, $s = 0, 1, 2, \ldots$, let $\sigma(t) = p \in \Gamma$. Choose Lyapunov function $V_{\mathrm{p}}(t) = \frac{1}{2}\sum_{i=1}^{n} \delta_i^2$. Since node 0 of every active topology involved in the switching sequence is globally reachable, then along the trajectory of system (10) one can obtain

$$\begin{aligned} \dot{V}_{\mathrm{p}}(t) \;=\; & \sum_{i=1}^{n} \delta_i \dot{\delta}_i = \sum_{i=1}^{n} \delta_i \Big[\sum_{v_j \in N_i(t)} a_{ij}(t)\varphi_1(\mathrm{sig}(\delta_j - \delta_i)^\alpha) \\ & - p_i(t)\varphi_2(\mathrm{sig}(\delta_i)^\alpha) \Big] \\ = \; & \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)(\delta_i - \delta_j)\varphi_1(\mathrm{sig}(\delta_j - \delta_i)^\alpha) \\ & - \sum_{i=1}^{n} p_i(t)\delta_i\varphi_2(\mathrm{sig}(\delta_i)^\alpha) \\ \le \; & \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)(\delta_i - \delta_j)\varphi_1(\mathrm{sig}(\delta_j - \delta_i)^\alpha) \\ \le \; & 0, \quad t \in [t_s, t_{s+1}),\ s = 0, 1, 2, \ldots,\ i \in I. \end{aligned}$$

Next, similar to the proof of Theorem 1, for any $p \in \Gamma$, $V_{\mathrm{p}}(t)$ reaches zero in finite time t_{p}^*, and $V_{\mathrm{p}}(t) = 0$ for all $t > t_{\mathrm{p}}^*$. That is to say, the origin is a globally finite-time stable equilibrium of system (10). Then by Lemma 3, system (8) over switching network topologies can achieve consensus in finite time. We complete the proof.

Noting that an undirected graph is balanced, then from Theorems 1 and 2, one can obtain the following results.

Corollary 1 Suppose that the network graph of the multi-agent system (4) is undirected, and the leader is globally reachable. Then under protocol (5) with $0 < \alpha < 1$, φ_k being a continuous odd function satisfying $x\varphi_k(x) > 0$ $(\forall\, x \neq 0)$, and $\varphi_k(x) = c_k x + o(x)$ around $x = 0$ for some constant $c_k > 0, k = 1, 2$, the leader-following finite-time consensus is achieved.

Corollary 2 Suppose that the network of system (4) is undirected and switched, and the leader is globally reachable in $\bar{\boldsymbol{G}}_{\sigma(t)}$. Then for any initial state and arbitrary switching signal $\sigma(t) : R^+ \to \Gamma$, under protocol (5) with $0 < \alpha < 1$, φ_k being a continuous odd function satisfying $x\varphi_k(x) > 0$ $(\forall\, x \neq 0)$, and $\varphi_k(x) = c_k x + o(x)$ around $x = 0$ for some constants $c_k > 0$, $k = 1, 2$, the leader-following finite-time consensus is achieved.

The proof of Corollary 1 and 2 is straightforward. Hence, it is omitted.

Remark 1 The function φ_k $(k=1,2)$ chosen in this paper is general. There are some special forms of them, such as $\varphi_k(x)=x$, $\varphi_k(x)=\mathrm{sat}(x)$ or $\varphi_k(x)=\tanh(x)$, where $\mathrm{sat}(\cdot)$ denotes the saturation function, $\tanh(\cdot)$ denotes the hyperbolic tangent function. The consensus protocol proposed in this paper can be easily designed by these functions. For example, protocol (5) can be chosen as

$$u_i(t)=\sum_{v_j\in N_i(t)} a_{ij}(t)\mathrm{sig}(x_j-x_i)^{\alpha}-p_i(t)\mathrm{sig}(x_i-x_0)^{\alpha},$$

or

$$u_i(t)=\sum_{v_j\in N_i(t)} a_{ij}(t)\mathrm{sat}(\mathrm{sig}(x_j-x_i)^{\alpha})-p_i(t)\mathrm{sat}(\mathrm{sig}(x_i-x_0)^{\alpha}),$$

or

$$u_i(t)=\sum_{v_j\in N_i(t)} a_{ij}(t)\tanh(\mathrm{sig}(x_j-x_i)^{\alpha})-p_i(t)\tanh(\mathrm{sig}(x_i-x_0)^{\alpha}).$$

Remark 2 In a multi-agent system, if the agent is considered as a node, the relation between the agents is considered as an edge, the multi-agent system can be seen as a simple individual behavior of complex dynamic network, which is called the multi-agent network. The macro cooperation behaviors can be produced by the interaction between the agents or the interaction between the agents and the environment, such as bird migration, ants foraging, bees choosing their nests, and so on. Although the multi-agent comes from the biological system, at present the concepts of the idea have been widely used in the field of artificial biological group systems, such as many mobile robot systems, sensor networks, distributed computing, distributed information processing, etc. Thus, the consensus is a typical group activity of the network, which requires all the nodes to converge to a common value. It is a basic and important problem in many actual applications.

4. Examples and simulations

In this section, some numerical simulations are performed to illustrate the effectiveness of the obtained theoretical results.

Example Consider the leader-following multi-agent system with five followers (denoted by the node i, $i=1,2,3,4,5$) and one leader (denoted by the node 0). Under the balanced graph $\bar{\boldsymbol{G}}_k$ with adjacency matrix $\boldsymbol{A}_k=[a_{ij}]_{5\times5}$ and $\boldsymbol{P}_k=(p_1,p_2,p_3,p_4,p_5)$, $k=1,2,3,4$, as

$$\boldsymbol{A}_1=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\end{bmatrix},\ \boldsymbol{P}_1=(1,0,0,0,0);$$

$$\boldsymbol{A}_2=\begin{bmatrix}0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1&0&0&1&0\end{bmatrix},\ \boldsymbol{P}_2=(0,0,0,0,1);$$

$$\boldsymbol{A}_3=\begin{bmatrix}0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&1\\1&0&0&0&0\end{bmatrix},\ \boldsymbol{P}_3=(0,1,0,0,0);$$

$$\boldsymbol{A}_4=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\1&0&0&0&1\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix},\ \boldsymbol{P}_4=(0,0,1,0,0).$$

The initial state of the followers is $\boldsymbol{x}(0)=(x_1(0),x_2(0),x_3(0),x_4(0),x_5(0))^{\mathrm{T}}=(-10,-5,0,10,15)$. Moreover, the initial state of the leader is $x_0(t)=x_0(0)=3$ for all $t\geq0$.

Note that the directed graph $\boldsymbol{G}_k$ is balanced and the node 0 is globally reachable in graph $\bar{\boldsymbol{G}}_k$, $k=1,2,3,4$. Design the following consensus protocol:

$$u_i(t)=\sum_{v_j\in N_i(t)} a_{ij}(t)\mathrm{sig}(x_j(t)-x_i(t))^{\alpha}-p_i(t)\mathrm{sig}(x_i(t)-x_0(t))^{\alpha},\quad i=1,2,3,4,5. \tag{11}$$

In all the simulations, the blue curve denotes the state of the leader, and the other curves denote the states of the followers.

Under protocol (1) with $\alpha=0.5$ and $\alpha=0.8$, the simulations of the agents' states under the graph $\bar{\boldsymbol{G}}_1$ are given in Figs. 1 and 2, and the simulations of the agents' states under the graph $\bar{\boldsymbol{G}}_2$ are given in Figs. 3 and 4. For the system over switching network topologies, which switch between the four coupling topologies $\bar{\boldsymbol{G}}_1$, $\bar{\boldsymbol{G}}_2$, $\bar{\boldsymbol{G}}_3$, $\bar{\boldsymbol{G}}_4$ sequentially with switching period (i.e., the dwell time) $T=10$, under protocol (11) with $\alpha=0.5$ and $\alpha=0.8$, the simulations of the agents' states are given in Figs. 5 and 6, respectively.

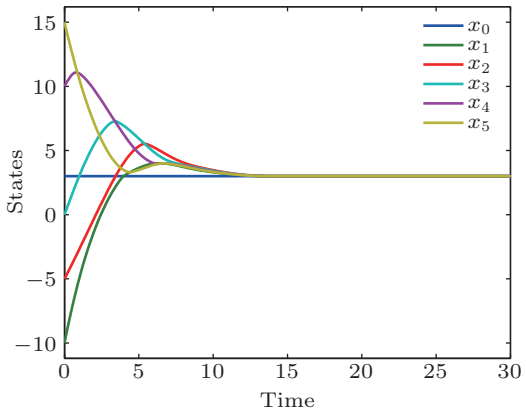


Fig. 1. (color online) States of the agents over $\bar{\boldsymbol{G}}_1$ and $\alpha=0.5$.

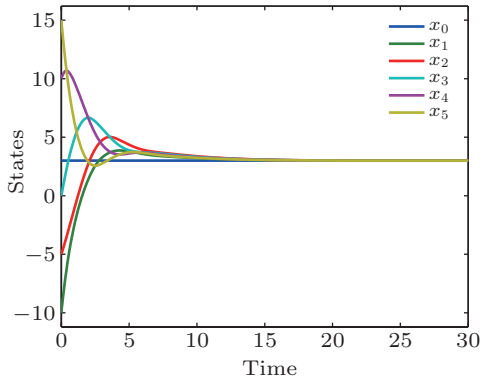


Fig. 2. (color online) States of the agents over $\bar{G}_1$ and $\alpha = 0.8$.

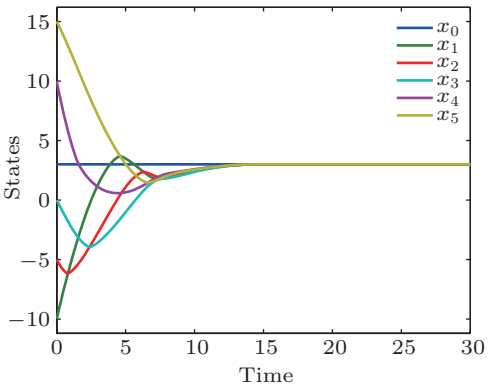


Fig. 3. (color online) States of the agents over $\bar{G}_2$ and $\alpha = 0.5$.

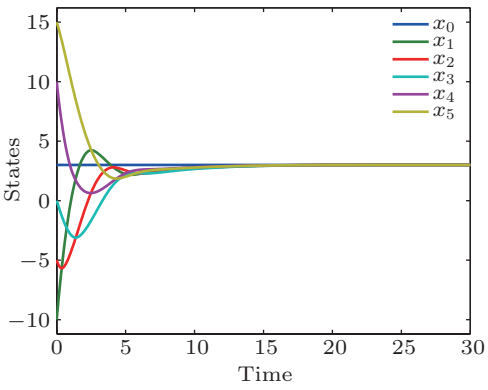


Fig. 4. (color online) States of the agents over $\bar{G}_2$ and $\alpha = 0.8$.

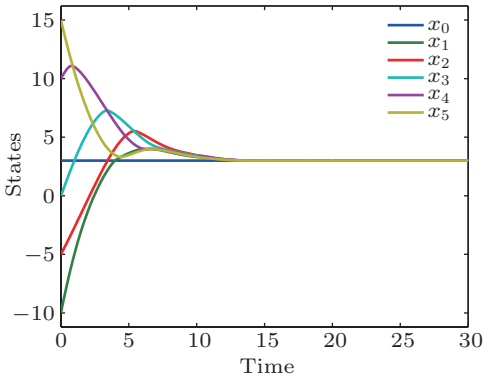


Fig. 5. (color online) States of the agents over switching network and $\alpha = 0.5$.

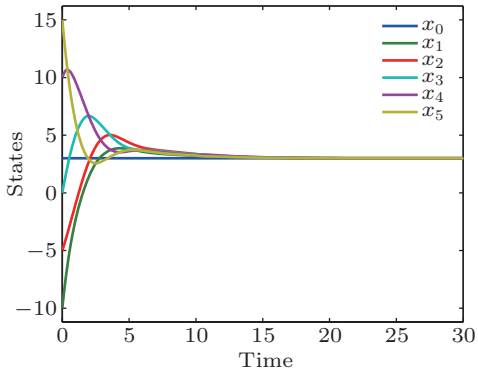


Fig. 6. (color online) States of the agents over switching network and $\alpha = 0.8$.

Figures 1–6 show that, under protocol (11) with $0 < \alpha < 1$, if the leader of the system is globally reachable, then all the states of the followers, either under fixed or switching directed balanced network, can converge to the state of the leader in finite time. This verifies the correctness of the obtained theoretical results.

5. Conclusions

The finite-time consensus for the leader-following multi-agent system under balanced and switching network topologies is studied in this paper. Based on the graph theory, homogeneity with dilation and LaSalle's invariance principle, the finite-time consensus protocols are proposed. Moreover, some simulation results are given to illustrate the effectiveness of the obtained theoretical results. The state of the leader considered in this paper is invariant. However, the leader is usually variable in the real world. Hence, the finite-time consensus for the leader-varying multi-agent system is worth investigating in the future.

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