Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method

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In this paper, we use the fractional complex transform and the (G'/G)-expansion method to study the nonlinear fractional differential equations and find the exact solutions. The fractional complex transform is proposed to convert a partial fractional differential equation with Jumarie's modified Riemann–Liouville derivative into its ordinary differential equation. It is shown that the considered transform and method are very efficient and powerful in solving wide classes of nonlinear fractional order equations.

Keywords: (G'/G)-expansion method, time-fractional Burgers equation, fractional-order biological population model, space–time fractional Whitham–Broer–Kaup equations

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1. Introduction

Recently, the nonlinear fractional differential equations in mathematical physics have played a major role in various areas. These equations appear in a great variety of contexts, such as physics, biology, engineering, fluid flow, signal processing, system identification, control theory, finance, and fractional dynamics. The fractional partial differential equations have been investigated by many researchers.[1–3] Finding the exact solutions of the fractional differential equations is an important work. Consequently, a large amount of papers have been published concerning the solution of the fractional differential equations in nonlinear phenomena. Wide classes of analytical methods have been proposed for solving the fractional differential equations, such as the fractional sub-equation method,[4–6] the first integral method,[7] and the (G'/G)-expansion method,[8,9] which can be used to construct the exact solutions for some time and space fractional differential equations. Based on these methods, a variety of fractional differential equations have been investigated and solved.

In this paper, we will use the (G'/G)-expansion method[10–13] to solve nonlinear fractional partial differential equations in the sense of modified Riemann–Liouville derivative by Jumarie.[14] The Jumarie's modified Riemann–Liouville derivative of order α is defined by the following expression:

$$\mathrm{D}_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) \mathrm{d}\xi, \\ \quad 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha - n)}, \\ \quad n \le \alpha < n+1, \quad n \ge 1. \end{cases} \tag{1}$$

Some important properties of the fractional modified Riemann–Liouville derivative have been summarized and some useful formulas of them are[15]

$$\mathrm{D}_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{2}$$

$$\mathrm{D}_t^{\alpha}(f(t)g(t)) = g(t)\mathrm{D}_t^{\alpha} f(t) + f(t)\mathrm{D}_t^{\alpha} g(t), \tag{3}$$

$$\mathrm{D}_t^{\alpha} f\,[g(t)] = f_g'[g(t)]\mathrm{D}_t^{\alpha} g(t) = \mathrm{D}_g^{\alpha}[g(t)](g'(t))^{\alpha}. \tag{4}$$

We consider the following general nonlinear fractional differential equations:

$$P(u, \mathrm{D}_t^{\alpha} u, \mathrm{D}_x^{\beta} u, \mathrm{D}_y^{\psi}, \mathrm{D}_t^{\alpha}\mathrm{D}_t^{\alpha} u, \mathrm{D}_t^{\alpha}\mathrm{D}_x^{\beta} u, \mathrm{D}_x^{\beta}\mathrm{D}_x^{\beta} u, \mathrm{D}_x^{\beta}\mathrm{D}_y^{\psi} u, \mathrm{D}_y^{\psi}\mathrm{D}_y^{\psi} u, \ldots) = 0, \quad 0 < \alpha, \beta, \psi < 1, \tag{5}$$

where u is an unknown function, and P is a polynomial of u. In this equation, the partial fractional derivatives involving the highest order derivatives and the nonlinear terms are included.

Li and He[16,17] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODE), so all analytical methods which are devoted to the advanced calculus can be easily applied to the fractional calculus. By using the traveling wave variable

$$u(x, y, t) = U(\xi), \tag{6}$$

$$\xi = \frac{\tau x^{\beta}}{\Gamma(1+\beta)} + \frac{\delta y^{\psi}}{\Gamma(1+\psi)} + \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}, \tag{7}$$

where τ, δ, and λ are non zero arbitrary constants, we can rewrite Eq. (5) as the following nonlinear ODE:

$$Q(U, U', U'', U''', \ldots) = 0, \tag{8}$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate Eq. (8) term by term one or more times.

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Suppose that the solution of Eq. (8) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \quad a_m \neq 0, \tag{9}$$

where a_i $(i = 0, \pm1, \pm2, \ldots, \pm m)$ are constants, while $G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{10}$$

with λ and μ being constants. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (8). By substituting Eq. (9) into Eq. (8) and using Eq. (10), we collect all terms with the same order of (G'/G). By equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for a_i $(i = 0, \pm1, \pm2, \ldots, \pm m)$, λ, μ, τ, δ, and λ. By solving the equation system and substituting a_i $(i = 0, \pm1, \pm2, \ldots, \pm m)$, $\lambda, \mu, \tau, \delta, \lambda$, and the general solutions of Eq. (10) into Eq. (9), we can obtain a variety of exact solutions of Eq. (5).

The rest of this paper is organized as follows. In Sections 2–4, we use this method to obtain the exact solutions for the time-fractional Burgers equation, the fractional-order biological population model, and the space–time fractional Whitham–Broer–Kaup equations respectively. Some conclusions are given in the last section.

2. Time-fractional Burgers equation

We consider the following fractional time fractional Burgers equation:[18]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0,\ 0 < \alpha \leq 1, \tag{11}$$

where α is a parameter describing the order of the fractional time derivative.

For our purpose, we introduce the following transformations:

$$u(x,t) = U(\xi), \tag{12}$$

$$\xi = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{13}$$

where k and c are constants.

By substituting Eq. (13) into Eq. (11), Eq. (11) is reduced into an ODE

$$-cU' + k\varepsilon UU' - k^2 \nu U'' = 0, \tag{14}$$

where $U' = \mathrm{d}U/\mathrm{d}\xi$. By integrating once, we find

$$\xi_0 - cU + k\varepsilon \frac{U^2}{2} - k^2 \nu U' = 0, \tag{15}$$

where ξ_0 is an integration constant. By using the ansatz (15), for the linear term of highest order U' with the highest order and the nonlinear term U^2, balancing U' with U^2 in Eq. (15) gives

$$m + 1 = 2m, \tag{16}$$

so

$$m = 1. \tag{17}$$

Suppose that the solutions of Eq. (15) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{18}$$

By using Eq. (10), from Eq. (18) we have

$$U'(\xi) = -a_1 \left(\frac{G'}{G}\right)^2 - a_1 \lambda \left(\frac{G'}{G}\right) - a_1 \mu, \tag{19}$$

$$U^2(\xi) = a_1^2 \left(\frac{G'}{G}\right)^2 + 2a_0 a_1 \left(\frac{G'}{G}\right) + a_0^2. \tag{20}$$

By substituting Eqs. (18)–(20) into Eq. (15), collecting the coefficients of $(G'/G)^i$ $(i = 0, \ldots, 2)$, and setting them to be zero, we obtain the system

$$\begin{aligned} &\frac{1}{2} k\varepsilon a_1^2 + k^2 \nu a_1 = 0, \\ &-ca_1 + k\varepsilon a_0 a_1 + k^2 \nu a_1 \lambda = 0, \\ &-ca_0 + \frac{1}{2} k\varepsilon a_0^2 + k^2 \nu a_1 \mu + \xi_0 = 0. \end{aligned} \tag{21}$$

By solving this system with Maple, we obtain

$$\begin{aligned} &a_0 = a_0, \quad a_1 = a_1, \quad \nu = \nu, \\ &c = -\frac{k^2 \nu (2a_0 - a_1 \lambda)}{a_1}, \quad \varepsilon = -\frac{2k\nu}{a_1}, \\ &k = k, \quad \xi_0 = -\frac{k^2 \nu (a_0^2 - a_0 a_1 \lambda + a_1^2 \mu)}{a_1}, \end{aligned} \tag{22}$$

where λ and μ are arbitrary constants. By using Eq. (22), expression (18) can be written as

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \tag{23}$$

where $\xi = kx + \frac{k^2 \nu (2a_0 - a_1 \lambda)}{a_1 \Gamma(1+\alpha)} t^\alpha$. By substituting the general solutions of Eq. (20) into Eq. (23), we have three types of travelling wave solutions of the time-fractional Burgers equation as follows.

When $\lambda^2 - 4\mu > 0$,

$$U_1(\xi) = a_0 + a_1 \left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi} \right) \right], \tag{24}$$

where $\xi = kx + \frac{k^2\nu(2a_0-a_1\lambda)}{a_1\Gamma(1+\alpha)}t^\alpha$.

When $\lambda^2 - 4\mu < 0$,

$$U_2(\xi) = a_0 + a_1\left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi - C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)\right], \tag{25}$$

where $\xi = kx + \frac{k^2\nu(2a_0-a_1\lambda)}{a_1\Gamma(1+\alpha)}t^\alpha$.

When $\lambda^2 - 4\mu = 0$,

$$U_3(\xi) = a_0 + a_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right), \tag{26}$$

where $\xi = kx + \frac{k^2\nu(2a_0-a_1\lambda)}{a_1\Gamma(1+\alpha)}t^\alpha$.

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0$, and $\mu = 0$, then U_1 becomes

$$\begin{aligned} u_1(x,t) &= a_0 - \frac{\lambda a_1}{2} + \frac{\lambda a_1}{2} \\ &\quad \times \coth\frac{\lambda}{2}\left(kx + \frac{k^2\nu(2a_0-a_1\lambda)}{a_1\Gamma(1+\alpha)}t^\alpha\right), \end{aligned} \tag{27}$$

and U_2 becomes

$$\begin{aligned} u_2(x,t) &= a_0 - \frac{\lambda a_1}{2} + \frac{\lambda a_1}{2} \\ &\quad \times \cot\frac{\lambda}{2}\left(kx + \frac{k^2\nu(2a_0-a_1\lambda)}{a_1\Gamma(1+\alpha)}t^\alpha\right), \end{aligned} \tag{28}$$

which are the exact solutions of the time fractional Burgers equation.

3. Fractional-order biological population model

We consider a nonlinear fractional-order biological population model of the form[19,20]

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) + h(u^2 - r) = 0, \\ &\quad t > 0,\ 0 < \alpha \leq 1, \end{aligned} \tag{29}$$

where u denotes the population density, $h(u^2 - r)$ represents the population supply due to births and deaths, h and r are constants, and α is a parameter describing the order of the fractional time derivative.[4]

For our purpose, we introduce the following transformations:

$$u(x,y,t) = U(\xi), \tag{30}$$

$$\xi = x + \mathrm{i}y - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{31}$$

where c is a constant and $\mathrm{i}^2 = -1$. By substituting Eq. (31) into Eq. (30), Eq. (29) is reduced into an ODE

$$cU' + hU^2 - hr = 0, \tag{32}$$

where $U' = \mathrm{d}U/\mathrm{d}\xi$.

By the same procedure as illustrated in Section 2, we can determine the value of m by balancing U^2 and U' in Eq. (31). We find $m = 1$. We can suppose that the solution of Eq. (29) is in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{33}$$

By using Eqs. (33) and (10), we have

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu, \tag{34}$$

$$U^2(\xi) = a_1^2\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + a_0^2. \tag{35}$$

By substituting Eqs. (33)–(35) into Eq. (29), collecting the coefficients of $(G'/G)^i$ $(i = 0,\ldots,2)$, and setting them to zero, we obtain the equation system

$$\begin{aligned} &-ca_1 + ha_1^2 = 0, \\ &-ca_1\lambda + 2ha_0a_1 = 0, \\ &-ca_1\mu + ha_0^2 - hr = 0. \end{aligned} \tag{36}$$

We can solve this system by Maple and obtain

$$\begin{aligned} a_0 &= \pm\lambda\sqrt{\frac{r}{\lambda^2-4\mu}}, \\ a_1 &= \pm 2\sqrt{\frac{r}{\lambda^2-4\mu}}, \\ c &= \pm 2h\sqrt{\frac{r}{\lambda^2-4\mu}}, \end{aligned} \tag{37}$$

where λ and μ are arbitrary constants. By using Eq. (37), expression (33) can be written as

$$U(\xi) = \pm\sqrt{\frac{r}{\lambda^2-4\mu}}\left(\lambda + 2\left(\frac{G'}{G}\right)\right), \tag{38}$$

where $\xi = x + \mathrm{i}y \pm \frac{2h}{\Gamma(1+\alpha)}\sqrt{\frac{r}{\lambda^2-4\mu}}t^\alpha$.

By substituting the general solutions of Eq. (10) into Eq. (38), we have two types of exact solutions of the nonlinear fractional-order biological population model as follows.

When $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} &U_{1,2}(\xi) \\ &= \pm\sqrt{r}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right), \end{aligned} \tag{39}$$

where $\xi = x + \mathrm{i}y \pm \frac{2h}{\Gamma(1+\alpha)}\sqrt{\frac{r}{\lambda^2-4\mu}}t^\alpha$.

When $\lambda^2-4\mu<0$,

$$U_{3,4}(\xi) = \pm\sqrt{r}\left(\frac{-C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right), \quad (40)$$

where $\xi = x+\mathrm{i}y\pm\frac{2h}{\Gamma(1+\alpha)}\sqrt{\frac{r}{\lambda^2-4\mu}}t^\alpha$.

In particular, if $C_1\neq 0$, $C_2=0$, $\lambda>0$, and $\mu=0$, then $U_{1,2}$ becomes

$$u_{1,2}(x,y,t) = \pm\sqrt{r}\tanh\frac{\lambda}{2}\left(x+\mathrm{i}y\pm\frac{2h}{\Gamma(1+\alpha)}\sqrt{\frac{r}{\lambda^2-4\mu}}t^\alpha\right), \quad (41)$$

and $U_{3,4}$ becomes

$$u_{3,4}(x,y,t) = \pm\sqrt{r}\cot\frac{\lambda}{2}\left(x+\mathrm{i}y\pm\frac{2h}{\Gamma(1+\alpha)}\sqrt{\frac{r}{\lambda^2-4\mu}}t^\alpha\right). \quad (42)$$

which are the exact solutions of the fractional-order biological population model.

4. Space–time fractional Whitham–Broer–Kaup equations

Now we consider the space–time fractional Whitham–Broer–Kaup equations[20]

$$\begin{aligned} &\mathrm{D}_t^\alpha u+u\mathrm{D}_x^\alpha u+\mathrm{D}_x^\alpha v+\beta\mathrm{D}_x^{2\alpha}u=0,\\ &\mathrm{D}_t^\alpha v+\mathrm{D}_x^\alpha(uv)-\beta\mathrm{D}_x^{2\alpha}v+\gamma\mathrm{D}_x^{3\alpha}u=0, \end{aligned} \quad (43)$$

where $u=u(x,t)$ is the field of horizontal velocity, $v=v(x,t)$ is the height deviating from the equilibrium position of liquid, β and γ are real constants that represent different diffusion powers, and α is a parameter describing the order of the fractional space–time derivative.

For our purpose, we use the following transformation:

$$\begin{aligned} &u(x,t)=U(\xi), \quad \xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)},\\ &v(x,t)=V(\xi), \quad \xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \quad (44)$$

where $c\neq 0$ and $k\neq 0$ are constants. Then by using Eq. (44), Eq. (43) can be turned into an ODE

$$-cU'+kUU'+kV'+k^2\beta U''=0, \quad (45)$$

$$-cV'+k(UV)'-k^2\beta V''+k^3\gamma U'''=0, \quad (46)$$

where $U'=\mathrm{d}U/\mathrm{d}\xi$. By integrating once and setting the constants of integration to be zero, we obtain

$$-cU+k\frac{U^2}{2}+kV+k^2\beta U'=0, \quad (47)$$

$$-cV+kUV-k^2\beta V'+k^3\gamma U''=0. \quad (48)$$

By considering the homogeneous balance between $U^2/2$ and U' in Eq. (47), we obtain

$$2m=m+1, \quad (49)$$

so

$$m=1. \quad (50)$$

Similarly, by considering the homogeneous balance between UV and U'' in Eq. (48), we obtain

$$m+n=m+2, \quad (51)$$

which gives

$$n=2. \quad (52)$$

Suppose that the solutions of Eqs. (50) and (52) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi)=a_0+a_1\left(\frac{G'}{G}\right), \quad a_1\neq 0, \quad (53)$$

$$V(\xi)=b_0+b_1\left(\frac{G'}{G}\right)+b_2\left(\frac{G'}{G}\right)^2, \quad b_2\neq 0, \quad (54)$$

where $G=G(\xi)$ satisfies the second order ODE in the form of

$$G''+\lambda G'+\mu G=0, \quad (55)$$

and a_0, a_1, b_0, b_1, b_2, λ, and μ are constants that will be determined later. By using Eqs. (53)–(55), we obtain

$$U^2(\xi) = a_1^2\left(\frac{G'}{G}\right)^2+2a_0a_1\left(\frac{G'}{G}\right)+a_0^2, \quad (56)$$

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2-a_1\lambda\left(\frac{G'}{G}\right)-a_1\mu, \quad (57)$$

$$\begin{aligned} U''(\xi) = {}& 2a_1\left(\frac{G'}{G}\right)^3+3a_1\lambda\left(\frac{G'}{G}\right)^2\\ &+(2a_1\mu+a_1\lambda^2)\left(\frac{G'}{G}\right)+a_1\lambda\mu, \end{aligned} \quad (58)$$

$$\begin{aligned} V'(\xi) = {}& -2b_2\left(\frac{G'}{G}\right)^3-2b_2\lambda\left(\frac{G'}{G}\right)^2-2b_2\mu\left(\frac{G'}{G}\right)\\ &-b_1\left(\frac{G'}{G}\right)^2-b_1\lambda\left(\frac{G'}{G}\right)-b_1\mu. \end{aligned} \quad (59)$$

By substituting Eqs. (56)–(59) into Eqs. (47) and (48), collecting the coefficients of $(G'/G)^i$ $(i=0,\ldots,3)$, and setting them to zero, we obtain the following equation system:

$$\begin{aligned} &2: && -k^2\beta a_1+\frac{1}{2}ka_1^2+kb_2=0,\\ &1: && -ca_1-k^2\beta a_1\lambda+kb_1+ka_0a_1=0,\\ &0: && -ca_0+\frac{1}{2}ka_0^2+kb_0-k^2\beta a_1\mu=0,\\ &3: && ka_1b_2+2k^2\beta b_2+2k^3\gamma a_1=0,\\ &2: && 3k^3\gamma a_1\lambda+k^2\beta b_1+ka_0b_2\\ &&& +2k^2\beta b_2\lambda+ka_1b_1-cb_2=0,\\ &1: && k^2\beta b_1\lambda-cb_1+ka_1b_0+ka_0b_1+k^3\gamma a_1\lambda^2\\ &&& +2k^3\gamma a_1\mu+2k^2\beta b_2\mu=0,\\ &0: && ka_0b_0-cb_0+k^2\beta b_1\mu+k^3\gamma a_1\lambda\mu=0. \end{aligned} \quad (60)$$

By solving the above algebraic equations, we obtain

$$\begin{aligned}
a_0 &= \frac{\lambda \pm \sqrt{\lambda^2-4\mu}}{2}a_1, \quad a_1 = a_1,\\
b_0 &= \frac{\mu a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right),\\
c &= \pm\frac{ka_1}{2}\sqrt{\lambda^2-4\mu},\\
b_1 &= \frac{\lambda a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right), \quad k = k,\\
b_2 &= \pm\frac{a_1}{2}\sqrt{a_1^2-4k^2\gamma}-\frac{1}{2}a_1^2,\\
\beta &= \pm\frac{1}{2k}\sqrt{a_1^2-4k^2\gamma},
\end{aligned} \tag{61}$$

where λ, μ, β, and γ are arbitrary constants. Substituting Eq. (61) into Eqs. (53) and (54) yields

$$\begin{aligned}
U(\xi) &= \frac{\lambda \pm \sqrt{\lambda^2-4\mu}}{2}a_1 + a_1\left(\frac{G'}{G}\right), \qquad (62)\\
V(\xi) &= \frac{\mu a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right)\\
&\quad + \frac{\lambda a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right)\left(\frac{G'}{G}\right)\\
&\quad \pm \left(\frac{a_1}{2}\sqrt{a_1^2-4k^2\gamma}-\frac{1}{2}a_1^2\right)\left(\frac{G'}{G}\right)^2, \qquad (63)
\end{aligned}$$

where $\xi = \frac{kt^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{\lambda^2-4\mu}t^\alpha}{2\Gamma(1+\alpha)}$.

By substituting the general solutions of Eq. (55) into Eqs. (62) and (63), we have two types of solutions of the space–time fractional Whitham–Broer–Kaup equations as follows.

When $\lambda^2-4\mu > 0$,

$$U_{1,2}(\xi) = \pm\frac{a_1\sqrt{\lambda^2-4\mu}}{2} + \frac{a_1\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right), \tag{64}$$

$$\begin{aligned}
V_{1,2}(\xi) &= \frac{\mu a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right) + \frac{\lambda a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right)\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)\\
&\quad \pm \left(\frac{a_1}{2}\sqrt{a_1^2-4k^2\gamma}-\frac{1}{2}a_1^2\right)\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi - C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2,
\end{aligned} \tag{65}$$

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{\lambda^2-4\mu}t^\alpha}{2\Gamma(1+\alpha)}$.

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, and $a_1 = 1$, then $U_{1,2}$ and $V_{1,2}(\xi)$ become

$$\begin{aligned}
u_{1,2}(x,t) &= \frac{\lambda}{2}\coth\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{\lambda^2-4\mu}t^\alpha}{2\Gamma(1+\alpha)}\right) \pm \frac{\lambda}{2},\\
v_{1,2}(x,t) &= -\frac{\lambda}{2} \pm \frac{\lambda}{2}\sqrt{1-4k^2\gamma}\\
&\quad \times \coth\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{\lambda^2-4\mu}t^\alpha}{2\Gamma(1+\alpha)}\right)\\
&\quad \pm \left(\frac{1}{2}\sqrt{1-4k^2\gamma}-\frac{1}{2}\right)\\
&\quad \times \coth^2\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{\lambda^2-4\mu}t^\alpha}{2\Gamma(1+\alpha)}\right).
\end{aligned} \tag{66}$$

When $\lambda^2-4\mu < 0$,

$$\begin{aligned}
U_{3,4}(\xi) &= \pm\frac{a_1\sqrt{\lambda^2-4\mu}}{2} + \frac{a_1\sqrt{\lambda^2-4\mu}}{2}\\
&\quad \times \left(\frac{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi - C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right),
\end{aligned} \tag{67}$$

$$\begin{aligned}
V_{3,4}(\xi) &= \frac{\mu a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right)\\
&\quad + \frac{\lambda a_1}{2}\left(-a_1 \pm \sqrt{a_1^2-4k^2\gamma}\right)\\
&\quad \times \left(\frac{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi - C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)\\
&\quad \pm \left(\frac{a_1}{2}\sqrt{a_1^2-4k^2\gamma}-\frac{1}{2}a_1^2\right)\\
&\quad \times \left(\frac{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi - C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)^2,
\end{aligned} \tag{68}$$

where $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{\lambda^2-4\mu}t^\alpha}{2\Gamma(1+\alpha)}$.

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, and $a_1 = 1$, then $U_{3,4}$ and $V_{3,4}(\xi)$ become

$$\begin{aligned}
u_{3,4}(x,t) &= \frac{\lambda}{2}\cot\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{4\mu-\lambda^2}t^\alpha}{2\Gamma(1+\alpha)}\right) \pm \frac{\lambda}{2},\\
v_{3,4}(x,t) &= -\frac{\lambda}{2} \pm \frac{\lambda}{2}\sqrt{1-4k^2\gamma}\\
&\quad \times \cot\frac{\lambda}{2}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{4\mu-\lambda^2}t^\alpha}{2\Gamma(1+\alpha)}\right)\\
&\quad \pm \left(\frac{1}{2}\sqrt{1-4k^2\gamma}-\frac{1}{2}\right)
\end{aligned}$$

$$\times \cot^2 \frac{\lambda}{2}\left(\frac{kx^{\alpha}}{\Gamma(1+\alpha)} \pm \frac{ka_1\sqrt{4\mu-\lambda^2}t^{\alpha}}{2\Gamma(1+\alpha)}\right), \quad (69)$$

which are the exact solutions of the space–time fractional Whitham–Broer–Kaup equations.

5. Conclusion

In this paper, we successfully use the (G'/G)-expansion method to solve fractional nonlinear partial differential equations with Jumarie's modified Riemann–Liouville derivative. This method is reliable and efficient and gives new solutions.

Now, we briefly summarize the results in this paper. Firstly, the fractional complex transform is extremely simple but effective for solving nonlinear fractional differential equations. Secondly, the (G'/G)-expansion method for nonlinear fractional differential equations with fractional complex transform has its own advantages: direct, succinct, basic; and it can be used for many other nonlinear equations. Thirdly, to our knowledge, the solutions obtained in this paper have not been reported in the literature so far.

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