The effect of fractional thermoelasticity on a two-dimensional problem of a mode I crack in a rotating fiber-reinforced thermoelastic medium

Ahmed E. Abouelregal[a)c)] and Ashraf M. Zenkour[b)d)†]

[a)] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura* 35516, *Egypt*
[b)] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box* 80203, *Jeddah* 21589, *Saudi Arabia*
[c)] *Department of Mathematics, College of Science and Arts, University of Aljouf, El-Qurayat, Saudi Arabia*
[d)] *Department of Mathematics, Faculty of Science, Kafrelsheikh University, Kafr El-Sheikh* 33516, *Egypt*

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This article is concerned with the effect of rotation on the general model of the equations of the generalized thermoelasticity for a homogeneous isotropic elastic half-space solid, whose surface is subjected to a Mode-I crack problem. The fractional order theory of thermoelasticity is used to obtain the analytical solutions for displacement components, force stresses, and temperature. The boundary of the crack is subjected to a prescribed stress distribution and temperature. The normal mode analysis technique is used to solve the resulting non-dimensional coupled governing equations of the problem. The variations of the considered variables with the horizontal distance are illustrated graphically. Some particular cases are also discussed in the context of the problem. Effects of the fractional parameter, reinforcement, and rotation on the variations of different field quantities inside the elastic medium are analyzed graphically. Comparisons are made between the results in the presence and those in the absence of fiber-reinforcing, rotating and fractional parameters.

Keywords: fiber-reinforced, mode-I crack, fractional order thermoelasticity theory, rotating medium, normal mode analysis

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1. Introduction

The governing field equations in classical dynamic coupled thermoelasticity (CTE) are of wave-type (hyperbolic) equations of motion and a diffusion-type (parabolic) equation of heat conduction. Therefore, it is seen that part of the solution of the energy equation extends to infinity, implying that if a homogeneous isotropic elastic medium is subjected to thermal or mechanical disturbance, the effects of temperature and displacement fields are felt at an infinite distance from the source of disturbance. This shows that part of the disturbance has an infinite velocity of propagation, which is physically impossible. With this drawback in mind, some researchers, such as Lord and Shulman[1] and Green and Lindsay,[2] modified Fourier's law and constitutive relations so as to obtain a hyperbolic equation for heat conduction. These investigations include the time needed to accelerate the heat flow and take into account the coupling between temperature and strain fields for an isotropic material. The theory of thermoelasticity without energy dissipation is another generalized theory and was formulated by Green and Naghdi.[3] It includes the "thermal displacement gradient" among its independent constitutive variables, and differs from the previous theories in that it does not accommodate the dissipation of thermal energy.[4]

The theory of including the effect of temperature change, known as the theory of thermoelasticity, has also been well established. According to the theory, the temperature field is coupled with the elastic strain field. In thermoelasticity, for the classical heat transfer, Fourier's conduction equation is extensively used in many engineering applications. The classical theory of thermoelasticity Nowacki,[5,6] rests upon the hypothesis of the Fourier law of heat conduction, in which the temperature distribution is governed by a parabolic-type partial differential equation. Consequently, the theory predicts that a thermal signal is felt instantaneously everywhere in a body. This implies that the speed of propagation of the thermal signal is infinite, which is impractical from the physical point of view, particularly for short-time. Thus, the use of Fourier's equation may result in discrepancies under some special conditions, such as low-temperature heat transfer, high frequency or ultrahigh heat flux heat transfer, etc.

The idea of introducing a continuous self-reinforcement at every point of an elastic solid was given by Belfield *et al.*[7] The model was later applied to the rotation of a tube by Verma and Rana.[8] Sengupta and Nath[9] discussed the problem of the surface waves in fiber-reinforced anisotropic elastic media. Hashin and Rosen[10] gave the elastic moduli for fiber-reinforced materials. The problem of reflection of plane waves at the free surface of a fiber-reinforced elastic half-space was discussed by Singh and Singh.[11] Singh[12] discussed the wave propagation in incompressible transversely-

†Corresponding author. E-mail: zenkour@gmail.com
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isotropic fiber-reinforced elastic media. Singh[13] studied the effects of anisotropy on reflection coefficients of plane waves in fiber-reinforced thermoelastic solid. Kumar and Gupta[14] investigated a source problem in fiber-reinforced anisotropic generalized thermoelastic solid under an acoustic fluid layer. Ailawalia and Budhiraja[15] discussed the effect of hydrostatic initial stress on a fiber-reinforced generalized thermoelastic medium. Abbas and Abd-Alla[16] studied the effect of initial stress on a fiber-reinforced anisotropic thermoelastic thick plate. Lotfy[17] introduced different thermoelasticity theories to study the influences of a magnetic field and rotation on a two-dimensional problem of fiber-reinforced thermoelasticity.

Fractional calculus is a natural extension of classical mathematics. In fact, since the founding of the differential calculus, the generalization of the concept of derivative and integral to a non-integer order has been the subject of distinct approaches. Due to this reason there are several definitions[18–20] which are proved to be equivalent. Fractional calculus has been used in many fields, ranging from statistical physics, chemistry to biological sciences and economics. In recent years, there has been a great deal of interest in fractional differential equations. Several definitions of the fractional derivative have been proposed. The history and classic transform rules of this subject are well covered in the monograph by Podlubny.[21]

During recent years, fractional calculus has also been introduced into the field of thermoelasticity. Povstenko[22] has constructed a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with a fractional order time derivative. He used Caputo's fractional derivative[23] and obtained the stress components corresponding to the fundamental solution of Cauchy's problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. In 2010, a new theory of generalized thermoelasticity in the context of a new consideration of heat conduction with a fractional order has been proposed by Youssef.[24] In the same year, Sherief *et al.*[25] has constructed another model in generalized thermoelasticity theory by using fractional time derivatives. Abouelregal[26] used the generalized thermoelasticity theory based on a fractional order model, to solve a one-dimensional boundary value problem of a semi-infinite piezoelectric medium. Zenkour and Abouelregal[27] used the state-space approach for an infinite medium with a spherical cavity based on two-temperature generalized thermoelasticity theory and fractional heat conduction.

Paria[28] and Wilson[29] investigated the propagation of magneto–thermoelastic waves in a non-rotating medium. These studies, based on the theory of classical coupled thermoelasticity, were essentially concerned with the interaction among the electromagnetic field, the thermal field, and the elastic field, as well as the dispersion relation. In a paper by Schoenberg and Censor,[30] the propagation of plane harmonic waves in a rotating elastic medium has been investigated in some detail. It has been shown that the rotation causes the elastic medium to be dispersive and anisotropic. This study included some discussion on the free-surface phenomenon in a rotating half-space. Results concerning slowness surfaces, energy flux, reflected waves, and generalized Rayleigh waves have been obtained. Later on, several authors[31–35] studied plane waves in rotating thermoelastic and magneto–thermoelastic media in the context of generalized theories. It seems from the above discussion that little attention has been paid to the study of propagation of thermoelastic plane waves in a rotating medium in the presence of an external magnetic field based on the generalized thermoelasticity. In view of the fact that most large bodies like the Earth, the Moon, and other planets each have an angular velocity, it is important to consider the propagation of magneto–thermoelastic plane waves in an electrically conducting, rotating viscoelastic medium under the action of an external magnetic field.

The aim of this paper is to determine the displacements, stresses, and temperature distributions in a fiber-reinforced generalized thermoelastic medium. This medium is rotating with a uniform angular velocity, and the effect of fractional order thermoelasticity is presented. The normal mode analysis is used to obtain the exact expressions for the displacement components, force stresses, and temperature. The variations of the considered variables with the horizontal distance are illustrated graphically. Comparisons are made between the results in the presence and those in the absence of fiber-reinforcing, rotating, and fractional parameters. The results reduce to corresponding classical results when the reinforced elastic parameters tend to zero and the medium becomes isotropic. The effects of fractional parameter, reinforcement, and rotation on the variations of different field quantities inside the medium are analyzed graphically. Results carried out in this paper can be used to design various fiber-reinforced anisotropic thermoelastic elements under thermal load to meet special engineering requirements.

2. Basic equations

Consider the propagation of plane thermoelastic waves in a homogeneous thermoelastic medium with density ρ at uniform initial temperature T_0. The medium is rotating with an angular velocity $\boldsymbol{\Omega} = \Omega \boldsymbol{n}$ where $\boldsymbol{n}$ is a unit vector representing the direction of the axis of rotation. The displacement equation of motion in the rotating frame of reference has two additional terms: centripetal acceleration $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})$ due to the time-varying motion only and the Coriolis acceleration $2(\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})$ where $\boldsymbol{u}$ is the dynamic displacement vector. These terms do not appear in a non-rotating medium. The dynamic displacement vector is actually measured from a steady-

state deformed position, and the deformation is assumed to be small.

The principle of balance of linear momentum leads to the following equations of motion:

$$\sigma_{ij,j} = \rho\left[\ddot{u}_i + (\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}) + 2(\boldsymbol{\Omega}\times\dot{\boldsymbol{u}}))_i\right], \tag{1}$$

where σ_{ij} are the components of stress tensor, ρ is the material density, u_i are the components of displacement vector and $ij = 1, 2, 3$ refer to general coordinates. The comma notation is used for spatial derivatives and the superimposed dot represents time differentiation.

The modified fractional order heat conduction equation in the absence of heat sources (assuming that the rotation of the body has no effect on heat conduction of the medium) has the form[24]

$$KT_{,ii} = \rho C_E\left(\delta + t_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial T}{\partial t} + \gamma T_0\left(\delta + t_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial u_{i,i}}{\partial t}, \quad 0 < \alpha \le 1, \tag{2}$$

where K is the thermal conductivity, t_0 is a constant with the dimensions of time that acts as a relaxation time, α is the fractional parameter, C_E is the specific heat at constant strain, T_0 is the temperature of the medium in its natural state, on the assumption that $|(T-T_0)/T| \ll 1$, and $\gamma = (3\lambda + 2\mu_{\mathrm{T}})\alpha_t$, in which α_t is the thermal expansion coefficient and λ and μ_{T} are Lame's coefficients.

The constitutive equations for a fiber-reinforced linearly elastic medium with respect to the reinforcement direction $\boldsymbol{b} \equiv (b_1, b_2, b_3)$ with $b_1^2 + b_2^2 + b_3^2 = 1$ are

$$\begin{aligned}\sigma_{ij} = {} & \lambda e_{kk}\delta_{ij} + 2\mu_{\mathrm{T}}e_{ij} + \xi(b_k b_m e_{km} + b_i b_j e_{kk}) \\ & + 2(\mu_{\mathrm{L}} - \mu_{\mathrm{T}})(b_k b_i e_{kj} + b_k b_j e_{ki}) \\ & + \beta b_k b_m e_{km} b_i b_j - \gamma(T - T_0),\end{aligned} \tag{3}$$

where $e_{ij} = (u_{i,j} + u_{j,i})/2$ are the components of strain tensor, ξ, β, and $(\mu_{\mathrm{L}} - \mu_{\mathrm{T}})$ are the reinforcement parameters.

3. Formulation of the problem

In the present article, a problem of a fiber-reinforced half-space $(x \ge 0)$ is considered. All functions will depend on time t and coordinates (x, y, z). For the two-dimensional problem, the components of the displacement vector are assumed to be in the form of $u_1 = u(x, y, t)$, $u_2 = v(x, y, t)$, and $u_3 = 0$. Also, it is assumed that the solutions are independent of z, i.e., $\partial(\cdot)/\partial z = 0$. The fiber-direction is chosen such that $\boldsymbol{b} = (1, 0, 0)$, so the preferred direction is the x axis. Then Eq. (3) yields

$$\begin{aligned}\sigma_{xx} &= A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} - \gamma(T - T_0), \\ \sigma_{yy} &= A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} - \gamma(T - T_0), \\ \sigma_{zz} &= A_{12}\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} - \gamma(T - T_0), \\ \sigma_{xy} &= \mu_{\mathrm{L}}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \sigma_{xz} = \sigma_{yz} = 0,\end{aligned} \tag{4}$$

where $A_{11} = \lambda + 2(\xi + \mu_{\mathrm{T}}) + 4(\mu_{\mathrm{L}} - \mu_{\mathrm{T}}) + \beta$, $A_{12} = \xi + \lambda$, and $A_{22} = \lambda + 2\mu_{\mathrm{T}}$. Using the summation convection and substituting Eq. (4) into Eq. (1), the equations of motion become

$$\begin{aligned}A_{11}\frac{\partial^2 u}{\partial x^2} + \mu_{\mathrm{L}}\frac{\partial^2 u}{\partial y^2} + A_{13}\frac{\partial^2 v}{\partial x \partial y} - \gamma\frac{\partial T}{\partial x} &= \rho(\ddot{u} - \Omega^2 u - 2\Omega\dot{v}), \\ A_{22}\frac{\partial^2 u}{\partial y^2} + \mu_{\mathrm{L}}\frac{\partial^2 u}{\partial x^2} + A_{13}\frac{\partial^2 u}{\partial x \partial y} - \gamma\frac{\partial T}{\partial y} &= \rho(\ddot{v} - \Omega^2 v + 2\Omega\dot{u}),\end{aligned} \tag{5}$$

where $A_{13} = A_{12} + \mu_{\mathrm{L}}$. For convenience, the following non-dimensional variables are used:

$$\begin{aligned}& x' = c_0\eta x,\ y' = c_0\eta y,\ u' = c_0\eta u,\ v' = c_0\eta v,\ t' = c_0^2\eta t, \\ & t_0' = c_0^2\eta t_0,\ \theta = \frac{\gamma(T - T_0)}{A_{22}},\ \sigma_{ij}' = \frac{\sigma_{ij}}{\gamma T_0},\ \Omega' = \frac{\Omega}{\eta c_0^2}, \\ & \eta = \frac{\rho C_E}{K},\ c_0^2 = \frac{A_{22}}{\rho}.\end{aligned} \tag{6}$$

In terms of non-dimensional quantities defined in Eq. (6), the above governing equations reduce to (dropping the superscript prime)

$$\begin{aligned}h_1\frac{\partial^2 u}{\partial x^2} + h_2\frac{\partial^2 u}{\partial y^2} + h_3\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial\theta}{\partial x} &= \frac{\partial^2 u}{\partial t^2} - \Omega^2 u - 2\Omega\frac{\partial v}{\partial t}, \\ \frac{\partial^2 u}{\partial y^2} + h_2\frac{\partial^2 u}{\partial x^2} + h_3\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial\theta}{\partial y} &= \frac{\partial^2 v}{\partial t^2} - \Omega^2 v + 2\Omega\frac{\partial u}{\partial t},\end{aligned} \tag{7}$$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \left(\delta + t_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial\theta}{\partial t} + \varepsilon\frac{\partial}{\partial t}\left(1 + t_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{8}$$

$$\begin{aligned}\sigma_{xx} &= \frac{1}{\mu_{\mathrm{T}}}\left(A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} - A_{22}\theta\right), \\ \sigma_{yy} &= \frac{1}{\mu_{\mathrm{T}}}\left(A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} - A_{22}\theta\right), \\ \sigma_{zz} &= \frac{1}{\mu_{\mathrm{T}}}\left(A_{12}\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} - A_{22}\theta\right), \\ \sigma_{xy} &= \frac{\mu_{\mathrm{L}}}{\mu_{\mathrm{T}}}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \sigma_{xz} = \sigma_{yz} = 0,\end{aligned} \tag{9}$$

where

$$h_1 = \frac{A_{11}}{A_{22}},\ h_2 = \frac{\mu_{\mathrm{L}}}{A_{22}},\ h_3 = \frac{A_{13}}{A_{22}},\ \varepsilon = \frac{\gamma^2 T_0}{\rho C_E A_{22}}. \tag{10}$$

4. Normal mode analysis

The normal mode analysis gives exact solutions without any restrictions on temperature, displacement, and stress distributions. It is applied to a wide range of problems in different branches. It can be applied to boundary-layer problems, which are described by the linearized Navier–Stokes equations in electro hydrodynamics. The normal mode expansion method has been proposed by Cheng *et al.*[36] for modeling

the thermoelastic generation process of elastic waveforms in an isotropic plate. Allam *et al.*[37] used the normal mode analysis to study the two-dimensional problem of electromagneto–thermoelasticity for a homogeneous isotropic perfectly conducting thick plate subjected to a time-dependent heat source in the context of Green and Naghdi's theory of thermoelasticity.

The solution of the considered physical variable can be decomposed in terms of normal modes and are given in the following form:

$$\{u,v,\theta,\sigma_{ij}\}(x,y,t)=\left\{u^*,v^*,\theta^*,\sigma_{ij}^*\right\},(x)\,\mathrm{e}^{\omega t+\mathrm{i}ay}, \quad (11)$$

where ω is the (complex) frequency, $\mathrm{i}=\sqrt{-1}$, a is the wave number in the y direction, and $u^*(x)$, $v^*(x)$, $\theta^*(x)$, and $\sigma_{ij}^*(x)$ are the amplitudes of the field quantities. The normal mode analysis in fact, comprises seeking the solution in Fourier's transformed domain. Assuming that all field quantities are sufficiently smooth on the real line such that the normal mode analysis of these quantities should exist. Equations (8)–(10), with the aid of Eq. (11), take the forms

$$\begin{aligned}
&[h_1\mathrm{D}^2-A_1]u^*+(\mathrm{i}ah_3+2\Omega\omega)\mathrm{D}v^*=\mathrm{D}\theta^*,\\
&[h_1\mathrm{D}^2-A_2]v^*+(\mathrm{i}ah_3-2\Omega\omega)\mathrm{D}u^*=\mathrm{i}a\theta^*,\\
&[\mathrm{D}^2-A_3]\theta^*=A_4(\mathrm{D}u^*+\mathrm{i}av^*),
\end{aligned} \quad (12)$$

$$\begin{aligned}
&\mu_{\mathrm{T}}\sigma_{xx}^*=A_{11}\mathrm{D}u^*+\mathrm{i}aA_{12}v^*-A_{22}\theta^*,\\
&\mu_{\mathrm{T}}\sigma_{yy}^*=A_{12}\mathrm{D}u^*+\mathrm{i}aA_{22}v^*-A_{22}\theta^*,\\
&\mu_{\mathrm{T}}\sigma_{zz}^*=A_{12}\mathrm{D}u^*+\mathrm{i}a\lambda v^*-A_{22}\theta^*,\\
&\mu_{\mathrm{T}}\sigma_{xy}^*=\mu_{\mathrm{L}}(\mathrm{i}au^*+Dv^*),
\end{aligned} \quad (13)$$

where $\mathrm{D}=\mathrm{d}()/\mathrm{d}x$ and

$$\begin{aligned}
&A_1=\omega^2-\Omega^2+a^2h_2,\ A_2=\omega^2-\Omega^2+a^2,\\
&A_3=a^2+\omega\left(\delta+t_0\omega^\alpha\right),\ A_4=\varepsilon\omega\left(\delta+t_0\omega^\alpha\right).
\end{aligned} \quad (14)$$

Eliminating $\theta^*(x)$ and $v^*(x)$ in Eq. (12), one obtains the following sixth-order differential equation for $u^*(x)$:

$$\left[\mathrm{D}^6-A\mathrm{D}^4+B\mathrm{D}^2-C\right]u^*(x)=0, \quad (15)$$

where the following definitions have been used:

$$\begin{aligned}
A&=\frac{A_2}{h_2}+\frac{A_1+A_4}{h_1}-\frac{h_3^2a^2}{h_1h_2}+A_3,\\
C&=\frac{A_1A_2A_3+A_1A_4a^2+4\omega^2\Omega^2A_3}{h_1h_2},\\
B&=\left\{A_1A_2+A_3\left(h_1A_2+h_2A_1-h_3^2a^2\right)\right.\\
&\quad\left.+A_4\left(h_1a^2+A_2-2h_3^2a^2\right)+4\omega^2\Omega^2\right\}/(h_1h_2).
\end{aligned} \quad (16)$$

Introducing k_n $(n=1,2,3)$ into Eq. (15), one finds

$$(\mathrm{D}^2-k_1^2)(\mathrm{D}^2-k_2^2)(\mathrm{D}^2-k_3^2)u^*(x)=0, \quad (17)$$

where k_i^2 are the roots of the characteristic equation $k^6-Ak^4+Bk^2-C=0$. These roots are given by

$$\begin{aligned}
k_1^2&=\frac{1}{3}(2s\sin q+A),\\
k_2^2&=-\frac{s}{3}(\sqrt{3}\cos q+\sin q)+\frac{A}{3},\\
k_3^2&=\frac{s}{3}(\sqrt{3}\cos q-\sin q)+\frac{A}{3},
\end{aligned} \quad (18)$$

in which

$$s=\sqrt{A^2-3B},\ \ q=\frac{1}{3}\sin^{-1}R,\ \ R=-\frac{2A^3-9AB+27C}{2s^3}. \quad (19)$$

The series solution of Eq. (17), which is bounded at $x\to\infty$, is given by

$$u^*(x)=\sum_{n=1}^{3}M_n(a,\omega)\,\mathrm{e}^{-k_nx}, \quad (20)$$

where M_n is a specific function of a and ω. In a similar manner, one obtains

$$\{v^*(x),\theta^*(x)\}=\sum_{n=1}^{3}\{M_n'(a,\omega),M_n''(a,\omega)\}\,\mathrm{e}^{-k_nx}, \quad (21)$$

where M_n' and M_n'' are additional specific functions depending on a and ω. Substituting Eqs. (20) and (21) into Eq. (13), one obtains the following relations:

$$\begin{aligned}
M_n'(a,\omega)&=\frac{\mathrm{i}a(h_3-h_1)k_n^2+\mathrm{i}aA_1+2\omega\Omega k_n}{k_n^2-A_3}M_n(a,\omega)\\
&=H_{1n}M_n(a,\omega),\\
M_n''(a,\omega)&=\frac{A_4(-k_n+\mathrm{i}aH_{1n})}{k_n^2-A_3}M_n(a,\omega)\\
&=H_{2n}M_n(a,\omega),\quad n=1,2,3.
\end{aligned} \quad (22)$$

Thus,

$$\{v^*(x),\theta^*(x)\}=\sum_{n=1}^{3}\{H_{1n},H_{2n}\}M_n(a,\omega)\,\mathrm{e}^{-k_nx}. \quad (23)$$

Substituting Eqs. (20) and (23) into Eq. (13), one obtains

$$\begin{aligned}
\{\sigma_{xx}^*,\sigma_{yy}^*,\sigma_{zz}^*,\sigma_{xy}^*\}&=\sum_{n=1}^{3}\{H_{3n},H_{4n},H_{5n},H_{6n}\}M_n(a,\omega)\\
&\quad\times\mathrm{e}^{-k_nx},
\end{aligned} \quad (24)$$

where

$$\begin{aligned}
&\mu_{\mathrm{T}}H_{3n}=-A_{11}k_n+\mathrm{i}aA_{12}H_{1n}-A_{22}H_{2n},\\
&\mu_{\mathrm{T}}H_{4n}=-A_{12}k_n+\mathrm{i}aA_{22}H_{1n}-A_{22}H_{2n},\\
&\mu_{\mathrm{T}}H_{5n}=-A_{12}k_n+\mathrm{i}a\lambda H_{1n}-A_{22}H_{2n},\\
&\mu_{\mathrm{T}}H_{6n}=\mu_{\mathrm{L}}(\mathrm{i}a-k_nH_{1n}).
\end{aligned} \quad (25)$$

5. Applications

In this section the parameters M_n $(n=1,2,3)$ should be determined. In the physical problem, the positive exponentials that are unbounded at infinity should be suppressed. The plane boundary is subjected to an instantaneous normal point force, and the boundary surface is isothermal. The boundary conditions at the vertical plane $y=0$ and in the beginning of the

crack at $x = 0$ are

$$\left\{\sigma_{xx}, \sigma_{yy}, \theta, \frac{\partial \theta}{\partial y}\right\}(x,y,t) = \{-P, 0, F, 0\}, \tag{26}$$

where P and F are the magnitudes of mechanical force and thermal source, respectively. Substituting the expressions of the variables considered into the above boundary conditions yields readily the following equations satisfied by the parameters:

$$\sum_{n=1}^{3} \{H_{3n}, H_{2n}, H_{6n}\} M_n(a,\omega) = \{-P, F, 0\}. \tag{27}$$

The resulting problem amounts to a system of three equations in the form $\{M_1, M_2, M_3\}^{\mathrm{T}} = \boldsymbol{H}^{-1}\{-P^*, F^*, 0\}^{\mathrm{T}}$. After applying the inverse of matrix method, one can obtain the values of the three constants M_n. Hence, the expressions for the displacement, the temperature distribution, and other physical quantities of the medium can be obtained.

6. Particular and special cases

A number of special cases are pertinent to document here.

6.1. Equation of coupled thermoelasticity with reinforcement and rotation

The equations of the coupled thermoelasticity (CTE) theory are obtained when $\alpha \to 0$, $\delta = 0$, and $t_0 = 0$.

6.2. Lord–Shulman's theory with reinforcement and rotation

The equations of Lord–Shulman's (L-S) theory are retrieved when $\alpha \to 0$, $\delta = 1$, and $t_0 > 0$.

6.3. Equations of generalized thermoelasticity with reinforcement and rotation and without energy dissipation

The equations of the generalized thermoelasticity without energy dissipation (the linearized G-N theory of type II) are obtained when $\alpha \to 1$, $\delta = 0$, and $t_0 = 1$. Also all results may reduce to the classical isotropic results when the anisotropic parameters for the fiber-reinforced medium tend to zero (if necessary, write $\xi = 0$, $\beta = 0$, and consider $|\mu_{\mathrm{L}} - \mu_{\mathrm{T}}| \to 0$). Neglecting the angular velocity (i.e. $\Omega = 0$), one obtains the transformed components of displacement, stress forces, and temperature distribution in a non-rotating generalized thermoelasticity medium.

7. Numerical results

To illustrate the theoretical results obtained in the preceding section and to compare various theories of thermoelasticity formulated earlier, some numerical results are presented for the physical quantities. The results depict the variations of normal displacement, normal force stress, and temperature distribution in the context of fractional order thermoelasticity theory. To study the effect of reinforcement on wave propagation, we use the following physical constants for generalized fiber-reinforced thermoelastic material:

$$\lambda = 7.59\times10^9\ \mathrm{N/m^2},\ \ \mu_{\mathrm{T}} = 1.89\times10^9\ \mathrm{N/m^2},$$
$$\mu_{\mathrm{L}} = 2.45\times10^9\ \mathrm{N/m^2},\ \ \alpha = -1.28\times10^9\ \mathrm{N/m^2},$$
$$\beta = 0.32\times10^9\ \mathrm{N/m^2},\ \ \rho = 7800\ \mathrm{kg/m^3},$$
$$\alpha_t = 1.78\times10^{-5}\ \mathrm{K^{-1}},\ \ C_E = 383.1\ \mathrm{J\cdot kg^{-1}\cdot K^{-1}},$$
$$T_0 = 293\ \mathrm{K},\ \ K = 386\ \mathrm{W\cdot mK^{-1}}.$$

The calculations are carried out for the non-dimensional time $t = 0.15$ s and relaxation time $t_0 = 0.02$ s in the range $0 < x < 5$. In addition, the parameters $a = 1$, $F = 1$, $P = 2$, $\omega = \omega_0 + \mathrm{i}\xi$, $\omega_0 = 2$, and $\xi = 1$ are used. The distributions of the real parts of the thermal temperature θ, the displacements u and v, the distributions of stresses σ_{xx}, σ_{yy}, and σ_{xy} with non-dimensional distance x are shown in Figs. 1–12. Here all the variables are taken in the non-dimensional forms.

Figures 1–6 are drawn to give comparison of the results obtained for displacements, temperature, and stresses against the thickness x for different values of α at $y = 1$. Three values of α ($\alpha = 1, 0.5$, and $\alpha = 0.25$, i.e., in the absence and the presence of the fiber reinforcement) are considered. It can be found from Figs. 1–6 that the physical quantities depend not only on time and space, but also on the characteristic parameter of fractional order thermoelasticity theory. It should be noted that the results for the temperature, displacement, and stress distributions, when the fractional order parameter is included in the heat equation, are distinctly different from those without the fractional order parameter in the heat equation. The different values of the parameter with a wide range $0 < \alpha \leq 1$ cover the two cases of the conductivity: $0 < \alpha < 1$ for weak conductivity and $\alpha = 1$ for normal conductivity (ordinary heat conduction equation).

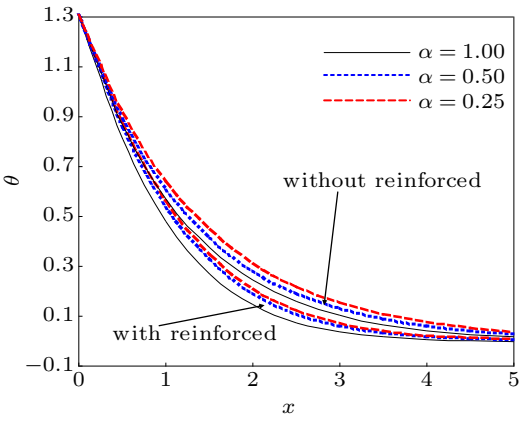


Fig. 1. (color online) Dependences of temperature θ on distance for different values of fractional order parameter α in the absence and the presence of fiber-reinforcement.

Figure 1 shows that temperature θ has a single maximum value at the boundary (which represents the beginning of the crack at the same time) and then decreases to its minimum values according to the value of α. The value of temperature quantity converges to zero with the increase of distance x. The effect of the fractional order parameter on temperature reduces the value of the amplitude of θ. In addition, the inclusion of reinforcement also reduces the temperature.

As shown in Fig. 2, the horizontal displacement u increases near the crack edge in the absence of the reinforcement, then smoothly decreases again to reach its minimum magnitude as x increases. However, u has its minimum value in the presence of reinforcement and increases as x increases. The values of u for $\alpha = 0.5$ and $\alpha = 0.25$ are small compared with those for $\alpha = 1$. Figure 3 shows that the values of v for $\alpha = 0.25$ and $\alpha = 0.5$ are large compared with those for $\alpha = 1$. Also, at the beginning of crack, it reaches the maximum value in the presence of reinforcement. It reaches to its minimum as x increases. Also, in the absence of fiber reinforcement, v is increasing with the increase of x. Displacements u and v show different behaviors, because the elasticity of the solid tends to resist vertical displacements in the problem under investigation.

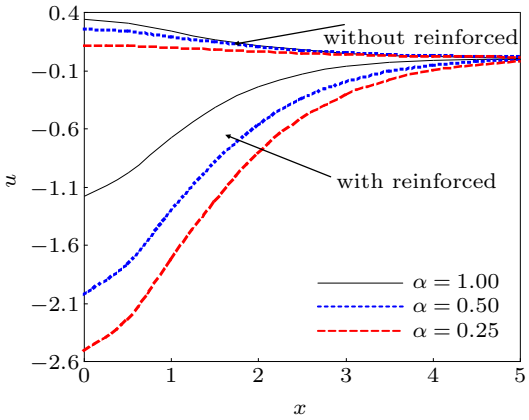


Fig. 2. (color online) Dependences of horizontal displacement u on distance for different values of fractional order parameter α in the absence and the presence of fiber-reinforcement.

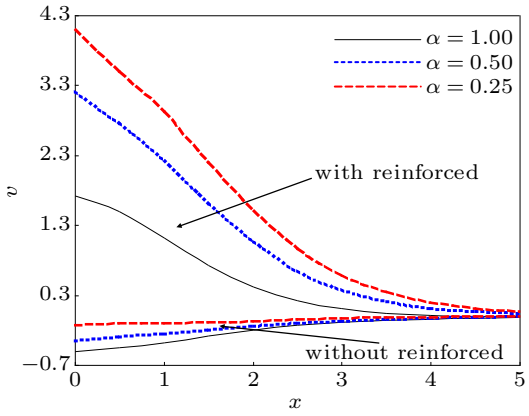


Fig. 3. (color online) Dependences of vertical displacement v on distance for different values of fractional order parameter α in the absence and the presence of fiber-reinforcement.

The stress component σ_{xx} reaches coincidence with a negative value (Fig. 4). It reaches the minimum value with the increase of x. The behaviors of σ_{xx} for the two cases (with and without fiber reinforcement) are opposite. The fractional orders reduce the amplitudes of the stress for the two cases. Figure 5 gives the same behavior of σ_{yy} as that found in Fig. 4. Figure 6 shows that stress component σ_{xy} satisfies the boundary condition at $x = 0$ and has a different behavior compared with that of σ_{xx}. These trends obey elastic and thermoelastic properties of the solid under investigation.

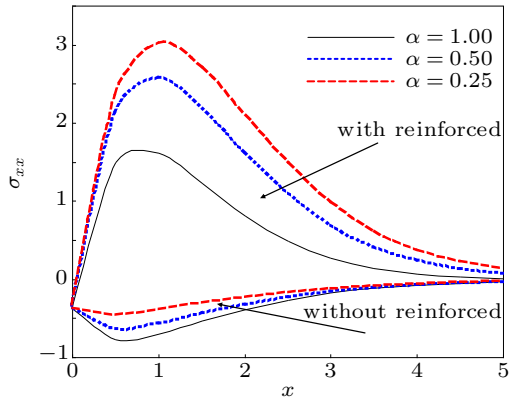


Fig. 4. (color online) Dependences of stress σ_{xx} on distance for different values of fractional order parameter α in the absence and the presence of fiber-reinforcement.

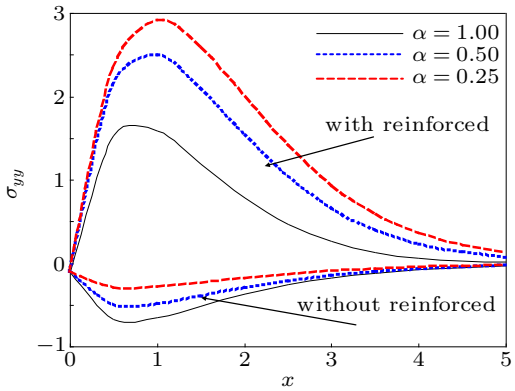


Fig. 5. (color online) Dependences of stress σ_{yy} on distance for different values of fractional order parameter α in the absence and the presence of fiber-reinforcement.

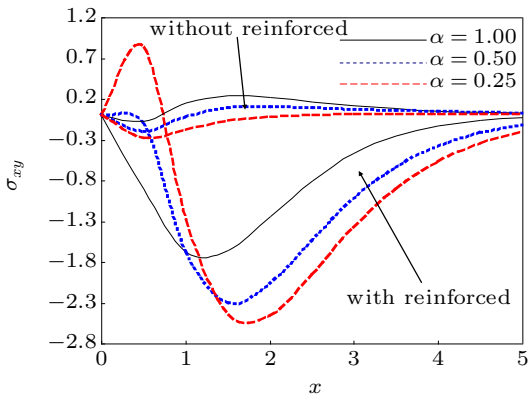


Fig. 6. (color online) Dependences of stress σ_{xy} on distance for different values of fractional order parameter α in the absence and the presence of fiber-reinforcement.

Two cases with and without fiber reinforcement are included in Figs. 1–6. In the case without fiber reinforcement one should set $\xi = 0$, $\beta = 0$, and $\mu_L - \mu_T = 0$. The values of u, θ, and σ_{xy} are evidently smaller in the case with fiber reinforcement than those in the absence of fiber reinforcement. The values of thermal stresses σ_{xx}, σ_{yy}, and the displacement v are evidently larger in the case with fiber reinforcement than those in the absence of fiber reinforcement. It is clear from the above investigation that the surface waves in the fiber-reinforced medium are affected by the reinforced parameters.

Figures 7–12 depicts six representative curves for the distributions of the thermal temperature θ, the displacements u and v, the distributions of stresses σ_{xx}, σ_{yy}, and σ_{xy} both in the absence and the presence of rotation. It is noticed that the results for the temperature, the displacement, and stress distributions with the rotation of the medium included in the equation of motion are distinctly different from those without the rotation, i.e., the thermoelastic wave is affected by rotation. It can be seen from these figures that, at constant $\alpha = 0.5$, the rotation acts to reduce the magnitudes of the real parts of u, θ, and σ_{xy} and increase the magnitudes of the real parts of the thermal stresses σ_{xx}, σ_{yy}, and the displacement v.

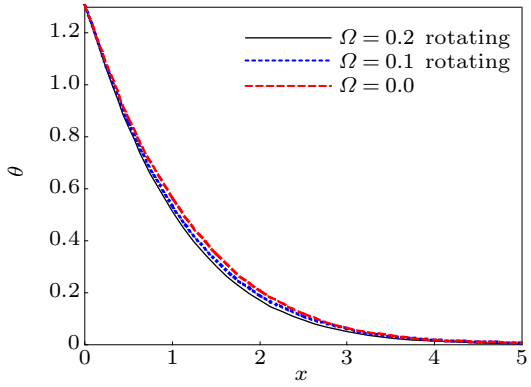


Fig. 7. (color online) Temperature distributions θ in the cases of material with fractional order thermoelasticity in the absence and the presence of fiber-reinforcement.

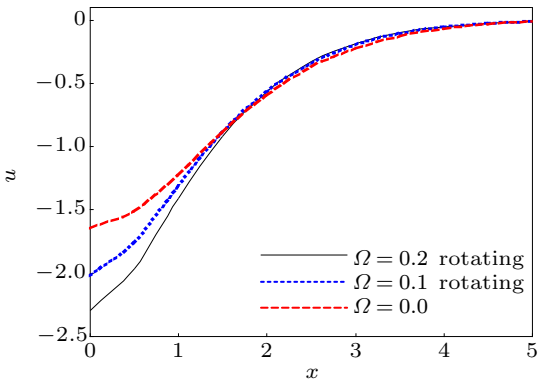


Fig. 8. (color online) Distributions of u in the cases of material with fractional order thermoelasticity in the absence and the presence of rotation.

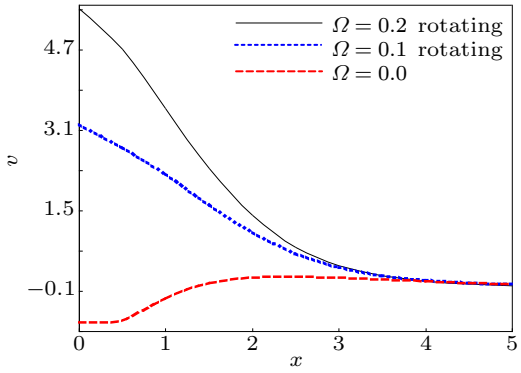


Fig. 9. (color online) Distributions of v in the cases of material with fractional order thermoelasticity in the absence and the presence of rotation.

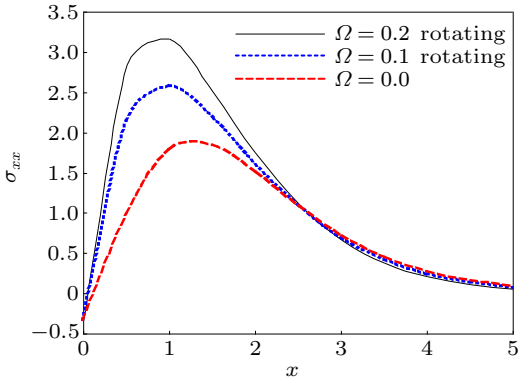


Fig. 10. (color online) Distributions of σ_{xx} in the cases of material with fractional order thermoelasticity in the absence and the presence of rotation.

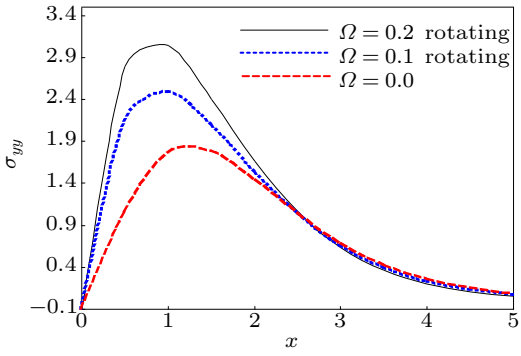


Fig. 11. (color online) Distributions of σ_{yy} in the cases of material with fractional order thermoelasticity in the absence and the presence of rotation.

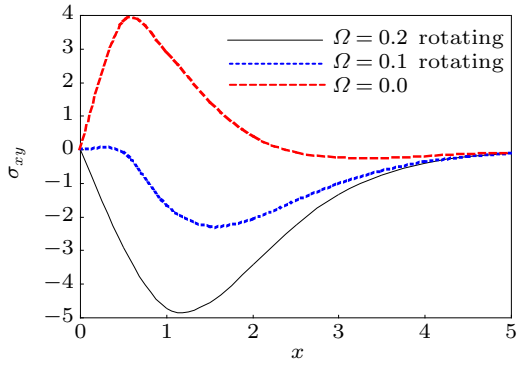


Fig. 12. (color online) Distributions of σ_{xy} in the cases of material with fractional order thermoelasticity in the absence and the presence of rotation.

8. Conclusions

In this work the effect of rotation on the thermoelastic interactions in an infinite thermally conducting medium is investigated in the context of the theory of thermoelasticity with fractional order. The plate is rotating with uniform angular velocity normal to its faces. A linear opening mode-I crack is investigated for the copper solid. The stress distribution and the temperature are evaluated each as a function of the distance from the crack edge. The cases of the presence and the absence of the reinforcement and rotating are addressed. Analytical solutions based on normal mode analysis for thermoelasticity with fractional order in solid are developed and utilized. From the present analysis the following conclusions may be made.

(i) The physical quantities are affected both by rotation and by the fractional parameter. Therefore, the presence of fractional order parameter and rotation in the current model is of significance.

(ii) The values of all physical quantities are continuous and converge to zero with an increase in distance x.

(iii) The fiber-reinforcement has an important role in determining the distributions of the field quantities.

(iv) The method which is used in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity.

(v) The theories of coupled thermoelasticity and generalized thermoelasticity can be obtained as limited cases.

(vi) Deformation of a body depends on the nature of the force applied as well as the type of boundary conditions.

(vii) Crack dimensions are significant to elucidate the mechanical structure of the solid. Cracks are stationary, and the external stress is demanded to propagate such cracks. A change of volume is accompanied by a change of the temperature, while the effect of the deformation on the temperature distribution is the subject of the theory of thermoelasticity.

(viii) According to the numerical results and their graphs, a new theory of thermoelasticity is constructed. The results provide a motivation to investigating conducting materials as a new class of applicable materials.

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