New homotopy analysis transform method for solving the discontinued problems arising in nanotechnology

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We present a new reliable analytical study for solving the discontinued problems arising in nanotechnology. Such problems are presented as nonlinear differential–difference equations. The proposed method is based on the Laplace transform with the homotopy analysis method (HAM). This method is a powerful tool for solving a large amount of problems. This technique provides a series of functions which may converge to the exact solution of the problem. A good agreement between the obtained solution and some well-known results is obtained.

Keywords: discretized mKdV lattice equation, nonlinear differential–difference equations, Laplace transform, homotopy analysis transform method

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1. Introduction

Many different analytical methods have recently been introduced to solve nonlinear problems, such as homotopy analysis method (HAM),[1–5] Adomian decomposition method (ADM),[6–9] and variational iteration method.[10,11] The HAM, which was devised by Liao in 1992, is capable of solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term, and it reduces the size of calculations because its interactions are direct and straightforward. This method has been successfully applied to solve many types of nonlinear problems in science and engineering.[12–16]

In this work, we aim to effectively employ a new homotopy analysis transform method (HATM) to establish the analytical solutions for the nonlinear differential–difference equations arising in nanotechnology. This method is based on the homotopy analysis method and the Laplace transform method. By using the presented method, numerical results can be obtained with a few iterations.[1] Moreover, the HATM contains an auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of the solution series.[2] Therefore, the HATM can handle linear and nonlinear problems without any assumption and restriction. Many authors have considered the proposed problem, for example, Mokhtari[10] used the variational iteration method to solve the nonlinear differential–difference equations, and Shun *et al.*[17] used the homotopy perturbation method to solve the discontinued problems arising in nanotechnology.

According to the E-infinity theory,[18–20] space at the quantum scale is not a continuum, and it is clear that nanotechnology possesses a considerable richness which bridges the gap between the discrete and the continuum.[19] On the nanoscale, He *et al.* [21] found experimentally an uncertainty phenomenon similar to Heisenberg's uncertainty principle in quantum mechanics. The continuum hypothesis therefore becomes invalid on the nanoscale. He and Zhu[22] suggested some differential–difference models describing the fascinating phenomena arising in heat/electron conduction and flow in carbon nanotubes, among which we will study the following model:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = (u_{n+1} - u_{n-1}) \sum_{i=1}^{m} \left(a_i + b_i (u_n)^i\right), \tag{1}$$

where a_i and b_i are constants. A physical interpretation is given in Ref. [22]. Equation (1) includes the well-known discretized mKdV lattice equation[23]

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = (\alpha - u_n^2)(u_{n+1} - u_{n-1}), \tag{2}$$

where $u_n(t)$ is an unknown function, and the subscript n in Eq. (1) represents the n-th lattice. Previously, such equations were solved by the exp-function method[24] and the variational iteration method.[10]

The main aim of this paper is using the homotopy analysis transform method to establish an approximate solution of the nonlinear differential–difference equation arising in nanotechnology (Eq. (2)). A comparison of our results with those obtained using the Adomian decomposition transform method is given.

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Definition 1 The Laplace transform of function $u(t)$ is defined as

$$\mathrm{L}[u(t)] := \int_0^\infty \mathrm{e}^{-st}u(t)\,\mathrm{d}t.$$

Definition 2 The Laplace transform of the derivative $u^{(n)}(t)$ is given by

$$\mathrm{L}[u^{(n)}(t)] = s^n\mathrm{L}[u(t)] - \sum_{k=0}^{n-1} s^{n-k-1}u^{(k)}(0).$$

2. Solution of HATM for differential–difference equations

In this section, we implement the HATM[25,26] to problem (2) of differential–difference equations. For simplicity, we ignore all boundary and initial conditions, which can be treated in a similar way. Now, the first step of the proposed method consists of applying the Laplace transform on both sides of Eq. (2), we obtain

$$\mathrm{L}\left[\frac{\mathrm{d}u_n}{\mathrm{d}t}\right] = \mathrm{L}[(\alpha - u_n^2)(u_{n+1} - u_{n-1})]. \tag{3}$$

Using definition 2 for the Laplace transform of the derivative, we have

$$\mathrm{L}[u_n(t)] = s^{-1}u_n(0) + s^{-1}\mathrm{L}[(\alpha - u_n^2)(u_{n+1} - u_{n-1})]. \tag{4}$$

We choose the linear operator

$$\pounds[\Psi_n(t;p)] = \mathrm{L}[\Psi_n(t;p)], \tag{5}$$

with the property $\pounds[c] = 0$, where c is an integral constant to be determined by the initial condition.

We define the nonlinear operator

$$\begin{aligned} &N[\Psi_n(t;p), \Psi_{n-1}(t;p), \Psi_{n+1}(t;p)] \\ &= \mathrm{L}[\Psi_n(t;p)] - s^{-1}u_n(0) \\ &\quad - s^{-1}\mathrm{L}[(\alpha - \Psi_n^2)(\Psi_{n+1} - \Psi_{n-1})], \end{aligned} \tag{6}$$

where $p \in [0,1]$ is an embedding parameter, and $\Psi_n(t;p)$ is a real function of t and p.

2.1. Zeroth-order deformation equation

Liao[1] has constructed the so-called zeroth-order deformation equation

$$(1-p)\mathrm{L}[\Psi_n(t;p) - u_{n,0}(t)] = p\hbar H_n(t)N[\Psi_n(t;p)], \tag{7}$$

where $u_{n,0}(t)$ is an initial guess of $u_n(t)$, $\hbar \neq 0$ is an auxiliary parameter, $H_n(t)$ is an auxiliary function, and $\Psi_n(t;p)$ is the unknown function. Obviously, we have

$$\Psi_n(t;0) = u_{n,0}(t), \quad \Psi_n(t;1) = u_n(t). \tag{8}$$

Thus, as p increases from 0 to 1, the solution $\Psi_n(t;p)$ varies from $u_{n,0}(t)$ to $u_n(t)$. Expanding $\Psi_n(t;p)$ in Taylor's series with respect to the embedding parameter p, we have

$$\Psi_n(t;p) = u_{n,0}(t) + \sum_{m=1}^{\infty} u_{n,m}(t)p^m, \tag{9}$$

where

$$u_{n,m}(t) = \frac{1}{m!}\frac{\partial^m \Psi_n(t;p)}{\partial p^m}\bigg|_{p=0}, \quad n,m \in \mathbf{N}. \tag{10}$$

Assume that the initial guess, the auxiliary function $H_n(t)$, and the auxiliary parameter $\hbar$ are selected such that series (9) is convergent at $p=1$, then at $p=1$, and by using Eq. (8), series (9) becomes

$$u_n(t) = u_{n,0}(t) + \sum_{m=1}^{\infty} u_{n,m}(t), \tag{11}$$

which must be one of the solutions of the original nonlinear equations.

2.2. The m-th order deformation equation

Define a vector

$$\boldsymbol{u}_{n,m}(t) = [u_{n,0}(t), u_{n,1}(t), \ldots, u_{n,m}(t)], \quad n, m \in \mathbf{N}. \tag{12}$$

Differentiating Eq. (7) m times with respect to the embedding parameter p, then setting $p=0$ and dividing them by $m!$, finally using Eq. (8), we have the so-called m-th order deformation equations

$$\begin{aligned} &\mathrm{L}[u_{n,m}(t) - \delta_{n,m}u_{n,m-1}(t)] \\ &+ \hbar H_n(t)\Re_{n,m}(\boldsymbol{u}_{n,m-1}, \boldsymbol{u}_{n+1,m-1}, \boldsymbol{u}_{n-1,m-1}), \end{aligned} \tag{13}$$

where

$$\begin{aligned} &\Re_{n,m}(\boldsymbol{u}_{n,m-1}, \boldsymbol{u}_{n+1,m-1}, \boldsymbol{u}_{n-1,m-1}) \\ &= \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\Psi_n(t;p), \Psi_{n-1}(t;p), \Psi_{n+1}(t;p)]}{\partial p^{m-1}}\bigg|_{p=0}, \end{aligned} \tag{14}$$

$$\delta_{n,m} = \begin{cases} 0, & m \leq 1\ ,\ n \in \mathbf{N}, \\ 1, & m > 1\ ,\ n \in \mathbf{N}. \end{cases} \tag{15}$$

Applying the inverse Laplace transform on both sides, we obtain the solution of the m-th order deformation equations (13) for the corresponding auxiliary function $H_n(t) = 1$, $m \geq 1$,

$$\begin{aligned} u_{n,m}(t) &= \delta_{n,m}u_{n,m-1}(t) \\ &\quad + \hbar\mathrm{L}^{-1}[\Re_m(\boldsymbol{u}_{n,m-1}, \boldsymbol{u}_{n+1,m-1}, \boldsymbol{u}_{n-1,m-1})], \end{aligned} \tag{16}$$

where

$$\begin{aligned} &\Re_{n,m}(\boldsymbol{u}_{n,m-1}, \boldsymbol{u}_{n+1,m-1}, \boldsymbol{u}_{n-1,m-1}) \\ &= \mathrm{L}[\Psi_n(t;q)] - s^{-1}u_{n,0} \\ &\quad - s^{-1}\mathrm{L}[(\alpha - u_n^2)(u_{n+1} - u_{n-1})]. \end{aligned} \tag{17}$$

Therefore, the solution of the m-th order deformation equations (13) is

$$u_{n,m}(t) = (\delta_{n,m}+\hbar)u_{n,m-1}(t) - \hbar(1-\delta_{n,m})u_n(0) + \hbar \mathrm{L}^{-1}[s^{-1}\mathrm{L}[(\alpha - u_n^2)(u_{n+1}-u_{n-1})]]. \quad (18)$$

Now, in order to illustrate the effectiveness of the method, we consider the following initial condition $u_{n,0}(t) = \sqrt{\alpha}\tanh(d)\tanh(dn)$, where d is an arbitrary constant. Then, using the iterative scheme (18) gives the first few components of the approximate solution

$$\begin{aligned} u_{n,0}(t) &= \sqrt{\alpha}\tanh(d)\tanh(dn), \\ u_{n,1}(t) &= \hbar[t(\alpha - \alpha\tanh^2(d)\tanh^2(dn)) \\ &\quad \times (-\sqrt{\alpha}\tanh(d)\tanh(d(n-1)) \\ &\quad + \sqrt{\alpha}\tanh(d)\tanh(d(1+n)))], \\ &\quad \ldots \end{aligned}$$

other components of the approximate solution can be obtained in the same manner.

Numerical results of our proposed method (HATM) for different values of $n \in [-30, 30]$ and $\hbar$ $(-2 \le \hbar \le 2)$ are presented in Figs. 1 and 2. The solution obtained using the HATM and the exact solution are presented in Fig. 3. We note that there is a complete agreement between the computed results obtained by the presented algorithm and the exact solution. From these results we can see that the presented approach is more efficient than the other methods, regarding that the HATM only takes three components of the solution.

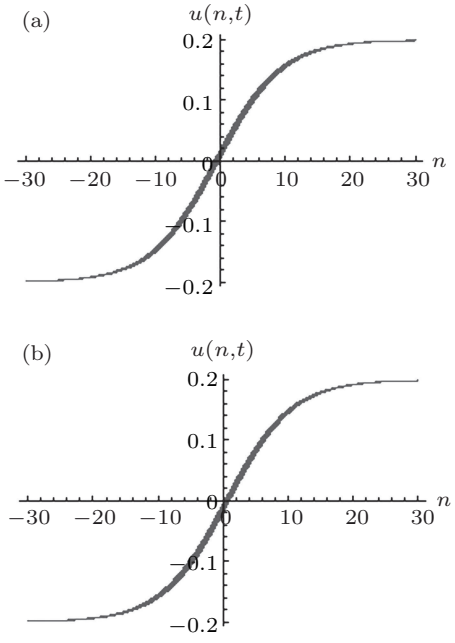


Fig. 1. The HATM solution at (a) $\hbar = -1.2, -1.4, -1.6, -1.8, -2.0$ and (b) $\hbar = 1.2, 1.4, 1.6, 1.8, 2.0$.

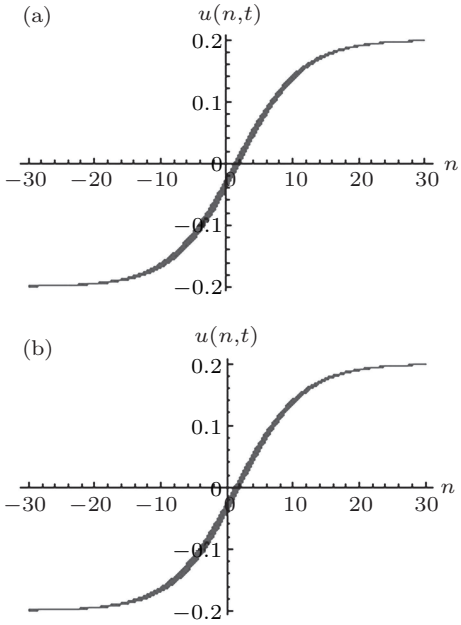


Fig. 2. The HATM solution at (a) $\hbar = -0.2, -0.4, -0.6, -0.8, -1.0$ and (b) $\hbar = 0.2, 0.4, 0.6, 0.8, 1.0$.

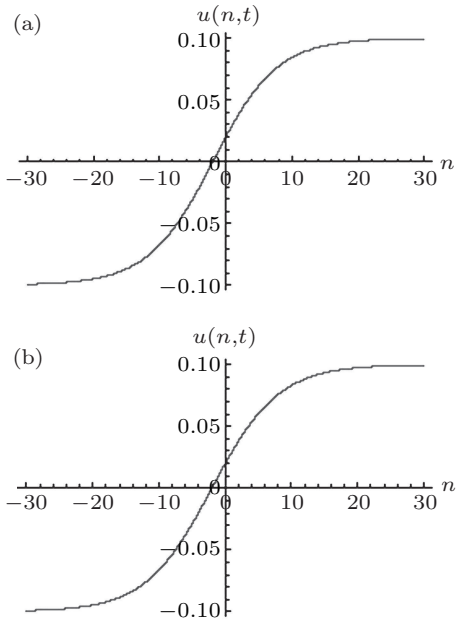


Fig. 3. (a) The HATM solution and (b) the exact solution.

It is noted that our approximate solution converges at $(-2 \le \hbar \le 2)$ (see Figs. 1 and 2). The explicit and analytic expression given by Eq. (18) contains the auxiliary parameter $\hbar$ which gives the convergence region and the rate of approximation for the HATM. However, the errors can be further reduced by including higher-order approximations. This shows that the HATM is a very useful method to get accurate analytic solutions to linear and strongly nonlinear differential–difference equations. It must be noted that the HATM used here gives the possibility of obtaining an analytical satisfactory solution for which the other techniques of calculation are more laborious and the results contain a great complexity.

3. Solution of ADTM for differential–difference equations

In this section, we implement the Adomian decomposition transformation method which is based on the Laplace transform with the Adomian decomposition method. This method is a powerful tool for solving a large amount of problems. Using this method, it is possible to find the exact solution or an approximate solution of the problem. This technique provides a series of functions, which may converge to the exact solution of the problem.

The procedure of this technique to solve the proposed problem (2) as follows.

The first step is applying the Laplace transform on both sides of Eq. (2) and using its properties to obtain

$$\begin{aligned} \mathrm{L}[u_n(t)] &= s^{-1}u_n(0)+s^{-1}\alpha\mathrm{L}[(u_{n+1}-u_{n-1})] \\ &\quad - s^{-1}\mathrm{L}[u_n^2(u_{n+1}-u_{n-1})]. \end{aligned}$$

We define the nonlinear operator $N(u_n,u_{n-1},u_{n+1}) = u_n^2(u_{n+1}-u_{n-1})$. Therefore, the above equation takes the following form:

$$\begin{aligned} \mathrm{L}[u_n(t)] &= s^{-1}u_n(0)+s^{-1}\alpha\mathrm{L}[(u_{n+1}-u_{n-1})] \\ &\quad - s^{-1}\mathrm{L}[N(u_n,u_{n-1},u_{n+1})]. \end{aligned} \tag{19}$$

The ADM defines the solution $u_n(t)$ by the following series:

$$u_n(t) = \sum_{m=0}^{\infty} u_{n,m}(t), \tag{20}$$

and the nonlinear operator $N(u_n,u_{n-1},u_{n+1})$ is represented by an infinite series of the so-called Adomian's polynomials

$$N(u_n,u_{n-1},u_{n+1}) = \sum_{m=0}^{\infty} A_{n,m}, \tag{21}$$

where $u_{n,m}(t), m \geq 0$ are the components of $u_n(t)$ that will be elegantly determined, and $A_{n,m}$ are called Adomian's polynomials and defined by

$$\begin{aligned} &A_{n,m} \\ &= \frac{1}{m!}\left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}N\left(\sum_{i=0}^{\infty}\lambda^i u_{n,i},\sum_{i=0}^{\infty}\lambda^i u_{n-1,i},\sum_{i=0}^{\infty}\lambda^i u_{n+1,i}\right)\right]_{\lambda=0}, \\ &m \geq 0. \end{aligned} \tag{22}$$

From the above considerations and by taking the inverse Laplace transform, the decomposition method defines the components $u_{n,m}(t)$ for $m \geq 0$ by the following recursive relationship:

$$\begin{aligned} u_{n,0}(t) &= \mathrm{L}^{-1}[s^{-1}u_n(0)], \\ u_{n,m+1}(t) &= \mathrm{L}^{-1}[s^{-1}\alpha\mathrm{L}[(u_{n+1,m}-u_{n-1,m})]] \\ &\quad - \mathrm{L}^{-1}[s^{-1}\mathrm{L}[A_{n,m}]], \quad m \geq 0. \end{aligned} \tag{23}$$

This will enable us to determine the components $u_{n,m}(t)$ recurrently. However, in many cases, the exact solution in a closed form may be obtained.

The first $A_{n,m}$ Adomian's polynomials that represent the nonlinear term $N(u_n,u_{n-1},u_{n+1})$ are given by

$$\begin{aligned} A_{n,0} &= u_{n,0}^2(u_{n+1,0}-u_{n-1,0}), \\ A_{n,1} &= u_{n,0}^2(u_{n+1,1}-u_{n-1,1}) \\ &\quad + 2u_{n,1}u_{n,0}(u_{n+1,0}-u_{n-1,0}), \\ &\quad \cdots \end{aligned}$$

From the recurrence relation (23), we can obtain the components of the solution $u_n(t)$ as follows:

$$\begin{aligned} u_{n,0}(t) &= \sqrt{\alpha}\tanh(d)\tanh(dn), \\ u_{n,1}(t) &= t\,(\alpha-\alpha\tanh^2(d)\tanh^2(dn)) \\ &\quad \times(-\sqrt{\alpha}\tanh(d)\tanh(d(n-1)) \\ &\quad + \sqrt{\alpha}\tanh(d)\tanh(d(1+n))), \\ &\quad \cdots \end{aligned}$$

For numerical comparison purpose, we construct the solution $u_n(t)$ such that

$$\lim_{m\to\infty} U_{n,m}(t) = u_n(t), \tag{24}$$

where

$$U_{n,m}(t) = \sum_{i=0}^{m-1} u_{n,i}(t), \quad m \geq 1.$$

For more details about the ADTM and its convergence, see Refs. [6] and [9].

Figure 4 presents a comparison between the solution using ADTM and the solution obtained from the HATM at $\hbar = -1$. From this figure, we can see that the solution

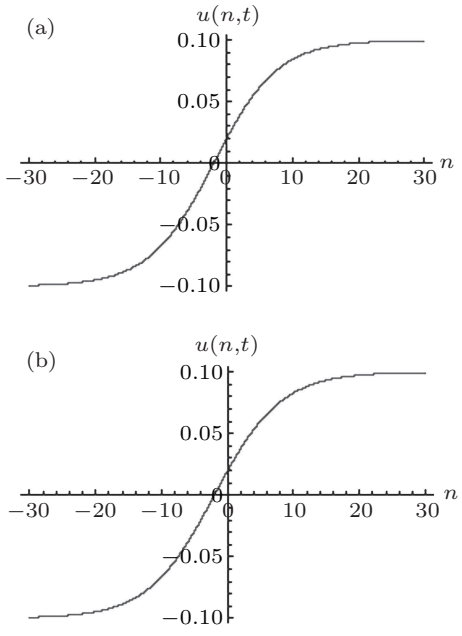


Fig. 4. Comparison between (a) ADTM solution and (b) HATM solution at $\hbar = -1$.

of the ADTM is in excellent agreement with the solution of the HATM, and this ensures that the ADTM is a special case of the HATM at $\hbar = -1$. The obtained results in Ref. [10] using the variational iteration method and in Ref. [17] using the homotopy perturbation method are also in excellent agreement with the solution of the HATM, and this ensures that the variational iteration method and the homotopy perturbation method are special cases of the HATM at $\hbar = -1$.

4. Conclusion

We developed an effective and new modification of HAM, which is based on the homotopy analysis method and the Laplace transform method and we studied its validity to find the solution of a nonlinear differential–difference model in a wide range, which arises in nanotechnology. The obtained solution by utilizing the new homotopy analysis method is very rapidly convergent due to the modification of the Laplace operator. It may be concluded that the HATM methodology is very powerful and efficient in finding approximate solutions as well as analytical solutions for many physical models. The presented numerical examples show that the results of the proposed method are in excellent agreement with those of the ADTM. It is noted that our approximate solution converges at $-2 \leq \hbar \leq 2$ (see Figs. 1 and 2). The explicit and analytic expression given by Eq. (18) contains an auxiliary parameter $\hbar$, which gives the convergence region and the rate of approximation for the HATM. The errors can be further reduced by calculating higher-order terms. This shows that the HATM is a very useful analytic method to obtain accurate analytic solutions for strongly nonlinear problems. The numerical results show that the proposed method has accuracy and reduced size of calculations compared with the VIM and HPM. It is also worth pointing out that the advantage of this method is a fast convergence of the solutions by means of the auxiliary parameter $\hbar = -1$ (see Figs. 1–4). The HATM can be very easily applied to both differential equations and nonlinear differential–difference equations. The advantage of the method is that it does not need a small parameter in the system, leading to wide application in nonlinear problems. In our work, we used the Mathematica package.

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