Random walks in generalized delayed recursive trees*

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Recently a great deal of effort has been made to explicitly determine the mean first-passage time (MFPT) between two nodes averaged over all pairs of nodes on a fractal network. In this paper, we first propose a family of generalized delayed recursive trees characterized by two parameters, where the existing nodes have a time delay to produce new nodes. We then study the MFPT of random walks on this kind of recursive tree and investigate the effect of the time delay on the MFPT. By relating random walks to electrical networks, we obtain an exact formula for the MFPT and verify it by numerical calculations. Based on the obtained results, we further show that the MFPT of delayed recursive trees is much shorter, implying that the efficiency of random walks is much higher compared with the non-delayed counterpart. Our study provides a deeper understanding of random walks on delayed fractal networks.

Keywords: mean first-passage time, random walk, delayed recursive tree

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1. Introduction

In recent years a great number of deterministic networked models have been proposed, such as Sierpinski fractals,[1,2] pseudofractal webs,[3] Apollonian networks,[4,5] Koch networks,[6] etc., including some deterministic uniform recursive trees[7,8] or the T-graph.[9,10] In general, deterministic networks[11–13] are defined against random networks where new nodes are related to existing nodes with a certain probability. Deterministic networks have an advantage with precise formulations for analysis and computation, which could be compared with experimental data from real and simulated networks.[14]

Many problems in physics and chemistry are related to random walks on fractal structures.[9] Random walks on networks are fundamental, which are related to several aspects of complex networks, such as network routing and reaction–diffusion processes.[15,16] A primary quantity characterizing diffusion is the first-passage time (FPT) defined as the expected time for a walker to first arrive at a given destination node starting with an equal probability to neighboring nodes from a random source node.[17,18] Lately, there has been aroused the interest in analytically calculating the mean FPT (MFPT) for some self-similar network models.[7,8,10,19,20]

Recently, a delayed pseudofractal network (DPFN)[21] has been studied, which has a special feature that newly-added edges have a delay in producing new nodes. Existing results show that the delay strengthens the scale-free and small-world characteristics of the underlying network. Numerical calculations reveal that the MFPT of DPFN is shorter, meaning that the efficiency of random walks is higher than that of the non-delayed counterpart. This is in accordance with the fact that sorting comparator network using Fibonacci numbers has fast routing.[22] This delay has significant influence on the topological and dynamical properties of the networks. However, an exact formula for the MFPT is not solvable at present. Hence, as one example for the motivation of the delayed nature of recursive networks, which we study in this work, is to take into account the time delay in data routing on computer networks. We try to formulate the delay on recursive trees in a mathematical manner, so that mathematical analysis tools can be used to obtain some properties that may be insightful as approximation of some real scenarios for better understanding of delays on some real networks such as data routing networks.

In this paper, we propose a family of generalized delayed recursive trees (DRTs), in which finite baby nodes connect to the existing root nodes with an equal probability. In the subsequent steps, baby nodes become adult nodes, and the adult nodes give birth to new baby nodes. For tree-like networks of this kind, through the relationship between random walks and electrical networks,[10] we obtain an exact formula of the MFPT and confirm it by extensive numerical calculations. The average distance of DRTs is in a linear relationship with the step time in the large limit of iteration. In addition, we compare the MFPT of DRTs and recursive trees (RTs) with the same network parameters and show that the MFPT of DRTs is shorter, implying that the efficiency of random walks for the DRTs is higher.

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2. Model description

The generalized delayed recursive trees are built in an iterative way. Denote $T_{r,m}(t)$ as the DRTs after t steps. At step $t = 1$, the number of adult nodes (also called root nodes) is r, which are connected into a string; at step $t = 2$, each of the r root nodes produces m baby nodes, which link to their root nodes; for $t \geq 3$, each of the existing baby nodes grows into adult nodes, while all the root nodes give birth to m baby nodes. The growing process is repeated until the trees reach a desired order. Compared with the recursive trees, the nodes in DRTs delay the production of new nodes, which is different from the iterative algorithms of the recursive trees introduced in Refs. [7] and [23]. Figure 1 shows the generating process of $T_{r,m}(t)$ with $r = 2, m = 2, t = 1, 2, 3, 4$.

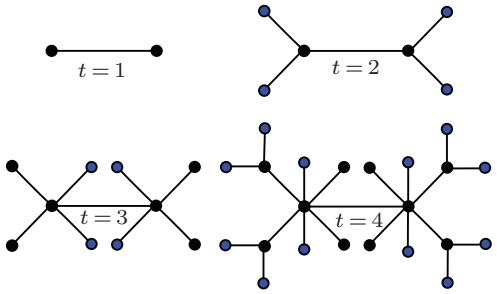


Fig. 1. $T_{r,m}(t)$ with $r = 2$, $m = 2$.

To facilitate the calculations of the following quantities, we first discuss $T_{r,m}(t)$ with $r = 1$, denoted by $T_m(t)$. According to the construction algorithm, the total number of nodes W_t in $T_m(t)$ satisfies the following iterative relationship:

$$W_{t+1} = W_t + mW_{t-1}, t \geq 2. \tag{1}$$

The characteristic equation of the difference Eq. (1) reads

$$\lambda^2 - \lambda - m = 0,$$

whose two roots are

$$\lambda_1 = \frac{1+\sqrt{4m+1}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{4m+1}}{2}.$$

Hence, the general solution of Eq. (1) is $W_t = c_1\lambda_1^t + c_2\lambda_2^t$. Using the initial conditions of $W_1 = 1$ and $W_2 = m+1$, we obtain

$$\begin{aligned} W_t &= c_1 r_1^t + c_2 r_2^t = \frac{1}{\sqrt{4m+1}}(\lambda_1^{t+1} - \lambda_2^{t+1}), \\ N_t &= rW_t, \end{aligned} \tag{2}$$

where

$$c_1 = (\sqrt{4m+1}+1)/2\sqrt{4m+1}$$

and

$$c_2 = (\sqrt{4m+1}-1)/2\sqrt{4m+1}.$$

N_t is the total number of nodes of $T_{r,m}(t)$. In the following sections, we numerically and analytically study the MFPT of the generalized delayed recursive trees $T_{r,m}(t)$.

3. Numerical calculation of the MFPT

In this section, we numerically calculate the MFPT of random walks on the delayed recursive trees. At each time step, the walker can move from its present position to any of its nearest neighbors with an equal probability. A key quantity characterizing the random walks is the first passage time (FPT). Here, we are interested in the MFPT between two distinct nodes, which is then averaged over all pairs. We use the pseudoinverse of the Laplacian matrix to calculate the FPT between two arbitrary nodes. The elements l_{ij}^t of the Laplacian matrix $\boldsymbol{L}_t$ are defined as follows: if there is a link between two different nodes i and j, then $l_{ij}^t = -1$; otherwise, $l_{ij}^t = 0$. The diagonal elements of $\boldsymbol{L}_t$ are defined as $l_{ii}^t = k_i$ (degree of node i). Thus, the pseudoinverse of the $\boldsymbol{L}_t$ is

$$\boldsymbol{L}_t^\flat = \left(\boldsymbol{L}_t - \frac{e_t e_t^{\mathrm{T}}}{N_t}\right)^{-1} + \frac{e_t e_t^{\mathrm{T}}}{N_t}, \tag{3}$$

where $e_t = (1, 1, \ldots, 1)^{\mathrm{T}}$.

The quantity $F_{ij}(t)$ is defined as the FPT for a walker starting from node i to node j for the first time, and can be expressed in terms of the elements $l_{ij}^{\flat,t}$ of $\boldsymbol{L}_t^\flat$ as that in Ref. [24],

$$F_{ij}(t) = \sum_{g=1}^{N_t} \left(l_{ig}^{\flat,t} - l_{ij}^{\flat,t} - l_{jg}^{\flat,t} + l_{jj}^{\flat,t}\right) l_{gg}^t, \tag{4}$$

where l_{gg}^t is the g-th entry of the diagonal of $\boldsymbol{L}_t$. Then, the sum $F_{\mathrm{tot}}(t)$ for the FPT between two arbitrary nodes over all node pairs is

$$F_{\mathrm{tot}}(t) = \sum_{i \neq j} \sum_{j=1}^{N_t} F_{ij}(t), \tag{5}$$

and the MFPT $\langle F \rangle_t$ reads

$$\langle F \rangle_t = \frac{F_{\mathrm{tot}}(t)}{N_t(N_t-1)} = \frac{1}{N_t(N_t-1)} \sum_{i \neq j} \sum_{j=1}^{N_t} F_{ij}(t). \tag{6}$$

The quantity of MFPT $\langle F \rangle_t$ is used to measure the efficiency of the random walks on $T_{r,m}(t)$: the smaller the MFPT value, the higher the efficiency is, and *vice versa*. The calculation of $\langle F \rangle_t$ requires determining the elements of the pseudoinverse matrix $\boldsymbol{L}_t^\flat$, which involves the inversion of an $N_t \times N_t$ matrix. From Eq. (2), the number of nodes N_t in $T_{r,m}(t)$ increases exponentially with step t size, so for larger t it is difficult to obtain $\langle F \rangle_t$ because of the computational complexity for calculating the pseudoinverse matrix. According to the construction algorithm of the considered recursive trees, and according to the connection between FPT and effective resistance, we can obtain an explicit formula for the MFPT as detailed below.

4. Exact solution for the MFPT

In this part, using the relationship between random walks and electrical networks, we derive a closed-form expression for the MFPT, which avoids calculating the inversion of the matrix in Eq. (3). Given a graph, we can obtain the underlying electrical network by replacing each edge with a unit resistor. The effective resistance is then defined as the voltage when a unit current enters into one node and leaves the other.[25] As is well known, the effective resistance between two nodes in a resistor network can be expressed as the FPT between the corresponding two nodes on the considered network, i.e., the effective resistance R_{ij} between a pair of nodes i and j is equal to $(F_{ij}+F_{ji})/2E$, where E is the total number of edges in the graph.[26]

Now, we take $T_{r,m}(t)$ as a resistor network by replacing each edge in $T_{r,m}(t)$ with a unit resistor. With regard to the relationship between the commuting time[27] and the FPT, i.e., $C_{ij}(t)=F_{ij}(t)+F_{ji}(t)$, we obtain $C_{ij}(t)=C_{ji}(t)=2E_tR_{ij}(t)$. Then, equations (5) and (6) are rewritten, respectively, as

$$F_{\rm tot}(t)=\frac{1}{2}\sum_{i\neq j}\sum_{j=1}^{N_t}C_{ij}(t)=E_t\sum_{i\neq j}\sum_{j=1}^{N_t}R_{ij}(t), \tag{7}$$

and

$$\langle F\rangle_t=\frac{F_{\rm tot}(t)}{N_t(N_t-1)}=\frac{1}{N_t}\sum_{i\neq j}\sum_{j=1}^{N_t}R_{ij}(t). \tag{8}$$

For a tree-like graph, the effective resistance R_{ij} between two nodes is exactly the usual shortest path distance between the corresponding nodes. Since $T_{r,m}(t)$ is a tree-like network, we can use these properties to obtain a rigorous expression for the MFPT.

From the above discussion, the determination of the MFPT means calculating the shortest path distance. To this end, we represent all the shortest path lengths of $T_{r,m}(t)$ with a matrix, where the entry d_{ij} is the shortest distance from node i to node j. Then, the average distance d_t of $T_{r,m}(t)$ is

$$d_t=\frac{\Phi_t}{N_t(N_t-1)/2}, \tag{9}$$

where

$$\Phi_t=\sum_{i,\,j\in T_{r,m}(t),\,i\neq j}d_{ij} \tag{10}$$

denotes the sum of all the shortest path lengths between two nodes over all pairs of nodes. Note that we only calculate d_{ij} or d_{ji}, not both.

4.1. Calculation of shortest path length matrix for $T_{r,m}(t)$ with $r=1$

Similarly, denote all the shortest path lengths of $T_m(t)$ by ϕ_t. According to the self-similar structures of $T_m(t)$, we can easily obtain the following relationship:

$$\phi_{t+1}=\phi_t+m\phi_{t-1}+\delta_t, \tag{11}$$

where δ_t represents the sum of path lengths of all the pairs of nodes lying in different parts.

As shown in Fig. 2, the distances among all pairs of nodes are divided into two parts, so that

$$\delta_t^{1,2}=\sum_{i\in T_m^{\rm root}(t),\,j\in T_m^1(t-1)}d_{ij},$$
$$\delta_t^{1,3}=\sum_{i\in T_m^1(t-1),\,j\in T_m^2(t-1)}d_{ij},$$

consequently,

$$\delta_t=m\,\delta_t^{1,2}+\frac{m(m-1)}{2}\,\delta_t^{1,3}. \tag{12}$$

In order to obtain $\delta_t^{1,2}$ and $\delta_t^{1,3}$, we define

$$S_t=\sum_{i\in T_m(t),\,i\neq Q}d_{iQ},$$

where Q is the root node. According to the self-similar network structure, we have

$$S_{t+1}=S_t+mS_{t-1}+mW_{t-1}, \tag{13}$$

with initial conditions of $S_1=0, S_2=m$.

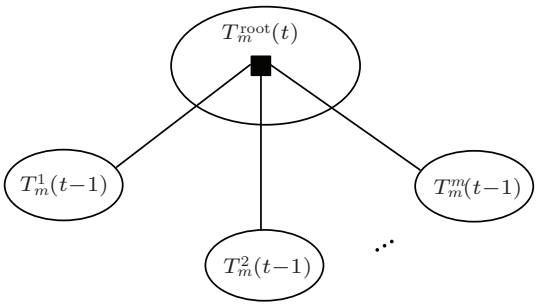


Fig. 2. A schematic illustration of $T_m(t+1)$, which is composed of $m+1$ parts and represented as $T_m^1(t-1),\ldots,T_m^m(t-1)$ and $T_m^{\rm root}(t)$, respectively. The black solid square denotes the root nodes.

We now calculate the exact solution of Eq. (13). We first obtain the special solutions of the following difference equations:

$$S_{t+1}^1=S_t^1+mS_{t-1}^1+\frac{m}{\sqrt{4m+1}}\lambda_1^t, \tag{14}$$
$$S_{t+1}^2=S_t^2+mS_{t-1}^2-\frac{m}{\sqrt{4m+1}}\lambda_2^t. \tag{15}$$

Inserting $S_t^1=A_1\,t\,\lambda_1^t$ into Eq. (14) gives

$$A_1=\frac{m\lambda_1}{(\lambda_1^2+m)\sqrt{4m+1}}.$$

Similarly, $S_t^2=A_2\,t\,\lambda_2^t$ with

$$A_2=-\frac{m\lambda_2}{(\lambda_2^2+m)\sqrt{4m+1}}.$$

Thus, the special solution S_t^* of Eq. (13) is

$$S_t^* = S_t^1 + S_t^2 = \frac{mt}{\sqrt{4m+1}}\left[\frac{\lambda_1^{t+1}}{(\lambda_1^2+m)} - \frac{\lambda_2^{t+1}}{(\lambda_2^2+m)}\right].$$

Then, the general solution of Eq. (13) is written as

$$S_t = c_1^*\lambda_1^t + c_2^*\lambda_2^t + \frac{mt}{\sqrt{4m+1}}\left[\frac{\lambda_1^{t+1}}{(\lambda_1^2+m)} - \frac{\lambda_2^{t+1}}{(\lambda_2^2+m)}\right]. \quad (16)$$

Using the initial conditions of $S_1 = 0$ and $S_2 = m$ gives

$$c_1^* = \frac{-m}{(4m+1)\sqrt{4m+1}} \quad \text{and} \quad c_2^* = \frac{m}{(4m+1)\sqrt{4m+1}}.$$

Based on the structures of the considered trees, the quantities $\delta_t^{1,2}$ and $\delta_t^{1,3}$ are expressed as

$$\begin{aligned}
\delta_t^{1,2} &= \sum_{i\in T_m^{\text{root}}(t),\, j\in T_m^1(t-1)} d_{ij} \\
&= \sum_{i\in T_m^{\text{root}}(t),\, j\in T_m^1(t-1)} (d_{iX} + d_{XY} + d_{Yj}) \\
&= W_{t-1}\sum_{i\in T_m^{\text{root}}(t)} d_{iX} + W_t W_{t-1} d_{XY} + W_t \sum_{j\in T_m^1(t-1)} d_{Yj} \\
&= W_{t-1}S_t + W_t W_{t-1} + W_t S_{t-1}, \quad (17) \\
\delta_t^{1,3} &= \sum_{i\in T_m^1(t-1),\, j\in T_m^2(t-1)} d_{ij} \\
&= \sum_{i\in T_m^1(t-1),\, j\in T_m^2(t-1)} (d_{iY} + d_{YZ} + d_{Zj}) \\
&= W_{t-1}\sum_{i\in T_m^1(t-1)} d_{iY} + W_{t-1}W_{t-1}d_{YZ} + W_{t-1}\sum_{j\in T_m^2(t-1)} d_{Zj} \\
&= 2W_{t-1}S_{t-1} + 2W_{t-1}W_{t-1}, \quad (18)
\end{aligned}$$

where X and $Y(Z)$ are the root nodes in $T_m(t)$ and $T_m(t-1)$, respectively. Inserting Eq. (17) and Eq. (18) into Eq. (12) gives

$$\delta_t = mW_{t-1}S_t + mW_{t-1}W_t + mW_t S_{t-1} + m(m-1)W_{t-1}S_{t-1} + m(m-1)W_{t-1}^2. \quad (19)$$

Substituting Eqs. (2) and (16) into Eq. (19) yields

$$\delta_t = \left[\Theta_1(\lambda_1\lambda_2)^t + \Theta_2\lambda_1^{2t} + \Theta_3\lambda_2^{2t}\right]t + \Theta_4(\lambda_1\lambda_2)^t + \Theta_5\lambda_1^{2t} + \Theta_6\lambda_2^{2t}, \quad (20)$$

where

$$\begin{aligned}
\Theta_1 &= -\frac{m^2}{a},\ (a = 4m+1); \\
\Theta_2 &= \frac{2m^2(m+\sqrt{a})}{a^2+a^{3/2}};\quad \Theta_3 = \frac{2m^2(m-\sqrt{a})}{a^2-a^{3/2}}; \\
\Theta_4 &= \frac{-4m^3+2m^2+m}{a^2}; \\
\Theta_5 &= \frac{m^2\left[(a+1)^2\sqrt{a}+4a^2\right]}{2(a+1)a^{5/2}+4a^3}; \\
\Theta_6 &= \frac{m^2\left[(a+1)^2\sqrt{a}-4a^2\right]}{2(a+1)a^{5/2}-4a^3}.
\end{aligned}$$

Then, equation (11) is recast to be

$$\phi_{t+1} = \phi_t + m\phi_{t-1} + \left[\Theta_1(\lambda_1\lambda_2)^t + \Theta_2\lambda_1^{2t} + \Theta_3\lambda_2^{2t}\right]t + \Theta_4(\lambda_1\lambda_2)^t + \Theta_5\lambda_1^{2t} + \Theta_6\lambda_2^{2t}. \quad (21)$$

Following the above method of calculating the general solution of S_t, we next derive the exact general solution of Eq. (21). Suppose that equation (21) has a special solution $\phi^* = \sum_{j=1}^6 \phi^{j*}$, where $\phi^{j*}\,(j = 1,2,\ldots,6)$ are the special solutions of the following six equations:

$$\begin{aligned}
\phi_{t+1} &= \phi_t + m\phi_{t-1} + \Theta_1 t\,(\lambda_1\lambda_2)^t, \\
\phi_{t+1} &= \phi_t + m\phi_{t-1} + \Theta_2 t\,\lambda_1^{2t}, \\
\phi_{t+1} &= \phi_t + m\phi_{t-1} + \Theta_3 t\,\lambda_2^{2t}, \\
\phi_{t+1} &= \phi_t + m\phi_{t-1} + \Theta_4(\lambda_1\lambda_2)^t, \\
\phi_{t+1} &= \phi_t + m\phi_{t-1} + \Theta_5\lambda_1^{2t}, \\
\phi_{t+1} &= \phi_t + m\phi_{t-1} + \Theta_6\lambda_2^{2t}.
\end{aligned}$$

Solving the above equations, we obtain

$$\begin{aligned}
&\phi^{1*} = (\alpha_1 t + \beta_1)(\lambda_1\lambda_2)^t, \quad \phi^{2*} = (\alpha_2 t + \beta_2)\lambda_1{}^{2t}, \\
&\phi^{3*} = (\alpha_3 t + \beta_3)\lambda_2{}^{2t}, \quad \phi^{4*} = \alpha_4(\lambda_1\lambda_2)^t, \\
&\phi^{5*} = \alpha_5\lambda_1{}^{2t}, \quad \phi^{6*} = \alpha_6\lambda_2{}^{2t},
\end{aligned}$$

where

$$\begin{aligned}
&\alpha_1 = -\frac{\Theta_1}{m},\quad \alpha_2 = \frac{\Theta_2\lambda_1^2}{\lambda_1^4-\lambda_1^2-m},\quad \alpha_3 = \frac{\Theta_3\lambda_2^2}{\lambda_2^4-\lambda_2^2-m}, \\
&\alpha_4 = -\frac{\Theta_4}{m},\quad \alpha_5 = \frac{\Theta_5\lambda_1^2}{\lambda_1^4-\lambda_1^2-m},\quad \alpha_6 = \frac{\Theta_6\lambda_2^2}{\lambda_2^4-\lambda_2^2-m}, \\
&\beta_1 = \frac{(m+1)\Theta_1}{m^2},\quad \beta_2 = -\frac{\Theta_2\lambda_1^2(m+\lambda_1^4)}{(\lambda_1^4-\lambda_1^2-m)^2}, \\
&\beta_3 = -\frac{\Theta_3\lambda_2^2(m+\lambda_2^4)}{(\lambda_2^4-\lambda_2^2-m)^2}.
\end{aligned}$$

Therefore, the general solution of Eq. (21) reads

$$\begin{aligned}
\phi_t &= \gamma_1\lambda_1^t + \gamma_2\lambda_2^t + \sum_{j=1}^6 \phi^{j*} \quad (22) \\
&= \gamma_1\lambda_1^t + \gamma_2\lambda_2^t + (\alpha_1 t + \alpha_4 + \beta_1)(\lambda_1\lambda_2)^t \\
&\quad + (\alpha_2 t + \alpha_5 + \beta_2)\lambda_1^{2t} + (\alpha_3 t + \alpha_6 + \beta_3)\lambda_2^{2t}.
\end{aligned}$$

Considering the initial conditions of $\phi_1 = 0$ and $\phi_2 = m^2$, we finally obtain

$$\gamma_1 = \frac{m^2 + m\lambda_2[\sqrt{a}\,\alpha_1 - \lambda_2(\alpha_4+\beta_1)] + \lambda_2^3[\sqrt{a}\,\alpha_3 + \lambda_1(\alpha_6+\beta_3)]}{m+\lambda_1^2} - \frac{\lambda_1^2[\lambda_1^2\alpha_2 + (m+\sqrt{a})(\alpha_2+\alpha_5+\beta_2)]}{m+\lambda_1^2},$$

$$\gamma_2 = \frac{m^2 - m\lambda_1[\sqrt{a}\,\alpha_1 + \lambda_1(\alpha_4+\beta_1)] - \lambda_1^3[\sqrt{a}\,\alpha_2 - \lambda_2(\alpha_5+\beta_2)]}{m+\lambda_2^2} - \frac{\lambda_2^2[\lambda_2^2\alpha_3 + (m-\sqrt{a})(\alpha_3+\alpha_6+\beta_3)]}{m+\lambda_2^2}.$$

4.2. Calculation of shortest path length matrix for $T_{r,m}(t)$

In this subsection, using the above result of ϕ_t, we obtain the Φ_t for $T_{r,m}(t)$. Since $T_{r,m}(t)$ is self-similar, $T_{r,m}(t)$ can be obtained by connecting root nodes of copies of $T_m(t)$ as shown in Fig. 3. According to the structures of $T_{r,m}(t)$, we obtain

$$\Phi_{t+1} = r\,\phi_{t+1} + \Delta_{t+1}, \tag{23}$$

and

$$\Delta_{t+1} = (r-1)\Delta_{1,2} + (r-2)\Delta_{1,3} + \cdots + \Delta_{1,r}, \tag{24}$$

where $\Delta_{i,j}$ $(i,j = 1,2,\ldots,r, i<j)$ denotes the sum over the lengths of all shortest paths, whose one end point lies in $Q_m^i(t+1)$ and another end is in $Q_m^j(t+1)$. From Fig. 3, it is obvious that the distance between the root node (denoted by ■) of $Q_m^i(t+1)$ and that of $Q_m^j(t+1)$ is $j-i$.

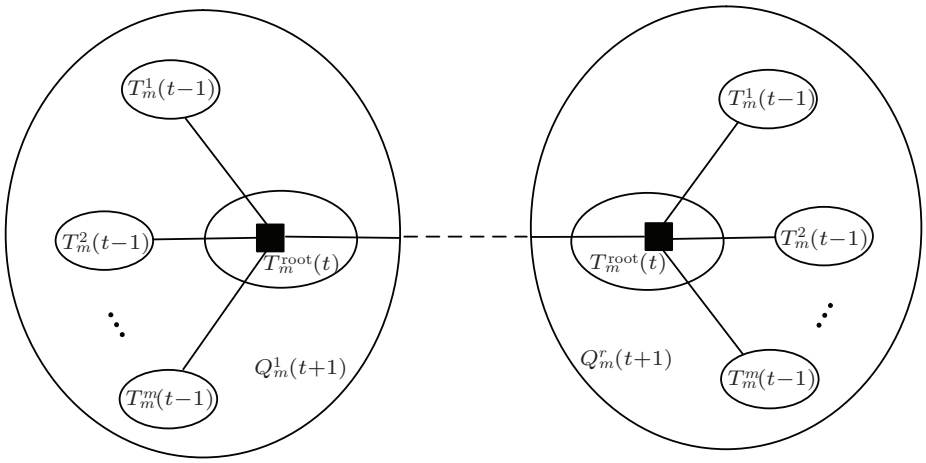


Fig. 3. $T_{r,m}(t+1)$ constructed by r parts of $T_m(t+1)$.

Let R_1, R_2, and R_3 be the root nodes of $Q_m^1(t+1)$, $Q_m^2(t+1)$, and $Q_m^3(t+1)$, respectively, then the quantities $\Delta_{1,2}$ and $\Delta_{1,3}$ will be expressed, respectively, as

$$\begin{aligned}\Delta_{1,2} &= \sum_{i\in Q_m^1(t+1),\, j\in Q_m^2(t+1)} (d_{iR_1} + d_{R_1R_2} + d_{R_2j}) \\ &= W_{t+1} \sum_{j\in Q_m^2(t+1)} d_{R_2j} + W_{t+1}\cdot W_{t+1} + W_{t+1} \sum_{i\in Q_m^1(t+1)} d_{iR_1} \\ &= 2W_{t+1}\cdot S_{t+1} + W_{t+1}^2,\end{aligned}$$

and

$$\begin{aligned}\Delta_{1,3} &= \sum_{i\in Q_m^1(t+1),\, j\in Q_m^3(t+1)} (d_{iR_1} + d_{R_1R_3} + d_{R_3j}) \\ &= W_{t+1} \sum_{j\in Q_m^3(t+1)} d_{R_3j} + 2W_{t+1}\cdot W_{t+1} + W_{t+1} \sum_{i\in Q_m^1(t+1)} d_{iR_1} \\ &= 2W_{t+1}\cdot S_{t+1} + 2W_{t+1}^2,\end{aligned}$$

where $d_{R_1R_2} = 1$ and $d_{R_1R_3} = 2$. Subsequently,

$$\Delta_{i,j} = 2W_{t+1}\cdot S_{t+1} + (j-i)W_{t+1}^2, \quad i,j = 1,2,\cdots,r, i<j. \tag{25}$$

Inserting Eq. (25) into Eq. (24) yields

$$\begin{aligned}\Delta_t &= (r-1)(2W_t\cdot S_t + W_t^2) + (r-2)(2W_t\cdot S_t + 2W_t^2) \\ &\quad + \cdots + (2W_t\cdot S_t + (r-1)W_t^2) \\ &= \sum_{s=1}^{r-1}[(r-s)(2W_t\cdot S_t + sW_t^2)] \\ &= \frac{r(r-1)\,g_1}{3}(\lambda_1\lambda_2)^t + \frac{r(r-1)\,g_2}{6}\lambda_1^{2t} + \frac{r(r-1)\,g_3}{6}\lambda_2^{2t},\end{aligned}$$

where

$$\begin{aligned}g_1 &= \frac{3m}{a}t + \frac{m(ar+a+3)}{a^2}, \\ g_2 &= \frac{6m\lambda_1^2}{a(\lambda_1^2+m)}t + \frac{(r+1)\lambda_1^2}{a} - \frac{6m\lambda_1}{a^2}, \\ g_3 &= \frac{6m\lambda_2^2}{a(\lambda_2^2+m)}t + \frac{(r+1)\lambda_2^2}{a} - \frac{6m\lambda_2}{a^2}.\end{aligned}$$

Substituting ϕ_t and Δ_t into Eq. (23) gives

$$\begin{aligned}\Phi_t &= r(\gamma_1\lambda_1^t + \gamma_2\lambda_2^t) \\ &\quad + \left[r(\alpha_1 t + \alpha_4 + \beta_1) + \frac{r(r-1)\,g_1}{3}\right](\lambda_1\lambda_2)^t \\ &\quad + \left[r(\alpha_2 t + \alpha_5 + \beta_2) + \frac{r(r-1)\,g_2}{6}\right]\lambda_1^{2t} \\ &\quad + \left[r(\alpha_3 t + \alpha_6 + \beta_3) + \frac{r(r-1)\,g_3}{6}\right]\lambda_2^{2t}.\end{aligned}$$

Then, the analytic form of the average distance of DRTs is

$$d_t = \frac{2\Phi_t}{N_t(N_t-1)}. \tag{26}$$

In the large limit of t, we have

$$d_t \approx \frac{6(\alpha_2 t + \alpha_5 + \beta_2) + (r-1)g_2}{3rc_1^2}.$$

As a result of the tree-like structure of $T_{r,m}(t)$, the shortest path distance is equal to the resistance distance, i.e., $d_{ij} = R_{ij}$. Using the exact results of Φ_t and W_t, we easily obtain the expression for the MFPT with only parameter t, as

$$\langle F\rangle_t = E_t d_t = \frac{2\Phi_t}{rW_t}. \tag{27}$$

5. Numerical analysis

In Fig. 4, we compare the analytical formula with numerical values of the MFPT of DRTs with $r = m = 2$. In the domain of $1 \le t \le 10$, the values of MFPT obtained from Eq. (27) are in complete agreement with the results derived by the pseudoinverse matrix method as shown in Eq. (3). On the other hand, this agreement is an independent test of our exact formula.

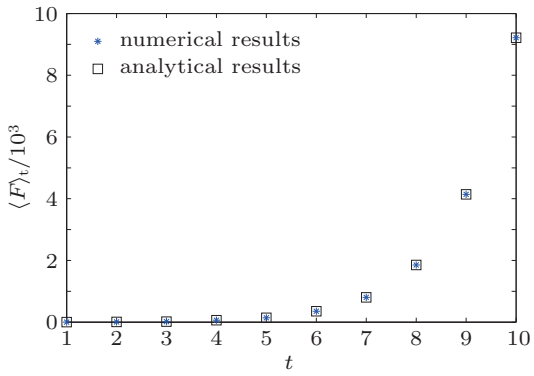


Fig. 4. (color online) MFPT $\langle F\rangle_t$ with respect to step time t. Numerical results (blue stars) obtained by Eqs. (4) and (6) accord with the exact values (black squares) from Eq. (27).

Figure 5 shows a comparison of the MFPT between DRTs and RTs. The recursive trees with parameters r and m were introduced in Ref. [28], and an exact formula of the MFPT was given. In the numerical simulation, we calculate the values of the MFPT for DRTs and RTs with same parameters $(r = m = 2)$. Numerical results show that the MFPT of DRTs is much shorter. In our model, the delay can cause the size of DRTs to be smaller than that of RTs. Using the relation between random walks and electrical networks, the effective resistance is equal to the shortest path length between each pair of nodes. From the numerical results, we know that the average path length of DRTs is smaller than that of RTs, which implies that the MFPT of DRTs is shorter. Note that in Ref. [10] Zhang *et al.* studied the MFPT for the T-graph, and obtained an exact formula for the MFPT. The values of MFPT of T-graph are larger than those of the delayed or non-delayed recursive trees. The discrepancy may be related to their fractal dimensions,[29] which would be investigated in the future.

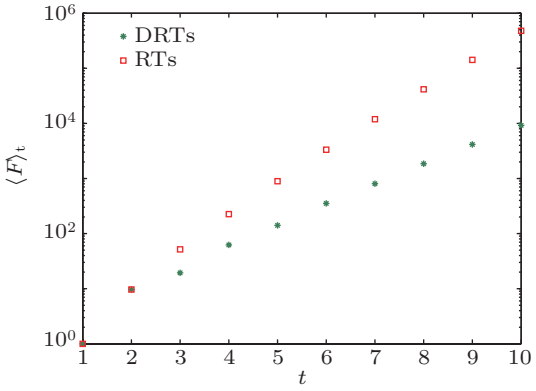


Fig. 5. (color online) Comparison of MFPT between DRTs and RTs with the same parameters $r = m = 2$ versus step time t on a semilogarithmic scale. The MFPT of DRTs is much shorter than that of RTs, which shows that the delay has stronger influence on the MFPT.

6. Conclusions

The present study on the determination of mean first-passage time (MFPT) of delayed fractal networks is rare. In this paper, we investigate the random walks on delayed recursive trees (DRTs) constructed by two parameters with novel features where the existing nodes delay the production of new nodes. By making use of the relationship between electrical networks and random walks, and by considering the self-similar network structure, we obtain a closed-form solution for the MFPT. Compared with the recursive trees (RTs), the MFPT of DRTs is much shorter, showing that the delay enhances the efficiency of random walks. In addition, we verify the numerical results for the MFPT obtained in Ref. [21]. Because of the delay, the new network framework will not have exactly the same size (numbers of nodes and edges) as the original framework, we only make a comparison of the efficiency of random walks between DRTs and RTs at the same step time. We expect that our results can motivate further study on potential applications in computer science using delayed recursive trees, such as implementing multi-computer systems and achieving fast routing in sorting comparator networks.

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