RAPID COMMUNICATION

Noether symmetry and conserved quantities of the analytical dynamics of a Cosserat thin elastic rod*

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In this paper, we investigate the Noether symmetry and Noether conservation law of elastic rod dynamics with two independent variables: time t and arc coordinate s. Starting from the Lagrange equations of Cosserat rod dynamics, the criterion of Noether symmetry with Lagrange style for rod dynamics is given and the Noether conserved quantity is obtained. Not only are the conservations of generalized moment and generalized energy obtained, but also some other integrals.

Keywords: analytical mechanics, Noether symmetry, conservation laws, Cosserat elastic rod dynamics

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1. Introduction

As mechanical models of DNA, fibers, undersea cables, and drill-strings, thin elastic rod problems are investigated intensively. Based on the Kirchhoff kinetic analogy, the equilibrium and stability of a thin elastic rod have been studied broadly.[1–8] However, because of the special slender and super-deformation characteristic, the equations of motion of an elastic rod are strongly nonlinear, which brings difficulties for solving them. However, if the symmetries of equations are revealed, it can be used to construct conserved quantities or exact solutions of the equations. In 1918, Noether first uncovered the connection between the symmetry of a Lagrangian dynamical system and conservation laws in her classical paper 'Invariant Variations Problems'.[9]

The symmetry method has been developed as a general method for solving the differential equations of mechanical systems. In addition to Noether symmetry,[9–18] Lie symmetry for mechanical systems has also been developed,[19–27] as well as Mei symmetry (form invariance),[28–31] unified symmetry,[32,33] conformal invariance,[34–37] and other symmetries.[38,39] The application of symmetry methods in mechanical systems is referred to several studies.[20,28,40]

However, there are a few applications of the symmetry methods in elastic rod mechanics. Maddocks[41] discussed the conservation laws of rod dynamics in vector form. Coleman[42] *et al.* discussed the conservation laws of a vector equation for planar motion rod dynamics and gave integration variational principles of rod dynamics. However, they did not draw support from analytical mechanical methods or symmetry methods. The authors in Refs. [41] and [42] got both the momentum integrals and energy conservation, besides the above two conserved quantities, and we will get two more first integrals in the present paper. Jung[43] gave out the discrete rod equation and discrete Noether theorem for discrete special Cosserat rod statics, which did not refer to the dynamics. Xue[44] discussed the conserved quantities of Cosserat elastic rod dynamics from mechanical theorems. In this paper, we will expand Noether symmetry to Cosserat rod dynamics in Lagrange form, which will be different from the above results.

In the present paper, firstly, we introduce the equation of motion for elastic rod dynamics. The analytical mechanical formation of a Kirchhoff elastic rod will be constructed,[45–47] Similarly, based on the Kirchhoff kinetic analogy, we will give out the variational principle and construct the Euler–Lagrange equation of the Cosserat rod dynamics.[48] Secondly, we give the criterion of Noether symmetry and the Noether conserved quantity of the Lagrange analytical dynamics of a Cosserat rod. Thirdly, we discuss the first integrals that are deduced from the Noether theorem. We will find that, besides the generalized momentum and energy conserved quantities, there are more first integrals obtained in the present paper. The first integrals of Cosserat elastic rod dynamics are in an integration form, which causes new problems for the research of elastic rod dynamics.

2. The equations of motion of elastic rod dynamics

We consider a thin elastic rod of length L. Its configuration can be discretized into the cross-section motion along the center line of the rod. We establish the inertial coordinate system $(O-\xi\eta\zeta)$ and the principal axes system $(P-xyz)$ that are fixed on the center of cross section. The unit vectors (e_x, e_y, e_z)

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are named Cosserat direct vectors, e_z is the normal direction of the cross section. The configuration of the cross section can be described by position vector $\boldsymbol{r} = \boldsymbol{OP}$ or array $r = (\xi, \eta, \zeta)^{\mathrm{T}}$, which comes from the inertial system origin O to point P of the center of the cross section, and the attitude can be described by the Euler angle array $\Phi = (\psi(s,t), \theta(s,t), \varphi(s,t))$. The equation of motion the of cross section of an elastic rod can be expressed by generalized coordinates

$$q_\alpha = q_\alpha(s,t), \quad (\alpha = 1, \ldots, 6), \tag{1}$$

where q_α stands for the generalized coordinates, and α_1 ($\alpha_1 = \xi, \eta, \zeta$ as $\alpha = 1,2,3$) denotes the displacement of the center of the cross section, and α_2 ($\alpha_2 = \psi(s,t), \theta(s,t), \varphi(s,t)$ as $\alpha = 4,5,6$) denotes the attitude of the cross section.

Suppose the strain vector of cross section is

$$\gamma = (\gamma_1, \gamma_2, \gamma_3)^{\mathrm{T}}, \tag{2}$$

where γ_1 and γ_2 are shear strains in plan (y,z) and (z,x) respectively, and γ_3 is axial strain along z axis. The deformation geometric equation of a thin elastic rod is[49]

$$\partial_s \boldsymbol{r} = \boldsymbol{e}_z + \boldsymbol{\gamma}, \tag{3}$$

where

$$\partial_s = \frac{\partial}{\partial s} + q'_\alpha \frac{\partial}{\partial q_\alpha} + q''_\alpha \frac{\partial}{\partial q'_\alpha} + \dot{q}'_\alpha \frac{\partial}{\partial \dot{q}_\alpha} + \cdots \tag{4}$$

is the partial derivative to arc coordinates s, and ∂_t denotes a partial derivative to time t, and q'_α denotes spatial derivative $\partial_s q_\alpha(s,t)$ and $\dot{q}_\alpha$ denotes temporal derivative $\partial_t q_\alpha(s,t)$, and $\gamma = 0$ for the Kirchhoff model.[46,47]

For a thin rod, consider the segment between points P and P', according to arc coordinates s and $s + \Delta s$. Assume that the resultant internal force and torque on the negative cross section of point P, are $-\boldsymbol{F}$ and $-\boldsymbol{M}$, and on the positive cross section of point P', are $\boldsymbol{F} + \Delta\boldsymbol{F}$ and $\boldsymbol{M} + \Delta\boldsymbol{M}$, $\boldsymbol{F}$ and $\boldsymbol{M}$ are bivariate continuous function to s and t. Making a reduction of all the forces to point P and reserving the first order a small amount, and based on the theorems of momentum and moment of momentum to a centroid, we can get

$$\begin{aligned} &\Delta\boldsymbol{F} = \rho A \Delta s \frac{\partial^2 \boldsymbol{r}}{\partial t^2}, \\ &\Delta\boldsymbol{M} + \Delta\boldsymbol{r} \times \boldsymbol{F} = \Delta s \frac{\partial}{\partial t}(\boldsymbol{J} \cdot \boldsymbol{\Omega}), \end{aligned} \tag{5}$$

where $\rho, A, \boldsymbol{J}$ are the density, area of the cross section, and the inertia tensor of the rod with unit length, respectively. $\boldsymbol{J}$ can be expressed as an array in a principal axes system $\boldsymbol{J} = \mathrm{diag}(J_1, J_2, J_3)$, and $J_i = \rho I_i / A$, I_1, I_2 are the inertial moment of the cross section to principal axes x, y, I_3 is polar inertial moment to principal axis z, and $I_1 + I_2 = I_3$. $\boldsymbol{\Omega} = \sum_4^6 \Xi_\alpha \dot{q}_\alpha$ is the angular velocity of the cross section, Ξ_α is the vector-value function of Euler angles.

The items in Eq. (5) divided by Δs and making $\Delta s \to 0$, the dynamical equations of the rod can be derived[4,5]

$$\begin{aligned} &\partial_s \boldsymbol{F} = \rho A \partial_t \boldsymbol{v}, \\ &\partial_s \boldsymbol{M} + \partial_s \boldsymbol{r} \times \boldsymbol{F} = \partial_t (\boldsymbol{J} \cdot \boldsymbol{\Omega}), \end{aligned} \tag{6}$$

where $\boldsymbol{v} = \partial_t \boldsymbol{r}$ is the velocity for the center of cross section.

3. Lagrange equation of motion of the Cosserat elastic rod dynamics

The differential variational principle of Cosserat thin elastic rod dynamics can be expressed as

$$\begin{aligned} \delta W &= (\partial_s \boldsymbol{F} - \rho A \partial_t \boldsymbol{v}) \cdot \delta \boldsymbol{r} \\ &\quad + (\partial_s \boldsymbol{M} + \partial_s \boldsymbol{r} \times \boldsymbol{F} - \partial_t (\boldsymbol{J} \cdot \boldsymbol{\Omega})) \cdot \delta \Phi \\ &= 0, \end{aligned} \tag{7}$$

where $\delta \boldsymbol{r}, \delta \Phi$ denotes the virtual translation with the center of cross section and virtual angle displacements to the center of section, respectively.

Suppose the elastic rod satisfies the following constitutive relation,[4] i.e.,

$$\begin{aligned} &\boldsymbol{F} = \boldsymbol{K} \cdot \boldsymbol{\gamma}, \\ &\boldsymbol{M} = \boldsymbol{B} \cdot \boldsymbol{\omega}, \end{aligned} \tag{8}$$

where $\boldsymbol{\omega} = \sum_4^6 \Xi_\alpha q'_\alpha$ are curvature-twisting vectors, $\boldsymbol{K} = \mathrm{diag}(K_1, K_2, K_3)$, $\boldsymbol{B} = \mathrm{diag}(B_1, B_2, B_3)$, K_1, K_2 and B_1, B_2 are shear and bending stiffness dyadic to x and y axes, K_3, B_3 are tensile compression rigidity and twisting rigidity to z axis, respectively. The variational principle (7) can be expressed as the Euler–Lagrange form

$$\begin{aligned} \delta W &= \left\{ \partial_s \left(\frac{\partial \Lambda}{\partial q'_\alpha} \right) + \partial_t \left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} \right) - \frac{\partial \Lambda}{\partial q_\alpha} \right\} \delta q_\alpha \\ &= 0, \quad (\alpha = 1, 2, \ldots, 6), \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\Lambda = T - V, \\ &T = \frac{1}{2} \rho \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \frac{1}{2} (\boldsymbol{J} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{\Omega}, \\ &V = \frac{1}{2} (\boldsymbol{K} \cdot \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma} + \frac{1}{2} (\boldsymbol{B} \cdot \boldsymbol{\omega}) \cdot \boldsymbol{\omega}, \end{aligned} \tag{10}$$

T and V are kinetic energy and stress energy with respect to the cross section.

For the free elastic rod, when the constraints only act on the two ends of the rod, we can get the Lagrange equation of elastic rod dynamics from variational principle (9)

$$\partial_s \left(\frac{\partial \Lambda}{\partial q'_\alpha} \right) + \partial_t \left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} \right) - \frac{\partial \Lambda}{\partial q_\alpha} = 0, \ (\alpha = 1, 2, \ldots, 6). \tag{11}$$

Equations (11) denote all the motions of the Cosserat elastic rod, and can be seen as the Euler–Lagrange equation of Cosserat elastic rod dynamics.

The terms in Eq. (11) have their mechanical sense,

$$\frac{\partial \Lambda}{\partial q'_{\alpha_1}} = -\frac{\partial V}{\partial q'_{\alpha_1}} = -\boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial q_{\alpha_1}},$$
$$\frac{\partial \Lambda}{\partial q'_{\alpha_2}} = -\frac{\partial V}{\partial q'_{\alpha_2}} = -\boldsymbol{M}\cdot\frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_{\alpha_2}} \tag{12}$$

are the generalized cross-section forces.

$$\frac{\partial \Lambda}{\partial \dot{q}_{\alpha_1}} = \frac{\partial T}{\partial \dot{q}_{\alpha_1}} = \rho\dot{\boldsymbol{r}}\frac{\partial \boldsymbol{r}}{\partial q_{\alpha_1}},$$
$$\frac{\partial \Lambda}{\partial \dot{q}_{\alpha_2}} = \frac{\partial T}{\partial \dot{q}_{\alpha_2}} = (\boldsymbol{J}\cdot\boldsymbol{\Omega})\cdot\frac{\partial \boldsymbol{\Omega}}{\partial \dot{q}_{\alpha_2}} \tag{13}$$

are the generalized momentums of cross section.

4. Noether symmetry and conservation laws of a Cosserat thin elastic rod in analytical dynamics form

Introduce in the one-parameter Lie point transformation group in the space (t,s,q_α)

$$t^* = t+\varepsilon\xi_t,\ \ s^* = s+\varepsilon\xi_s,\ \ q^*_\alpha = q_\alpha+\varepsilon\eta_\alpha, \tag{14}$$

where ξ_t, ξ_s, η_α are infinitesimal generators and are the function of (t,s,q_α), ε is an infinitesimal parameter. It has infinitesimal generator vector

$$X^{(0)} = \xi_t(t,s,q_\alpha)\frac{\partial}{\partial t}+\xi_s(t,s,q_\alpha)\frac{\partial}{\partial s} + \eta_\alpha(t,s,q_\alpha)\frac{\partial}{\partial q_\alpha}. \tag{15}$$

The first-order prolongation of an infinitesimal generator is

$$X^{(1)} = X^{(0)}+\eta^{(1)}_{t\alpha}(t,s,q_\alpha)\frac{\partial}{\partial \dot{q}_\alpha}+\eta^{(1)}_{s\alpha}(t,s,q_\alpha)\frac{\partial}{\partial q'_\alpha}, \tag{16}$$

where

$$\eta^{(1)}_{t\alpha} = \partial_t\eta_\alpha-(\partial_t\xi_t)\dot{q}_\alpha-(\partial_t\xi_s)q'_\alpha,$$
$$\eta^{(1)}_{s\alpha} = \partial_s\eta_\alpha-(\partial_s\xi_t)\dot{q}_\alpha-(\partial_s\xi_s)q'_\alpha. \tag{17}$$

The Noether identity can be written as

$$X^{(1)}(\Lambda)+\Lambda\dot{\xi}_t+\Lambda\xi'_s = 0. \tag{18}$$

Criterion 1 If the Lie point transformation group (14) satisfies the Noether identity (18), then it is the Noether symmetry transformation of elastic rod dynamics (11).

By defining the form of conservation laws as

$$\partial_t F(s,t,\dot{q}_\alpha,q'_\alpha) = -\partial_s G(s,t,\dot{q}_\alpha,q'_\alpha), \tag{19}$$

where F is the density and $-G$ is the flux, we can give the expression of a Noether conserved quantity for two independent variables elastic rod dynamics in Lagrangian form.

Theorem 1 If the infinitesimal generators are Noether symmetrical, the system has the following conservation laws

$$\partial_t\left\{\Lambda\xi_t+\frac{\partial\Lambda}{\partial\dot{q}_\alpha}(\eta_\alpha-\dot{q}_\alpha\xi_t-q'_\alpha\xi_s)\right\} + \partial_s\left\{\Lambda\xi_s+\frac{\partial\Lambda}{\partial q'_\alpha}(\eta_\alpha-\dot{q}_\alpha\xi_t-q'_\alpha\xi_s)\right\} = 0. \tag{20}$$

Proof Expanding Eq. (20) and making use of Noether identity (18), we can get

$$\left(\partial_s\left(\frac{\partial\Lambda}{\partial q'_\alpha}\right)+\partial_t\left(\frac{\partial\Lambda}{\partial\dot{q}_\alpha}\right)-\frac{\partial\Lambda}{\partial q_\alpha}\right) \times(\eta_\alpha-\dot{q}_\alpha\xi_t-q'_\alpha\xi_s) = 0. \tag{21}$$

Special case When the elastic rod does not move with time t, the Noether identity (18) reduces to

$$\frac{\partial\Lambda}{\partial s}\xi_s+\frac{\partial\Lambda}{\partial q_\alpha}\eta_\alpha+\frac{\partial\Lambda}{\partial q'_\alpha}(\eta'_\alpha-\xi'_sq'_s)+\Lambda\xi'_s = 0 \tag{22}$$

and the corresponding Noether conserved quantity of equation of elastic rod statics is

$$\partial_s\left\{\Lambda\xi_s+\frac{\partial\Lambda}{\partial q'_\alpha}(\eta_\alpha-q'_\alpha\xi_s)\right\} = 0. \tag{23}$$

Proof Expanding Eq. (23), and considering the Lagrange equation of elastic rod statics[43]

$$\partial_s\left(\frac{\partial\Lambda}{\partial q'_\alpha}\right)-\frac{\partial\Lambda}{\partial q_\alpha} = 0, \tag{24}$$

it is easy to verify the above result. The Noether theorem of rod dynamics can degenerate into that of elastic rod statics.

In fact, we can give a weaker condition for Noether symmetry than the Noether identity (18).

Criterion 2 If the Lie point transformation group (14) satisfies the following generalized Noether identity

$$X^{(1)}(\Lambda)+\Lambda\dot{\xi}_t+\Lambda\xi'_s+\partial_tG_1+\partial_sG_2 = 0, \tag{25}$$

where $G_1 = G_1(t,s,q_\alpha), G_2 = G_2(t,s,q_\alpha)$ are gauge functions, then it is the generalized Noether symmetry transformation of elastic rod dynamics (11). So we have the following generalized Noether theorem.

Theorem 2 If the infinitesimal generators are generalized Noether-symmetrical, the system has the following conservation laws

$$\partial_t\left\{\Lambda\xi_t+\frac{\partial\Lambda}{\partial\dot{q}_\alpha}(\eta_\alpha-\dot{q}_\alpha\xi_t-q'_\alpha\xi_s)+G_1\right\} + \partial_s\left\{\Lambda\xi_s+\frac{\partial\Lambda}{\partial q'_\alpha}(\eta_\alpha-\dot{q}_\alpha\xi_t-q'_\alpha\xi_s)+G_2\right\} = 0. \tag{26}$$

The above theorem can be verified by straightforward calculations.

5. Discussion

In this section we will illustrate the application of Noether symmetry and conservation quantity in elastic rod dynamics. We will see that the cyclic integral and energy integrals about time and arc coordinates in integration form, which bring new problems for analytical mechanics.

5.1. Cyclic integrals

For infinitesimal generator

$$X = \frac{\partial}{\partial q_\alpha}, \tag{27}$$

the Noether identity becomes

$$\frac{\partial \Lambda}{\partial q_\alpha} = 0. \tag{28}$$

Here q_α can be treated as a cyclic coordinate. The corresponding Noether conserved quantity is

$$\partial_t \left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} \right) + \partial_s \left(\frac{\partial \Lambda}{\partial q'_\alpha} \right) = 0. \tag{29}$$

Discussion 1 For an elastic rod with length l, if generalized internal forces about cyclic coordinate q_α acting on two ends of the rod are equal, in time interval $[0,t_1]$, i.e.,

$$\left(\frac{\partial \Lambda}{\partial q'_\alpha} \right)_{s=l} = \left(\frac{\partial \Lambda}{\partial q'_\alpha} \right)_{s=0}, \tag{30}$$

we can get the conservation of the generalized momentum of the elastic rod

$$\int_0^l \left[\left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} \right)_t \right] \mathrm{d}s = \int_0^l \left[\left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} \right)_{t=0} \right] \mathrm{d}s. \tag{31}$$

As $\alpha = \alpha_1$, considering Eqs. (12) and (13), equation (30) denotes the generalized internal force on the two ends of rod, and equation (31) denotes the total momentum of a rod conserved along the arc length. An elastic rod under the action of a equilibrant couple on the two ends[44] coincides with the condition (30), so it has conserved quantities (31). As $\alpha = \alpha_2$, equation (30) denotes the generalized moment of the cross section of the rod, and equation (31) denotes the moment of momentum of the rod's cross section. A corresponding example that accords with this case is a closed circle rod without constraints.[44]

Discussion 2 If the generalized momentums of a section are equal at two ends in the time interval $[0,t_1]$, i.e.,

$$\left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} \right)_{t_1} = \left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} \right)_{t=0}, \tag{32}$$

we can get impulse conservation of the generalized internal force of the elastic rod

$$\int_0^{t_1} \left[\left(\frac{\partial \Lambda}{\partial q'_\alpha} \right)_{s=l} \right] \mathrm{d}t = \int_0^{t_1} \left[\left(\frac{\partial \Lambda}{\partial q'_\alpha} \right)_{s=0} \right] \mathrm{d}t. \tag{33}$$

As $\alpha = \alpha_1$, equation (32) denotes that the momentums of rod at the first time are equal to that at last time, and equation (33) denotes that the impulse of the internal force is conserved in time interval $[0,t_1]$. For example, an elastic rod has no loads except the two ends, as its momentums are equal on the two ends, or performs the periodic motion,[44] as conserved quantities (33). As $\alpha = \alpha_2$, the corresponding example can be an elastic rod of periodic motion without constraints,[44] which has conserved quantities (33).

Equations (31) and (33) are cyclic integral at about q_α. For elastic rod statics, equation (32) is the identity $0 = 0$, and equation (33) can become the first integrals found in Ref. [31].

5.2. Energy integrals about arc coordinate *s*

For infinitesimal generator

$$X = \frac{\partial}{\partial s}, \tag{34}$$

the Noether identity becomes

$$\frac{\partial \Lambda}{\partial s} - f_\alpha q'_\alpha = 0. \tag{35}$$

If f_α is equal to zero or is gyroscopic force, i.e.,

$$\sum_{\alpha=1}^{6} f_\alpha q'_\alpha = 0, \tag{36}$$

the corresponding Noether conserved quantity is

$$\partial_s \left(\Lambda - \frac{\partial \Lambda}{\partial q'_\alpha} q'_\alpha \right) = \partial_t \left(\frac{\partial \Lambda}{\partial \dot{q}_\alpha} q'_\alpha \right). \tag{37}$$

Based on the Euler homogeneous theorem and the relations

$$\frac{\partial \boldsymbol{\omega}}{\partial q'_{\alpha_2}} = \frac{\partial \boldsymbol{\omega}'}{\partial q''_{\alpha_2}} = \frac{\partial \boldsymbol{\Omega}}{\partial \dot{q}_{\alpha_2}} = \frac{\partial \dot{\boldsymbol{\Omega}}}{\partial \ddot{q}_{\alpha_2}} = \Xi_{\alpha_2},$$

equation (37) becomes

$$\begin{aligned} &\partial_s (V + T + \boldsymbol{F} \cdot \boldsymbol{e}_z) \\ &= \partial_t \left(\sum_{\alpha=1}^{3} \frac{\partial T}{\partial \dot{q}_\alpha} q'_\alpha \right) \\ &= \partial_t [\rho \dot{\boldsymbol{r}} \cdot \boldsymbol{r}' + (\boldsymbol{J} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{\omega}]. \end{aligned} \tag{38}$$

Equation (38) is the energy equation of elastic rod dynamics about the arc coordinate, which denotes the rate of mechanical energy density along arc coordinates equal to the derivative of the power of momentum density of the arc coordinate to time.

Discussion 3 Under some conditions, two conservation laws about the time and the arc coordinate are presented.

(i) Between the time interval $[0,t_1]$, if the following condition is satisfied at the two ends of the elastic rod

$$(T + V + \boldsymbol{F} \cdot \boldsymbol{e}_z)_{s=0} = (T + V + \boldsymbol{F} \cdot \boldsymbol{e}_z)_{s=l}, \tag{39}$$

then the conserved quantity about the time will be

$$\int_0^l [\rho \dot{\boldsymbol{r}} \cdot \boldsymbol{r}' + (\boldsymbol{J} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{\omega}]_t \mathrm{d}s$$

$$= \int_0^l [\rho \dot{\boldsymbol{r}} \cdot \boldsymbol{r}' + (\boldsymbol{J} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{\omega}]_{t=0} \mathrm{d}s. \quad (40)$$

Equation (39) expresses the energy on the two ends of rod, and equation (40) expresses the energy of the rod cross section with the motion and twist in the space of time and arc coordinates. For example, the closed elastic rod.[44]

(ii) At the two ends of the time interval $[0, t_1]$, if the following condition

$$[\rho \dot{\boldsymbol{r}} \cdot \boldsymbol{r}' + (\boldsymbol{J} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{\omega}]_{t=t_1} = [\rho \dot{\boldsymbol{r}} \cdot \boldsymbol{r}' + (\boldsymbol{J} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{\omega}]_{t=0} \quad (41)$$

holds at any section s, then the conserved quantity about the arc coordinate will be

$$\int_0^{t_1} (T + V + \boldsymbol{F} \cdot \boldsymbol{e}_z)_s \mathrm{d}t = \int_0^{t_1} (T + V + \boldsymbol{F} \cdot \boldsymbol{e}_z)_{s=0} \mathrm{d}t. \quad (42)$$

The example for Eq. (2) can be found in Ref. [44]. Based on the Kirchhoff kinetic analogy, we will get that the conserved quantity (40) extends the classical kinetic energy.

Special case For elastic rod statics, equation (41) is the identity $0 = 0$, and equation (42) becomes

$$(V + \boldsymbol{F} \cdot \boldsymbol{e}_z)_l = (V + \boldsymbol{F} \cdot \boldsymbol{e}_z)_{s=0}, \quad (43)$$

which also can be deduced directly from the Noether conserved quantity of the elastic rod statics (23) for generator $X = \partial/\partial s$. Equation (43) is the energy integral.

5.3. Energy integrals about time *t*

For infinitesimal generator

$$X = \frac{\partial}{\partial t}, \quad (44)$$

the Noether identity becomes

$$\frac{\partial \Lambda}{\partial t} - f_\alpha \dot{q}_\alpha = 0. \quad (45)$$

If f_α equals zero or gyroscopic force, i.e.,

$$\sum_{\alpha=1}^{6} f_\alpha \dot{q}_\alpha = 0, \quad (46)$$

the corresponding Noether conserved quantity is

$$\partial_t \left(\Lambda - \frac{\partial \Lambda}{\partial \dot{q}_\alpha} \dot{q}_\alpha \right) = \partial_s \left(\frac{\partial \Lambda}{\partial q'_\alpha} \dot{q}_\alpha \right). \quad (47)$$

Based on the Euler homogeneous theorem, equation (47) becomes

$$\partial_t (V + T) = \partial_s \left(\sum_{\alpha=1}^{3} \frac{\partial V}{\partial q'_\alpha} \dot{q}_\alpha \right) = \partial_s [\boldsymbol{F} \cdot \dot{\boldsymbol{r}} + \boldsymbol{M} \cdot \boldsymbol{\Omega}]. \quad (48)$$

Discussion 4 Two conservation laws about the time and the arc coordinate are presented as follows.

i) Between the time interval $[0, t_1]$, if the following condition is satisfied at the two ends of the elastic rod

$$[\boldsymbol{F} \cdot \dot{\boldsymbol{r}} + \boldsymbol{M} \cdot \boldsymbol{\Omega}]_{s=0} = [\boldsymbol{F} \cdot \dot{\boldsymbol{r}} + \boldsymbol{M} \cdot \boldsymbol{\Omega}]_{s=l}, \quad (49)$$

then the conserved quantity about the time will be

$$\int_0^l (T + V)_t \mathrm{d}s = \int_0^l (T + V)_{t=0} \mathrm{d}s. \quad (50)$$

Equation (49) denotes that the power of the rod is equal on the two ends, then its mechanical energy (50) is invariant along the arc coordinate.

ii) At the two ends of the time interval $[0, t_1]$, if the following condition

$$(T + V)_{t_1} = (T + V)_{t=0} \quad (51)$$

holds at any section s, then the conserved quantity about the arc coordinate will be

$$\int_0^{t_1} \{\boldsymbol{F} \cdot \dot{\boldsymbol{r}} + \boldsymbol{M} \cdot \boldsymbol{\Omega}\}_s \mathrm{d}t = \int_0^{t_1} \{\boldsymbol{F} \cdot \dot{\boldsymbol{r}} + \boldsymbol{M} \cdot \boldsymbol{\Omega}\}_{s=0} \mathrm{d}t. \quad (52)$$

Case ii) expresses that if the mechanical energy is invariant in the time interval $[0, t_1]$, then the work of the general interval force is a conserved quantity along the arc coordinate. They are the same as the results in Ref. [44].

6. Conclusion

We have constructed the Noether theorem of Cosserat elastic rod dynamics with independent time variables and arc coordinates. The Lagrange equation and criterion of Noether symmetry for Cosserat rod dynamics are given, the Noether conserved quantity for rod dynamics is obtained. The discussion denotes that we can not only get the first integrals of momentum and energy from Noether symmetry, but also some more other first integrals. The conserved quantities of the integral form indicate the particularities of Cosserat elastic rod dynamics.

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