A necessary and sufficient condition for transforming autonomous systems into linear autonomous Birkhoffian systems*

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The problem of transforming autonomous systems into Birkhoffian systems is studied. A reasonable form of linear autonomous Birkhoff equations is given. By combining them with the undetermined tensor method, a necessary and sufficient condition for an autonomous system to have a representation in terms of linear autonomous Birkhoff equations is obtained. The methods of constructing Birkhoffian dynamical functions are given. Two examples are given to illustrate the application of the results.

Keywords: autonomous systems, linear autonomous Birkhoff's equations, non-Hamiltonian systems, Whittaker's equations

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1. Introduction

The Birkhoffian mechanics is the modern development of analytic mechanics. We can realize a direct universal representation of the inverse problem in terms of the Birkhoff equations.[1,2] Through comparing linear autonomous Birkhoff equations with nonautonomous Birkhoff equations, we can find that the former have a simpler structure and are easier to solve, and so they are the basis for investigating Birkhoff's equations, such as the constructive methods of Birkhoffian dynamical functions, integral theory, and structure preserving algorithms.[3–9]

The well-known Whittaker and Hojman-Urrutia equations[10] represent a kind of system with a simple first-order form, but without the traditional Hamiltonian representations. Thus they are of the linear autonomous non-Hamiltonian type. When searching for a Birkhoffian representation of such a system, we naturally expect that the corresponding Birkhoff equations are also the linear autonomous one. This contributes to both the integral theory and the algorithm realization for the Birkhoffian formulation of the systems.[11–19] Therefore, it becomes an important problem to determine whether a given system can be represented via the linear autonomous Birkhoff equations. This question will be discussed in this paper.

This paper is arranged as follows. In Section 2, starting from the normal first-order form of system, we define the specific form of the linear autonomous Birkhoff equations. In Section 3, a necessary and sufficient condition is obtained via the undetermined tensor method. Then, the methods of constructing the Birkhoffian and Birkhoffian functions are given.

2. Linear autonomous Birkhoff equations

Birkhoff's equations can be derived from a linear first-order variational principle in the same measure as that of Hamilton's equations, although in the most general way possible.[2] The form of the general Birkhoff equation is

$$\left(\frac{\partial R_\nu(t,a)}{\partial a^\mu}-\frac{\partial R_\mu(t,a)}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B(t,a)}{\partial a^\mu}+\frac{\partial R_\mu(t,a)}{\partial t}\right)=0, \quad (\mu,\nu=1,\ldots,2n), \tag{1}$$

where B is the Birkhoffian usually taken as the energy function of the system, R_μ are a set of Birkhoffian functions.

Birkhoff's equations (1) are said to be autonomous when R_μ and B functions do not depend explicitly on time, in which case the equations are assumed to be in the simplified form, i.e.,

$$\Omega_{\mu\nu}(a)\dot{a}^\nu-\frac{\partial B(a)}{\partial a^\mu}=0, \quad (\mu,\nu=1,\ldots,2n), \tag{2}$$

where

$$\Omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu} \tag{3}$$

is called Birkhoff's tensor, which is an antisymmetrical tensor and verifies the Jacobi law, i.e.,

$$\Omega_{\mu\nu}+\Omega_{\nu\mu}=0, \tag{4a}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\tau}+\frac{\partial \Omega_{\nu\tau}}{\partial a^\mu}+\frac{\partial \Omega_{\tau\mu}}{\partial a^\nu}=0. \tag{4b}$$

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In addition, the regularity condition $\det(\Omega_{\mu\nu}) \neq 0$ means that the $2n$ equations (2) are independent and can be transformed into the equivalent form

$$\dot{a}^{\mu} = \Omega^{\mu\nu}(a)\frac{\partial B(a)}{\partial a^{\nu}}, \quad (\mu,\nu = 1,\ldots,2n), \tag{5}$$

where $\Omega^{\mu\nu} = \Omega^{-1}_{\mu\nu}$.

Suppose that a given linear autonomous system is described by first-order ordinary differential equations

$$\dot{a}^{\mu} = A^{\mu}_{\tau}a^{\tau} + \alpha^{\mu}, \quad \alpha^{\mu} = \text{const.}, \quad (\mu,\tau = 1,\ldots,2n), \tag{6}$$

where A^{μ}_{τ} is the constant coefficient such that $\det\left(A^{\mu}_{\tau}\right) \neq 0$. Now, we shall discuss the forms of linear autonomous Birkhoff equations by combining the physical meaning of the Birkhoffian B. Obviously, if Eq. (5) is equivalent to Eq. (6), we have

$$\Omega^{\mu\nu}\frac{\partial B(a)}{\partial a^{\nu}} = A^{\mu}_{\tau}a^{\tau} + \alpha^{\mu}, \quad (\mu,\nu,\tau = 1,\ldots,2n), \tag{7}$$

or rewrite it as

$$\frac{\partial B(a)}{\partial a^{\nu}} = \Omega_{\mu\nu}\left(A^{\mu}_{\tau}a^{\tau} + \alpha^{\mu}\right), \quad (\mu,\nu,\tau = 1,\ldots,2n). \tag{8}$$

Note that the Birkhoffian B is usually taken as the energy function, which means that it should be a quadratic function in variable a. Then the tensor $\Omega_{\mu\nu}$ should be a constant when equation (7) or (8) is verified. Therefore, we restrict the form of the linear autonomous Birkhoff equations as follows: the tensor $\Omega_{\mu\nu}$ is constant in Eq. (2) or Eq. (5), and the function B is in the form of

$$B = \frac{1}{2}a^{\nu}P_{\nu\tau}a^{\tau} + \beta_{\nu}a^{\nu}, \quad \beta_{\nu} = \text{const.}, \quad (\nu,\tau = 1,\ldots,2n), \tag{9}$$

where the coefficients $P_{\nu\tau}$ and β_{ν} are constants, and satisfy the following conditions

$$P_{\nu\tau} = \Omega_{\mu\nu}\mathrm{A}^{\mu}_{\tau}, \quad \beta_{\nu} = \Omega_{\mu\nu}\alpha^{\mu}. \tag{10}$$

Next, substituting Eq. (9) into Eq. (2), we obtain the particular forms of linear autonomous Birkhoff equations

$$\Omega_{\mu\nu}\dot{a}^{\nu} - \left(P_{\mu\tau}a^{\tau} + \beta_{\mu}\right) = 0, \quad (\mu,\nu,\tau = 1,\ldots,2n). \tag{11}$$

Note the regularity condition of tensor $\Omega_{\mu\nu}$, we obtain

$$\dot{a}^{\mu} = \Omega^{\mu\nu}\left(P_{\nu\tau}a^{\tau} + \beta_{\nu}\right), \quad (\mu,\nu,\tau = 1,\ldots,2n). \tag{12}$$

We can easily see that linear autonomous Birkhoff equations (11) or (12) possess the following properties.

(i) The equations are easy to solve and the algorithm is implemented simply.

(ii) The Birkhoffian B is a quadratic function in variable a and is easy to construct. In addition, when $\beta_{\nu} = 0$ are satisfied, the function $B(a)$ is in the quadratic form and possesses the meaning of energy.

(iii) The Birkhoffian functions R_{μ} are easy to obtain via the method in Ref. [8].

Now, we specifically discuss the necessary and sufficient condition for transforming an autonomous system into linear autonomous Birkhoff equation (11) or (12).

3. A condition for transforming an autonomous system into linear autonomous Birkhoff equations

The authors have expatiated on the undetermined tensor method in Ref. [8], in which the problem for constructing the Birkhoffian representation of a nonautonomous system, i.e.,

$$\dot{a}^{\mu} = \Xi^{\mu}(t,a), \quad (\mu = 1,\ldots,2n) \tag{13}$$

is transformed into solving a first-order linear partial differential equation

$$\begin{aligned}&\frac{\partial\Omega_{\mu\nu}}{\partial t} + \frac{\partial\Omega_{\mu\nu}}{\partial a^{\rho}}\Xi^{\rho}(t,a)\\ &+\left(\Omega_{\mu\rho}\frac{\partial\Xi^{\rho}(t,a)}{\partial a^{\nu}} - \Omega_{\nu\rho}\frac{\partial\Xi^{\rho}(t,a)}{\partial a^{\mu}}\right) = 0,\\ &(\mu,\nu,\rho = 1,\ldots,2n),\end{aligned} \tag{14}$$

where $\Omega_{\mu\nu}$ is the unknown function.

When the nonautonomous system (13) degenerates into an autonomous system

$$\dot{a}^{\mu} = \Xi^{\mu}(a), \quad (\mu = 1,\ldots,2n), \tag{15}$$

equation (15) degenerates into the following equation

$$\begin{aligned}&\frac{\partial\Omega_{\mu\nu}}{\partial a^{\rho}}\Xi^{\rho}(a) + \left(\Omega_{\mu\rho}\frac{\partial\Xi^{\rho}(a)}{\partial a^{\nu}} - \Omega_{\nu\rho}\frac{\partial\Xi^{\rho}(a)}{\partial a^{\mu}}\right) = 0,\\ &(\mu,\nu,\rho = 1,\ldots,2n).\end{aligned} \tag{16}$$

We obtain $\Omega_{\mu\nu}$ after solving Eq. (16), then the Birkhoff equations may be either autonomous or non-autonomous.

As mentioned in the Introduction, we should take account of precedence in our search for an autonomous Birkhoff representation for an autonomous dynamical system. Because a nonautonomous system (13) does not have an autonomous Birkhoffian representation, we need only consider the autonomous system (15).

First of all, we analyze the version which must be verified when the autonomous system (15) can be transformed into linear autonomous Birkhoff equations. On the one hand, Birkhoff's tensor $\Omega_{\mu\nu}$ can be obtained by solving Eq. (16), which is derived from the undetermined tensor method. On the other hand, the tensor $\Omega_{\mu\nu}$ is a constant for linear autonomous Birkhoff equations. Therefore, if system (15) has a linear autonomous Birkhoffian representation, equation (16) ought to have a constant solution $\Omega_{\mu\nu}$. Conversely, if $\Omega_{\mu\nu}$ is a constant, we know that $\partial\Omega_{\mu\nu}/\partial a^{\rho} = 0$. This means that the constant solution must satisfy the algebra equations, namely,

$$\Omega_{\mu\rho}\frac{\partial\Xi^{\rho}}{\partial a^{\nu}} - \Omega_{\nu\rho}\frac{\partial\Xi^{\rho}}{\partial a^{\mu}} = 0, \quad (\mu,\nu,\rho = 1,\ldots,2n). \tag{17}$$

Secondly, we analyze the conditions which ensure that system (15) can be transformed into a linear autonomous Birkhoffian representation (11). Obviously, if system (15) is equivalent to the Birkhoff equations (11), then system (15) must be the linear autonomous system in the form as shown in Eq. (6). In this case, equation (17) degenerates into

$$\Omega_{\mu\rho}A^{\rho}_{\nu}-\Omega_{\nu\rho}A^{\rho}_{\mu}=0,\quad(\mu,\nu,\rho=1,\ldots,2n). \tag{18}$$

Hence, we have the following proposition.

Proposition 1 A necessary and sufficient condition for the autonomous system (15) to be represented in terms of the linear autonomous Birkhoff equations (11) is that the following two conditions are satisfied

i) System (15) must be a linear autonomous system coinciding with the form of Eq. (6).

ii) There exists a regular symplectic tensor $\Omega_{\mu\nu}$, in which each element is a real number and verifies Eq. (18).

The above conclusions are expressed in tensor language, they are suitable for theoretical derivations. But it is inconvenient to perform specific calculations because of the complicated relationships among indices. To make the conclusions more intuitive and suitable for the calculation, the matrix expression of Proposition 1 is given below.

Introducing a vector $\boldsymbol{a}=(a^1,\ldots,a^{2n})^{\mathrm{T}}$, then equation (6) can be expressed by

$$\dot{\boldsymbol{a}}=\boldsymbol{A}\boldsymbol{a}+\boldsymbol{\alpha}, \tag{19}$$

where $\boldsymbol{A}=(a_{ij})$ $(i,j=1,\ldots,2n)$ is a constant matrix and satisfies $\det(A)\neq 0$. In addition, $\boldsymbol{\alpha}$ is a column vector and equation (11) can be written as

$$\boldsymbol{\Omega}\dot{\boldsymbol{a}}-(\boldsymbol{P}\boldsymbol{a}+\beta)=0, \tag{20}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{P}$ are the matrix forms corresponding to the tensors $\Omega_{\mu\nu}$ and $P_{\nu\tau}$ $(\nu,\tau=1,\ldots,2n)$, respectively. The quantity β is a column vector. Therefore, the equivalence matrix form of Eq. (18) is

$$\boldsymbol{\Omega}\boldsymbol{A}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Omega}^{\mathrm{T}}=0\Leftrightarrow\boldsymbol{\Omega}\boldsymbol{A}=(\boldsymbol{\Omega}\boldsymbol{A})^{\mathrm{T}}, \tag{21}$$

where $\boldsymbol{A}^{\mathrm{T}}$ is the transposed matrix of A. Therefore, Proposition 1 can be reformulated as follows.

Proposition 2 A necessary and sufficient condition for the autonomous system (15) to be represented in terms of the linear autonomous Birkhoff equations (20) is that the following two conditions are satisfied.

(I) System (15) is a linear autonomous system coinciding with the form of Eq. (19).

(II) There exists a regular symplectic matrix $\boldsymbol{\Omega}$ in a real number field such that $\boldsymbol{\Omega}\boldsymbol{A}$ is a symmetry matrix.

It should be pointed out that because of the regularity of $\boldsymbol{\Omega}$ and $\boldsymbol{A}$, the matrix product $\boldsymbol{\Omega}\boldsymbol{A}$ is also regular. This illustrates that the value of $\boldsymbol{\Omega}\boldsymbol{A}$ is nonzero in condition (II).

Finally, we discuss the approach to the calculation of the functions B and R_μ. Substituting Eq. (19) into Eq. (20) and comparing the coefficients, we have

$$\boldsymbol{P}=\boldsymbol{\Omega}\boldsymbol{A},\quad\beta=\boldsymbol{\Omega}\boldsymbol{\alpha}. \tag{22}$$

Note that for Eq. (9), we know that the Birkhoffian B can be obtained by

$$B=\frac{1}{2}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{a}+\boldsymbol{a}^{\mathrm{T}}\beta=\frac{1}{2}\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{\Omega}\boldsymbol{A})\boldsymbol{a}+\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{\Omega}\boldsymbol{\alpha}), \tag{23}$$

and the Birkhoffian functions R_μ are given by[8]

$$R_\mu\equiv r_{\mu\nu}a^{\nu}=\begin{cases}-\Omega_{\mu\nu}a^{\nu}, & \mu<\nu,\\ 0, & \mu\geq\nu,\end{cases}\quad(\mu,\nu=1,\ldots,2n), \tag{24}$$

where $r_{\mu\nu}=r_{\mu\nu}(\Omega)$.

4. Illustrative examples

Example 1 Determine whether the one-dimensional damped oscillator system[3]

$$\ddot{x}+x+\gamma\dot{x}=0,\quad\gamma=\text{const.} \tag{25}$$

can be represented by linear autonomous Birkhoff equation (11) or (12).

Let

$$a^1=x,\quad a^2=\dot{x}, \tag{26}$$

then equation (25) can be transformed into the normal first-order form

$$\dot{a}^1=a^2,\quad\dot{a}^2=-a^1-\gamma a^2. \tag{27}$$

This is a linear autonomous system and can be expressed in the following matrix form

$$\begin{pmatrix}\dot{a}^1\\ \dot{a}^2\end{pmatrix}=\begin{pmatrix}0&1\\ -1&-\gamma\end{pmatrix}\begin{pmatrix}a^1\\ a^2\end{pmatrix},\quad\begin{pmatrix}0&1\\ -1&-\gamma\end{pmatrix}\equiv\boldsymbol{A}. \tag{28}$$

Assume the Birkhoff tensor to be

$$\boldsymbol{\Omega}=\begin{pmatrix}0&\Omega_{12}\\ -\Omega_{12}&0\end{pmatrix}. \tag{29}$$

Calculate the matrix product, i.e.,

$$\begin{aligned}\boldsymbol{\Omega}\boldsymbol{A}&=\begin{pmatrix}0&\Omega_{12}\\ -\Omega_{12}&0\end{pmatrix}\begin{pmatrix}0&1\\ -1&-\gamma\end{pmatrix}\\&=\begin{pmatrix}-\Omega_{12}&-\Omega_{12}\gamma\\ 0&-\Omega_{12}\end{pmatrix}.\end{aligned} \tag{30}$$

Obviously, $\boldsymbol{\Omega}\boldsymbol{A}$ is a symmetry matrix only when $\Omega_{12}=0$ is satisfied. In this case, we know that the regular condition will be broken from Proposition 2. Therefore, the linear autonomous Birkhoffian representation (11) of system (25) does not exist.

Example 2 Determine whether the Whittaker equations

$$\ddot{x}-x=0,\quad \ddot{y}-\dot{x}=0, \tag{31}$$

can be represented with linear autonomous Birkhoff equation (11) or (12).

Let

$$a^1=x,\ \ a^2=y,\ \ a^3=\dot{x},\ \ a^4=\dot{y}, \tag{32}$$

then equation (3) can be transformed into a normal first-order form

$$\dot{a}^1=a^3,\ \ \dot{a}^2=a^4,\ \ \dot{a}^3=a^1,\ \ \dot{a}^4=a^3. \tag{33}$$

This is a linear autonomous system and the correspondence matrix form is

$$\begin{pmatrix}\dot{a}^1\\ \dot{a}^2\\ \dot{a}^3\\ \dot{a}^4\end{pmatrix}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}a^1\\a^2\\a^3\\a^4\end{pmatrix},$$
$$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix}\equiv \boldsymbol{A}. \tag{34}$$

Assume the Birkhoff tensor to be

$$\boldsymbol{\Omega}=\begin{pmatrix}0&\Omega_{12}&\Omega_{13}&\Omega_{14}\\-\Omega_{12}&0&\Omega_{23}&\Omega_{24}\\-\Omega_{13}&-\Omega_{23}&0&\Omega_{34}\\-\Omega_{14}&-\Omega_{24}&-\Omega_{34}&0\end{pmatrix}. \tag{35}$$

Calculate the matrix product, i.e.,

$$\boldsymbol{\Omega A}=\begin{pmatrix}0&\Omega_{12}&\Omega_{13}&\Omega_{14}\\-\Omega_{12}&0&\Omega_{23}&\Omega_{24}\\-\Omega_{13}&-\Omega_{23}&0&\Omega_{34}\\-\Omega_{14}&-\Omega_{24}&-\Omega_{34}&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix}$$
$$=\begin{pmatrix}\Omega_{13}&0&-\Omega_{14}&\Omega_{12}\\\Omega_{23}&0&\Omega_{24}-\Omega_{12}&0\\0&0&\Omega_{34}-\Omega_{13}&-\Omega_{23}\\-\Omega_{34}&0&-\Omega_{14}&-\Omega_{24}\end{pmatrix}\equiv \boldsymbol{P}. \tag{36}$$

Clearly, when

$$\Omega_{14}=\Omega_{23}=0,\ \ \Omega_{12}=\Omega_{24}=-\Omega_{34}=\text{const.}\neq 0, \tag{37}$$

the matrix $\boldsymbol{P}=\boldsymbol{\Omega A}$ is symmetric. From Proposition 2, we know that the Whittaker equations can be expressed by linear autonomous Birkhoff equation (11) or (12). Take

$$\Omega_{13}=\Omega_{14}=\Omega_{23}=0,\ \ \Omega_{12}=\Omega_{24}=-\Omega_{34}=1, \tag{38}$$

then we shall obtain

$$\boldsymbol{P}=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&-1&0\\1&0&0&-1\end{pmatrix},$$

$$\boldsymbol{\Omega}=\begin{pmatrix}0&1&0&0\\-1&0&0&1\\0&0&0&-1\\0&-1&1&0\end{pmatrix}. \tag{39}$$

From Eq. (23), we obtain the Birkhoffian B as follows:

$$B=\frac{1}{2}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{a}=\frac{1}{2}\boldsymbol{a}^{\mathrm{T}}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&-1&0\\1&0&0&-1\end{pmatrix}\boldsymbol{a}. \tag{40}$$

Putting them in order, we have

$$B=-\frac{1}{2}\left[\left(a^3\right)^2+\left(a^4\right)^2\right]+a^1a^4. \tag{41}$$

The Birkhoffian functions R_μ can be constructed from Eq. (24) as

$$\begin{pmatrix}R_1\\R_2\\R_3\\R_4\end{pmatrix}=\begin{pmatrix}0&-\Omega_{12}&-\Omega_{13}&-\Omega_{14}\\0&0&-\Omega_{23}&-\Omega_{24}\\0&0&0&-\Omega_{34}\\0&0&0&0\end{pmatrix}\begin{pmatrix}a^1\\a^2\\a^3\\a^4\end{pmatrix}$$
$$=\begin{pmatrix}a^2\\-a^4\\a^4\\0\end{pmatrix}, \tag{42}$$

or rewritten as

$$R_1=a^2,\ \ R_2=-a^4,\ \ R_3=a^4,\ \ R_4=0. \tag{43}$$

Finally, the expected linear autonomous Birkhoff equations are

$$\begin{cases}\dot{a}^2-a^4=0,\\-\dot{a}^1+\dot{a}^4=0,\\-\dot{a}^4+a^3=0,\\-\dot{a}^2+\dot{a}^3-a^1+a^4=0.\end{cases} \tag{44}$$

5. Summary

This article focuses on whether a given autonomous system can be represented via linear autonomous Birkhoff equations. By combining the concrete form of linear autonomous Birkhoff equations and the undetermined tensor method, we obtained a necessary and sufficient condition expressed by Proposition 1 and Proposition 2. As can be seen from example 1, a given system (15) cannot be necessarily transformed into Eq. (11), while example 2 shows a specific application of the conditions and concrete steps to writing linear autonomous Birkhoff equations. Furthermore, the undetermined tensor method plays an important role in deriving Eq. (18). Therefore, this paper can also be regarded as an application of the undetermined tensor method in autonomous systems.

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