Invariance of Painlevé property for some reduced (1+1)-dimensional equations*

Zhi Hong-Yan(智红燕)[a)†] and Chang Hui(常 辉)[b)]

[a)] College of Science, China University of Petroleum, Qingdao 266580, China
[b)] College of science, Qingdao Binhai University, Qingdao 266510, China

(Received 9 December 2012; revised manuscript received 28 May 2013)

We study the Painlevé property of the (1+1)-dimensional equations arising from the symmetry reduction for the (2+1)-dimensional ones. Firstly, we derive the similarity reduction of the (2+1)-dimensional potential Calogero–Bogoyavlenskii–Schiff (CBS) equation and Konopelchenko–Dubrovsky (KD) equations with the optimal system of the admitted one-dimensional subalgebras. Secondly, by analyzing the reduced CBS, KD, and Burgers equations with Painlevé test, respectively, we find both the Painlevé integrability, and the number and location of resonance points are invariant, if the similarity variables include all of the independent variables.

Keywords: similarity reduction, Painlevé analysis, resonance point, (1+1)-dimensional reduced equation

PACS: 02.30.Sv, 02.30.Ik, 02.30.Jr **DOI:** 10.1088/1674-1056/22/11/110203

1. Introduction

In 1978, Ablowitz, Ramani, and Segur established the deep and important connection between nonlinear partial differential equations (PDEs) that were solved by inverse scattering transforms and nonlinear ODE(s) without movable critical points. The Ablowitz–Ramani–Segur (ARS) conjecture was presented[1,2] as "Every nonlinear ODE obtained by an exact reduction of a nonlinear PDE of inverse scattering transformation (IST) class is of P-type." In recent years, much attention has been paid to multidimensional integrable systems, especially the (2+1)-dimensional ones, such as exact solution, soliton solution, Bäcklund transformation, Painlevé analysis, similarity analysis, and so on. Lots of results have been obtained on the similarity reduction of the differential equations.[3–5] Is there a similar connection between the (2+1)-dimensional nonlinear equation(s) and the (1+1)-dimensional reductions like the ARS conjecture?

To answer the above questions, we study the (1+1)-dimensional equations arising from the similarity reduction of the (2+1)-dimensional potential Calogero–Bogoyavlenskii–Schiff (CBS) equation, Burgers equation, and Konopelchenko–Dubrovsky (KD) equations, respectively. Both the potential CBS and KD equations are Painlevé integrable. Whereas, the Burgers equation is non-Painlevé integrable.

The (2+1)-dimensional CBS reads[6–8]

$$w_t + ww_z + \frac{1}{2}w_x\partial_x^{-1}w_z + \frac{1}{4}w_{xxz} = 0, \tag{1}$$

where $\partial_x^{-1}f = \int f\,\mathrm{d}x$. Equation (1) was constructed by Bogoyavlenskii, Schiff, and Toda in three different ways. Bogoyavlenskii used the modified Lax formalism,[6] Schiff derived Eq. (1) by reducing the self-dual Yang–Mills equation.[7] Toda and Yu also derived Eq. (1) with the Calogero method.[8]

Although equation (1) arises in a nonlocal form, it can be written as the differential equations of the system

$$\begin{aligned} &2w_xv + w_{xxz} + 4w_t + 4ww_z = 0,\\ &w_z = v_x, \end{aligned} \tag{2}$$

as well as in the potential form

$$4u_{xt} + 4u_xu_{xz} + 2u_{xx}u_z + u_{xxxz} = 0, \tag{3}$$

where $w = u_x$. Zhang *et al.*[9] proved that equation (3) is integrable by Painlevé test. The classical and nonclassical methods were used to obtain symmetry reductions and exact solutions of Eq. (3).[10–12] Wazwaz[13,14] applied tanh–coth method and Hirota's bilinear method to Eq. (3). Ayati and Biazar[15] derived the generalized solitary solutions to Eq. (3) with exp-function method. Ruan[16] studied the coherent structures, and the generalized expansion method of Riccati equation was used to construct the soliton-like solutions[17] of Eq. (3). With the Hirota bilinear method, Wang[18] constructed the quasi-periodic wave solutions to Eq. (3).

If we set $u(x,z,t) = V(X,Y,T)$, and $X = -x$, $Y = -z$, $T = -t/4$, equation (3) can be transformed into the famous breaking soliton equation

$$V_{XT} - 4V_XV_{XY} - 2V_YV_{XX} + V_{XXXY} = 0, \tag{4}$$

which was used to describe the interaction of a Riemann wave propagating along the y axis with a long wave along the x axis.

*Project supported by the Natural Science Foundation of Shandong Province, China (Grant Nos. ZR2011AQ017 and ZR2010AM028) and the Fundamental Research Funds for the Central Universities of Ministry of Education of China (Grant No. 13CX02010A).

†Corresponding author. E-mail: zhihongyan@126.com

© 2013 Chinese Physical Society and IOP Publishing Ltd

Yan and Zhang[19] derived a new Bäcklund transformation and some exact solutions for Eq. (4). The cross soliton-like waves and Wronskian form of the N-soliton solution to Eq. (4) were presented in Refs. [20,21].

The Lax pair of Eq. (3) is

$$\phi_t = \frac{1}{4}u_{xz}\phi - \frac{1}{2}u_z\phi_x + \lambda\phi_z,$$
$$\phi_{xx} = -(u_x + \lambda)\phi.$$

Burgers equation is one of the simplest nonlinear wave equations. It is a special case of the famous Richards equation, which has lots of applications in the porous media field. The (2+1)-dimensional time-dependent Burgers equation is[22]

$$u_t + t^m u_{xx} + t^n u_{yy} + uu_x = 0, \tag{5}$$

which was introduced to illustrate the symmetry reduction under multi-dimensional (solvable) subalgebra.

The (2+1)-dimensional KD equations are

$$u_t - u_{xxx} - 6\beta uu_x + \frac{3}{2}\alpha^2 u^2 u_x - 3v_y + 3au_x v = 0,$$
$$u_y - v_x = 0, \tag{6}$$

which have been proposed by Konopelchenko and Dubrovsky[23] with the inverse scattering transformation method.

Xu *et al.*[24] studied the Painlevé integrability of KD equation. Many researchers have investigated Eq. (6) by some expansion methods and obtained traveling wave and soliton solutions (for more details, see Ref. [24] and the references therein). One of the authors[25] derived the symmetry reduction of Eq. (6).

Although, the similarity reductions of the potential CBS, Burgers, and KD equations have been derived, there are no reports on the Painlevé analysis of these reductions. In this paper, by analyzing these similarity reductions with Painlevé test,[26] we find both the Painlevé integrability, and the number and location of resonance points are invariant, if the similarity variables include all independent variables x, y (or z), t.

2. Optimal system of one-dimensional subalgebras and similarity reduction for potential CBS equation

With the classical Lie group method,[27,28] we obtain the admitted generators of Eq. (3) as follows: $v_1 = \partial_t$, $v_2 = \partial_z$, $v_3 = t\partial_z + x\partial_u$, $v_4 = z\partial_z + t\partial_t$, $v_5 = xt\partial_x + 2tz\partial_z + 2t^2\partial_t + (2xz - tu)\partial_u$, $v_6 = x\partial_x - 2z\partial_z - u\partial_u$, $v_7(\alpha) = \alpha(t)\partial_x + 2\alpha'(t)z\partial_u$, $v_8(\beta) = \beta(t)\partial_u$, where $\alpha(t)$, $\beta(t)$ are arbitrary C^∞ functions of t. The above results have been presented in Refs. [10]–[12]. Here, L_1 denotes the algebra spanned by $v_1, ..., v_6$, and L_2 denotes the algebra spanned by $v_1, ..., v_6, v_7(\alpha), v_8(\beta)$.

After constructing the admitted vector fields, one can derive the similarity reductions. However, it is impossible to use all the one-dimensional subalgebras to construct invariant solutions. Therefore, we use the standard procedure[27,28] to derive the optimal system of one-dimensional subalgebras with the adjoint representation. The adjoint representation $\mathrm{Ad}(\exp(\epsilon v_i))$ is defined by

$$\mathrm{Ad}(\exp(\epsilon v_i))v_j = v_j - \epsilon[v_i, v_j] + \frac{\epsilon^2}{2}[v_i, [v_i, v_j]] - \cdots.$$

The authors of the Refs. [10]–[12] have derived the optimal system of one-dimensional subalgebras of the algebra L_1, whose generators are

$$av_1 + v_6,\ v_6,\ v_4 + bv_6,\ av_1 + v_3,\ v_1 + bv_2,\ v_3, \tag{7}$$

where $a \in \mathbf{R}$, $a \neq 0$ and $b \in \mathbf{R}$. In fact, $v_4 + bv_3$ should be one of generators of the optimal system. Also, one can compute the optimal system of one-dimensional subalgebras of L_2, although it is infinite-dimensional algebra. Similarly, we compute the necessary adjoint representation:

$$\begin{aligned}
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha)))v_1 = v_1 + \epsilon v_7(\alpha'),\\
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha)))v_2 = v_2 + \epsilon v_8(2\alpha'),\\
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha)))v_3 = v_3 + \epsilon v_8(2t\alpha' - \alpha),\\
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha)))v_4 = v_4 + \epsilon v_7(t\alpha'),\\
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha)))v_5 = v_5 - \epsilon v_7(\alpha),\\
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha)))v_6 = v_6 + \epsilon v_7(2t\alpha' - t\alpha),\\
&\mathrm{Ad}(\exp(\epsilon v_8(\beta)))v_1 = v_1 + \epsilon v_8(\beta'),\\
&\mathrm{Ad}(\exp(\epsilon v_8(\beta)))v_2 = v_2,\\
&\mathrm{Ad}(\exp(\epsilon v_8(\beta)))v_3 = v_3,\\
&\mathrm{Ad}(\exp(\epsilon v_8(\beta)))v_4 = v_4 + \epsilon v_8(t\beta'),\\
&\mathrm{Ad}(\exp(\epsilon v_8(\beta)))v_5 = v_5 - \epsilon v_8(\beta),\\
&\mathrm{Ad}(\exp(\epsilon v_8(\beta)))v_6 = v_6 + \epsilon v_8(2t^2\beta' + t\beta),\\
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha_1)))v_7(\alpha_2) = v_7(\alpha_2),\\
&\mathrm{Ad}(\exp(\epsilon v_7(\alpha)))v_8(\beta) = v_8(\beta).
\end{aligned} \tag{8}$$

Together with Eqs. (7) and (8), we obtain the corresponding generators of the optimal system of one-dimensional subalgebras for L_2, which are

$$\begin{aligned}
&av_1 + v_6,\ v_6,\ v_4 + bv_6,\ av_1 + v_3,\ v_1 + bv_2,\\
&v_4 + bv_3,\ v_3 + v_7(\alpha),\ v_7(\alpha),
\end{aligned} \tag{9}$$

where $a \in \mathbf{R}$, $a \neq 0$ and $b \in \mathbf{R}$.

Reduction 1 Using the generator $av_1 + v_6$, we obtain the similarity variables and similarity solution

$$\xi = x\exp\left(-\frac{t}{a}\right),$$
$$\eta = z\exp\left(\frac{2t}{a}\right),$$
$$u = \exp\left(-\frac{t}{a}\right)F(\xi, \eta), \tag{10}$$

and the reduced PDE of Eq. (3) is

$$-8F_\xi - 4\xi F_{\xi\xi} + 8\eta F_{\xi\eta} + 4aF_\xi F_{\xi\eta} + 2aF_{\xi\xi}F_\eta + aF_{\xi\xi\xi\eta} = 0. \quad (11)$$

Reduction 2 Using the generator v_6, we obtain the similarity variables and similarity solution

$$\xi = x^2 z,\ \eta = t,\ u = \frac{1}{x}F(\xi,\eta), \quad (12)$$

and the reduced PDE of Eq. (3) is

$$-2F_\eta + 4\xi F_{\xi\eta} - 4\xi F F_{\xi\xi} + 2\xi F_\eta^2 + 12\xi^2 F_\xi F_{\xi\xi} + 3\xi F_{\xi\xi} + 4\xi^3 F_{\xi\xi\xi\xi} + 12\xi^2 F_{\xi\xi\xi} = 0. \quad (13)$$

Reduction 3 Using the generator $v_4 + bv_6$, we obtain the similarity variables and similarity solution

$$\xi = xt^{-b},\ \eta = zt^{2b-1},\ u = t^{-b}F(\xi,\eta), \quad (14)$$

and the reduced PDE of Eq. (3) is

$$8bF_\xi + 4b\xi F_{\xi\xi} + 4\eta(1-2b)F_{\xi\eta} - 4F_\xi F_{\xi\eta} - 2F_{\xi\xi}F_\eta - F_{\xi\xi\xi\eta} = 0. \quad (15)$$

Reduction 4 Using the generator $av_1 + v_3$, we obtain the similarity variables and similarity solution

$$\xi = x,\quad \eta = 2az - t^2,\ u = \frac{xt}{a} + xF(\xi,\eta), \quad (16)$$

and the reduced PDE of Eq. (3) is

$$2 + 4a^2 F F_\eta + 4a^2\xi F F_{\xi\eta} + 8a^2\xi F_\xi F_\eta + 4a^2\xi^2 F_\xi F_{\xi\eta} + 2a^2\xi^2 F_\eta F_{\xi\xi} + 3a^2 F_{\xi\xi\eta} + a^2\xi F_{\xi\xi\xi\eta} = 0. \quad (17)$$

Reduction 5 Using the generator $v_1 + bv_2$, we obtain the similarity variables and similarity solution

$$\xi = x,\ \eta = z - bt,\ u = F(\xi,\eta), \quad (18)$$

and the reduced PDE of Eq. (3) is

$$-4bF_{\xi\eta} + 4F_\xi F_{\xi\eta} + 2F_{\xi\xi}F_\eta + F_{\xi\xi\xi\eta} = 0. \quad (19)$$

Reduction 6 Using the generator $v_3 + v_7(\alpha)$, we obtain the similarity variables and similarity solution

$$\xi = xt - \alpha(t)z,\ \eta = t,\quad u = \frac{1}{2}\frac{x^2}{\alpha} - \frac{x^2 t\alpha'}{\alpha^2} + \frac{2xz\alpha'}{\alpha} + F(\xi,\eta), \quad (20)$$

and the reduced PDE of Eq. (3) is

$$8\xi\alpha\alpha'' + 8\xi\alpha'^2 - 12\xi\eta\alpha^2\alpha' F_{\xi\xi} - 4\eta\alpha^3 F_{\xi\eta} - 2\alpha^3 F_\xi - 12\eta\alpha^2\alpha' F_\xi + 6\eta^2\alpha^4 F_\xi F_{\xi\xi} + \eta^3\alpha^4 F_{\xi\xi\xi\xi} = 0. \quad (21)$$

Reduction 7 Using the generator $v_4 + bv_3$, we obtain the similarity variables and similarity solution

$$\xi = x,\ \eta = \frac{z}{t} - b\ln(t),\ u = bx\ln(t) + F(\xi,\eta), \quad (22)$$

and the reduced PDE of Eq. (3) is

$$-4b + 4(b+\eta)F_\eta - 4F_{\xi\eta}F_\xi - 2F_{\xi\xi}F_\eta - F_{\xi\xi\xi\eta} = 0. \quad (23)$$

Reduction 8 Using the generator $v_7(\alpha)$, we obtain the similarity variables and similarity solution

$$\xi = z,\ \eta = t,\ u = \frac{2xz\alpha'}{\alpha} + 2\alpha' F(\xi,\eta), \quad (24)$$

and the reduced PDE of Eq. (3) is

$$\alpha\alpha'' + \alpha'^2 = 0. \quad (25)$$

3. Painlevé analysis of reduced potential CBS equation

It is widely known that the Painlevé test in the sense of the Weiss–Tabor–Carnevale (WTC) method[26] is a powerful tool for investigating integrable equations. With the WTC method, Zhang *et al.*[9] have proved that equation (3) is integrable in the sense of the Painlevé test, the Painlevé expansion is

$$u(x,z,t) = \sum_{i=0}^{\infty} u_i(x,z,t)\phi^{i-1}(x,z,t), \quad (26)$$

and $u_0 = 2\phi_x$. The resonance points are -1, 1, 4, 6, and the recursion relation is

$$Q_1\phi_x^3\phi_z u_j(x,z,t) = M(u_{j-1},...,u_0,\phi),\ j = 1,2,3,..., \quad (27)$$

where $Q_1 = (j+1)(j-1)(j-4)(j-6)$, $M(u_{j-1},...,u_0,\phi)$ is a function of $u_{j-1},...,u_0,\phi$ and their derivatives of x, z, t.

Reduction 8 is easily solved, whose solution is $\alpha = 2c_0\sqrt{\eta + c_1}$. Here, c_0 and c_1 are arbitrary constants. The similarity variables ξ and η do not include the dependent variable x, so we do not use Painlevé test to Eq. (25). Applying the WTC method to Reductions 1–7 of Eq. (3), respectively, we derive the Painlevé expansion is

$$F(\xi,\eta) = \sum_{i=0}^{\infty} f_i(\xi,\eta)h^{i-1}(\xi,\eta), \quad (28)$$

the resonance points are $-1, 1, 4, 6$. The recursion relations for f_j and the expansion of f_0 are listed in Table 1.

Table 1. (1+1)-dimensional reduced CBS equation.

Reduction	Recursion relations	Expansion of f_0
1	$Q_1 a h_\xi^3 h_\eta f_j = M_1(f_{j-1},...,f_0,h)$	$f_0 = 2h_\xi$
2	$4Q_1\xi^3 h_\eta^4 f_j = M_2(f_{j-1},...,f_0,h)$	$f_0 = 4\xi h_\xi$
3	$Q_1 h_\xi^3 h_\eta f_j = M_3(f_{j-1},...,f_0,h)$	$f_0 = 2h_\xi$
4	$Q_1 a^2\xi h_\xi^3 h_\eta f_j = M_4(f_{j-1},...,f_0,h)$	$f_0 = \frac{2}{\xi}h_\xi$
5	$Q_1 h_\xi^3 h_\eta f_j = M_5(f_{j-1},...,f_0,h)$	$f_0 = 2h_\xi$
6	$Q_1\alpha^4(\eta)\eta^3 h_\xi^4 f_j = M_6(f_{j-1},...,f_0,h)$	$f_0 = 2\eta h_\xi$
7	$Q_1 h_\xi^3 h_\eta f_j = M_7(f_{j-1},...,f_0,h)$	$f_0 = 2h_\xi$

In Table 1, $M_i(f_{j-1},...,f_0,h)$, $(i=1,...,7)$ are some functions of $f_{j-1},...,f_0$, h and their derivatives. Then we can compute the coefficients at the non-resonances, and verify the compatibility condition at every positive resonance. For example, the recursion relations of Reduction 1 is

$$\begin{aligned}&(j+1)(j-1)(j-4)(j-6)ah_\xi^3 h_\eta f_j\\ &= M_1(f_{j-1},...,f_0,h).\end{aligned} \tag{29}$$

The numbers of the resonances correspond to arbitrary functions h, f_1, f_4, f_6. From Eq. (29), we obtain f_2, f_3, f_5, f_7, f_8, ...

Substituting the expression of f_0 into the right side of Eq. (29) for $j=1$, we obtain zero.

Substituting the expression of f_0 f_2, and f_3 into the right side of Eq. (29) for $j=4$, we obtain zero.

Substituting the expression of f_0 f_2, f_3, and f_5 into the right side of Eq. (29) for $j=6$, we obtain zero.

That is to say, for Reduction 1, the recurrence relations are consistently satisfied at the resonances. Then Reduction 1 possesses the Painlevé property. Similarly, we can check that Reductions 2–7 are all Painlevé integrable.

4. Painlevé test of reduction for other (2+1)-dimensional equations

4.1. Painlevé analysis of (1+1)-dimensional reduced Burgers equation

With WTC method, we can check the (2+1)-dimensional time-dependent Burgers equation (5) is non-Painlevé integrable in the sense of the Painlevé test, the Painlevé expansion is

$$u(x,y,t)=\sum_{i=0}^{\infty}u_i(x,y,t)\phi^{i-1}(x,y,t), \tag{30}$$

and $u_0=2(t^m\phi_x^2+t^n\phi_y^2)/\phi_x$. The resonance points are -1, 2, and the recursion relation is

$$Q_2(t^m\phi_x^2+t^n\phi_y^2)u_j(x,y,t)=M(u_{j-1},...,u_0,\phi),\ j=1,2,3,...$$

where $Q_2=(j+1)(j-2)$, $M(u_{j-1},...,u_0,\phi)$ is a function of $u_{j-1},...,u_0$, ϕ, and their derivatives. The reduced equation of the optimal system of one-dimensional subalgebra is as follows.[22]

Reduction I The similarity variables and similarity solution are $\xi=t^{-(m+1)/2}x$, $\eta=t^{-(n+1)/2}y$, $u=t^{(m-1)/2}F(\xi,\eta)$, and the reduction of Eq. (5) is

$$\begin{aligned}&2F_{\xi\xi}+2F_{\eta\eta}+2FF_\xi-(m+1)\xi F_\xi\\ &-(n+1)\eta F_\eta+(m-1)F=0.\end{aligned}$$

Reduction II The similarity variables and similarity solution are $\xi=(t+a_1)y-a_2x$, $\eta=t$, $u=x/(t+a_1)+F(\xi,\eta)$, and the reduction of Eq. (5) is

$$\begin{aligned}&(\eta+a_1)[a_2^2\eta^m+\eta^n(\eta+a_1)^2]F_{\xi\xi}-a_2(\eta+a_1)FF_\xi\\ &+\xi F_\xi+(\eta+a_1)F_\eta+F=0,\end{aligned}$$

where a_1 and a_2 are arbitrary constants.

Reduction III The similarity variables and similarity solution are $\xi=x-a_1y$, $\eta=t$, $u=F(\xi,\eta)$, and the reduction of Eq. (5) is

$$F_\eta+(\eta^m+a_1^2\eta^n)F_{\xi\xi}+FF_\xi=0,$$

where a_1 is an arbitrary constant.

Reduction IV $\xi=y$, $\eta=t$, $u=F(\xi,\eta)$,

$$F_\eta+\eta^nF_{\xi\xi}=0.$$

Reduction IV is a linear differential equation with respect to $F(\xi,\eta)$, whose solution is

$$\begin{cases}F=\exp\left(-\dfrac{c_1}{n+1}y^{n+1}\right)[c_2\exp(\sqrt{c_1}x)\\ \qquad +c_3\exp(-\sqrt{c_1}x)],\ \text{if } n\neq-1,\\ F=\exp\left(-\dfrac{1}{y^{c_1}}\right)[c_2\exp(\sqrt{c_1}x)\\ \qquad +c_3\exp(-\sqrt{c_1}x)],\ \text{if } n=-1.\end{cases}$$

Here, c_1, c_2, and c_3 are arbitrary constants. The similarity variables ξ and η do not include the independent variables x. Applying the WTC method to Reductions I–III of Eq. (5), respectively, we derived the Painlevé expansion is the same as Eq. (28) and the resonance points are $-1,2$. None of Reductions I–III is integrable in the sense of the Painlevé test. The recursion relations for f_j and the expansion of f_0 are listed in Table 2. In Table 2, $Q_2=(j+1)(j-2)$, $r_1=\eta^n(a_1+\eta)^2+\eta^ma_2^2$, $r_2=\eta^m+a_1^2\eta^n$.

Table 2. (1+1)-dimensional reduced Burgers equation.

Reduction	Recursion relations	Expansion of f_0
I	$-2Q_2(h_\xi^2+h_\eta^2)f_j=M_1(f_{j-1},...,f_0,h)$	$f_0=\dfrac{2}{h_\xi}(h_\xi^2+h_\eta^2)$
II	$Q_2r_1h_\xi^2(\eta+a_1)f_j=M_2(f_{j-1},...,f_0,h)$	$f_0=-\dfrac{2}{a_2}r_1h_\xi$
III	$Q_2r_2h_\xi^2f_j=M_3(f_{j-1},...,f_0,h)$	$f_0=2r_2h_\xi$

4.2. Painlevé analysis of (1+1)-dimensional reduced KD equations

Xu *et al.*[24] have proved that equation (6) are Painlevé integrable in the sense of Painlevé test. The Painlevé expansions are

$$\begin{aligned}u(x,y,t)&=\sum_{i=0}^{\infty}u_i(x,y,t)\phi^{i-1}(x,y,t),\\ v(x,y,t)&=\sum_{i=0}^{\infty}v_i(x,y,t)\phi^{i-1}(x,y,t),\end{aligned}$$

and two different Painlevé expansion branches are $\{u_0 = (2/\alpha)\phi_x,\ v_0 = (2/\alpha)\phi_y\}$, and $\{u_0 = -(2/\alpha)\phi_x,\ v_0 = -(2/\alpha)\phi_y\}$. The recursion relations are

$$\begin{pmatrix} (j+1)(j-3)(j-4)\phi_x^3 & 0 \\ (1-j)\phi_y & (j-1)\phi_x \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} M_1(u_{j-1},...,u_0,v_{j-1},...,v_0,\phi) \\ M_2(u_{j-1},...,u_0,v_{j-1},...,v_0,\phi) \end{pmatrix}. \quad (31)$$

The resonance points are -1, 1, 3, 4. One of the authors has derived the admitted Lis algebra and similarity reduction.[25] The optimal system of one-dimensional subalgebra is $\{X(p),Y(q),W(r)\}$, where p,q,r are arbitrary functions of t. According to the results presented in Ref. [25], the similarity reduction of the optimal system is as follows.

Reduction i For $X(p)$, the similarity variables are $\xi = Ax + A'y^2/6$, $\eta = A^2 y$, and the similarity solutions are

$$\begin{aligned} u &= -\frac{A'}{3\alpha A}y + \frac{2\beta}{\alpha^2}A + AF(\xi,\eta), \\ v &= -\frac{A'}{3\alpha A}x + \frac{A'}{3}yF(\xi,\eta) + \frac{A'^2 - AA''}{18\alpha A^2}y^2 \\ &\quad + \frac{2\beta^2}{\alpha^3} + A^2 G(\xi,\eta), \end{aligned} \quad (32)$$

where $p(t) = A^{-3}(t)$. The reductions of Eq. (6) are

$$\begin{aligned} &2F_{\xi\xi\xi} - 6\alpha F_\xi G - 3\alpha^2 F^2 F_\xi + 6G_\eta = 0, \\ &F_\eta - G_\xi = 0. \end{aligned} \quad (33)$$

Reduction ii For $Y(q)$, the similarity variables and similarity solutions are

$$\begin{aligned} &\xi = x - \frac{q'}{12q}y^2,\ \eta = t,\ u = \frac{q'}{6q}y + F(\xi,\eta), \\ &v = \frac{2qq'' - q'^2}{72\alpha q^2}y^2 + \frac{2\beta q' - \alpha^2 F(\xi,\eta)}{6\alpha^2 q}y + G(\xi,\eta), \end{aligned}$$

and the reductions of Eq. (6) are

$$\begin{aligned} &2\alpha^2 qF_\eta - 2\alpha^2 qF_{\xi\xi\xi} + (3\alpha^4 F^2 - 12\alpha^2\beta qF \\ &\quad + 6\alpha^3 qG)F_\xi + \alpha^2 q'F - 2\beta q' = 0, \\ &6\alpha qG_\xi - q' = 0. \end{aligned} \quad (34)$$

Reduction iii For $w(r)$, the similarity variables and similarity solutions are

$$\xi = y,\ \eta = t,\ u = F(\xi,\eta),\ v = \frac{r'}{3\alpha r}x + G(\xi,\eta),$$

and the reductions of Eq. (6) are

$$F_\eta - 3G_\xi = 0,\ F_\xi - \frac{r'}{3\alpha r} = 0. \quad (35)$$

Reduction iii of KD equations are linear differential equations and their solutions have been derived in Ref. [25]. The similarity variables do not include the independent variable x. Therefore, we do not use Painlevé test to Eq. (35).

By applying the WTC method to both Reductions i and ii, we find the Painlevé expansions are

$$\begin{aligned} F(\xi,\eta) &= \sum_{i=0}^{\infty} f_i(\xi,\eta)h^{i-1}(\xi,\eta), \\ G(\xi,\eta) &= \sum_{i=0}^{\infty} g_i(\xi,\eta)h^{i-1}(\xi,\eta). \end{aligned}$$

For Reduction i of KD equations, two different Painlevé expansion branches are

$$\left\{f_0 = \frac{2}{\alpha}h_\xi,\ g_0 = \frac{2}{\alpha}h_\eta\right\},$$

and

$$\left\{f_0 = -\frac{2}{\alpha}h_\xi,\ g_0 = -\frac{2}{\alpha}h_\eta\right\}. \quad (36)$$

The recursion relations are

$$\begin{pmatrix} 2(j+1)(j-3)(j-4)h_\xi^3 & 0 \\ (1-j)h_\eta & (j-1)h_\xi \end{pmatrix} \begin{pmatrix} f_j \\ g_j \end{pmatrix} = \begin{pmatrix} M_1(f_{j-1},...,f_0,g_{j-1},...,g_0,h) \\ M_2(f_{j-1},...,f_0,g_{j-1},...,g_0,h) \end{pmatrix}. \quad (37)$$

For Reduction ii of KD equations, two different Painlevé expansion branches are

$$\left\{f_0 = \frac{2}{\alpha}h_\xi,\ g_0 = 0\right\},\ \text{and}\ \left\{f_0 = -\frac{2}{\alpha}h_\xi,\ g_0 = 0\right\}. \quad (38)$$

The recursion relations are

$$\begin{pmatrix} -2(j+1)(j-3)(j-4)\alpha^2 q(\eta)h_\xi^3 & 0 \\ 0 & 6(1-j)\alpha q(\eta)h_\xi \end{pmatrix} \times \begin{pmatrix} f_j \\ g_j \end{pmatrix} = \begin{pmatrix} M_1(f_{j-1},...,f_0,g_{j-1},...,g_0,h) \\ M_2(f_{j-1},...,f_0,g_{j-1},...,g_0,h) \end{pmatrix}. \quad (39)$$

Both Reductions i and ii are Painlevé integrable and the number and location of the resonance points are the same as the original KD equations.

If we set

$$\begin{aligned} F(\xi,\eta) &= \frac{f_0(\xi,\eta)}{h(\xi,\eta)} + f_1(\xi,\eta), \\ G(\xi,\eta) &= \frac{g_0(\xi,\eta)}{h(\xi,\eta)} + g_1(\xi,\eta), \end{aligned} \quad (40)$$

where $f_0(\xi,\eta)$ and $g_0(\xi,\eta)$ are given in Eq. (36), $f_1(\xi,\eta)$ and $g_1(\xi,\eta)$ are seed solutions to Eq. (33), we can obtain the Painlevé truncated solution. For example, substituting

$$F(\xi,\eta) = \frac{2h_\xi}{\alpha h} + c_0,\ G(\xi,\eta) = \frac{2h_\eta}{\alpha h} + c_1, \quad (41)$$

into the first equation of Eq. (33), we can obtain a differential system with respect to h as follows:

$$\begin{aligned}&-2h^2h_{\xi\xi\xi\xi\xi}+8hh_\xi h_{\xi\xi\xi\xi}-12(h_\xi)^2h_{\xi\xi}+6h{h_{\xi\xi}}^2\\&+12hh_{\xi\xi}h_\eta 6\alpha c_1h^2h_{\xi\xi}-12(h_\xi)^2h_\eta-6\alpha c_1h(h_\xi)^2\\&+12\alpha c_0hh_\xi h_{\xi\xi}-12\alpha c_0(h_\xi)^3+3\alpha^2c_0^2h^2h_{\xi\xi}\\&-3\alpha^2c_0^2h(h_\xi)^2-6h^2h_{\eta\eta}+6h(h_\eta)^2=0.\end{aligned}\tag{42}$$

Substituting Eq. (41) into the second equation of Eq. (33), we obtain an identical equation. Solving Eq. (42), we obtain the Painlevé truncated solution. It is easy to check that equation (42) has a solution

$$h=c_2+\exp(k\xi+l\eta+c_3),\tag{43}$$

where c_0, c_1, c_2, and c_3 are arbitrary constants,

$$k=\frac{\sqrt{3\alpha^2c_0^2+12\alpha c_1}-3\alpha c_0}{4},$$
$$l=\frac{\alpha c_0\sqrt{3\alpha^2c_0^2+12\alpha c_1}-6\alpha c_1}{8}.$$

Together with Eqs. (32) and (43), we can obtain the new soliton-like solution

$$\begin{aligned}u&=-\frac{A'}{3\alpha A}y+\frac{2\beta}{\alpha^2}A+\frac{2kA\exp(\omega)}{\alpha(c_2+\exp(\omega))}+c_0A,\\v&=-\frac{A'}{3\alpha A}x+\frac{A'^2-AA''}{18\alpha A^2}y^2+\frac{2\beta^2}{\alpha^3}\\&\quad+\frac{1}{3}yAA'\left[\frac{2k\exp(\omega)}{\alpha(c_2+\exp(\omega))}+c_0\right]\\&\quad+A^2\left[\frac{2l\exp(\omega)}{\alpha(c_2+\exp(\omega))}+c_0\right],\end{aligned}\tag{44}$$

where $\omega=k(Ax+A'y^2/6)+lA^2y$, $A=A(t)$ is an arbitrary function of t. Equation (44) is different from solution (41) presented in Ref. [24].

5. Conclusion

In this paper, we derived the optimal system of one-dimensional subalgebra admitted by the (2+1)-dimensional potential CBS and KD equations, and reduced them into (1+1)-dimensional ones. By applying the Painlevé analysis with WTC method to the reductions of potential CBS, Burgers and KD equations, we found some interesting phenomena.

i) Both potential CBS and KD equations are Painlevé integrable, and all of the (1+1)-dimensional reductions are integrable in the sense of Painlevé test, if the similarity variables include all the independent variables x, y (or z), t.

ii) The time-dependent Burgers equation is non-Painlevé integrable, and the (1+1)-dimensional reductions are also non-integrable in the sense of Painlevé test, if the similarity variables include all the independent variables x, y, t.

iii) The number and location of resonance points of the reduction are invariable, whether the original equation is Painlevé integrable or not.

What will happen to the other reduced (1+1)-dimensional equations arising from the symmetry reduction of the (2+1)-dimensional ones? That is worth our further investigation.

References

[1] Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo. Cimento* **23** 333

[2] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: Society for Industrial and Applied Mathematics)

[3] Tang X Y and Lou S Y 2001 *Chin. Phys.* **10** 897

[4] Zhang Y F and Zhang H Q 2002 *Chin. Phys.* **11** 319

[5] Xie S Y and Lin J 2010 *Chin. Phys. B* **19** 050201

[6] Bogoyavlenskii O I 1990 *Math. USSR-Izv.* **34** 245

[7] Schiff J 1992 *NATO ASI Ser. B* **278** 393

[8] Toda K and Yu S J 2000 *J. Math. Phys.* **41** 4747

[9] Zhang H P, Chen Y and Li B 2009 *Acta Phys. Sin.* **58** 7393 (in Chinese)

[10] Bruzon M, Gandarias M L, Muriel C, Ramirez J, Saez A and Romero F R 2003 *Theor. Math. Phys.* **137** 1367

[11] Gandarias M and Bruzon M 2008 *79th Annual Meeting of the International Association of Applied Mathematics and Mechanics*, May 28, 2008, Bremen, Germany p. 10591

[12] Gandarias M and Bruzon M 2008 *American Conference on Applied Mathematics (math'08)*, March 24–26, Harvard, Massachusetts, USA, p. 153

[13] Wazwaz A M 2008 *Appl. Math. Comput.* **196** 363

[14] Wazwaz A M 2008 *Appl. Math. Comput.* **203** 592

[15] Ayati Z and Biazar J 2009 *Iran. J. Optim.* **1** 173

[16] Ruan H Y 2002 *J. Phys. Soc. Jpn.* **71** 453

[17] Chen Y, Li B and Zhang H Q 2003 *Commun. Theor. Phys.* **40** 137

[18] Wang J M and Yang X 2012 *Nonlinear Anal.* **75** 2256

[19] Yan Z Y and Zhang H Q 2002 *Comput. Math. Appl.* **44** 1439

[20] Cui K 2012 *Chin. Phys. Lett.* **29** 060508

[21] Lu Z S, Duan L X and Xie F D 2010 *Chin. Phys. Lett.* **27** 070502

[22] Ivanova N M, Sophocleous C and Tracina R 2012 *Z. Angew. Math. Phys.* **61 793**

[23] Konopelchenko B G and Dubrovsky V G 1984 *Phys. Lett. A* **102** 15

[24] Xu P B, Gao Y T, Yu X, Wang L and Lin G D 2011 *Commun. Theor. Phys.* **55** 1017

[25] Zhi H Y 2008 *Appl. Math. Comput.* **203** 931

[26] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522

[27] Olver P J 1986 *Application of Lie Group to Differential Equations* (Berlin: Springer)

[28] Bluman G and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer)