The Wigner distribution function of a super Lorentz–Gauss SLG_{11} beam through a paraxial *ABCD* optical system*

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An orthonormal beam family of super Lorentz–Gauss (SLG) beam model is proposed to describe the higher-order mode beams with high divergence, which are generated by a high power diode laser. Here we consider the simplest case of the SLG beams, where there are four mutually orthogonal SLG beams, namely SLG_{00}, SLG_{01}, SLG_{10}, and SLG_{11} beams. The SLG_{00} beam is just the Lorentz–Gauss beam. Based on the Collins integral formula and the Hermite–Gaussian expansion of a Lorentz function, an analytical expression for the Wigner distribution function (WDF) of an SLG_{11} beam through a paraxial *ABCD* optical system is derived. The properties of the WDF of an SLG_{11} beam propagating in free space are demonstrated. The normalized WDFs of an SLG_{11} beam at the different spatial points are depicted in several observation planes. The influence of the beam parameter on the WDF of an SLG_{11} beam in free space is analyzed at different propagation distances. The second-order moments of the WDF of an SLG_{11} beam in free space are also examined. This research reveals the propagation properties of an SLG_{11} beam from another perspective. The WDFs of SLG_{01} and SLG_{10} beams can be easily obtained by using the WDFs of Lorentz–Gauss beam and the SLG_{11} beam.

Keywords: Wigner distribution function, super Lorentz–Gauss beam, paraxial *ABCD* optical system

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1. Introduction

Due to the high divergence, Lorentz–Gauss beams are introduced to describe the radiation emitted by a diode laser.[1,2] The properties of Lorentz–Gauss beams have been extensively investigated.[3–10] The researches show that Lorentz–Gauss beams are only applicable to the description of the fundamental mode beams with high divergence. Therefore, an orthonormal beam family of the super Lorentz–Gauss (SLG) beam model has been proposed to describe the higher-order mode beams with high divergence, which are generated by a high power diode laser.[11] The propagations of SLG beams through a paraxial *ABCD* optical system,[12] in a turbulent atmosphere,[13] and in uniaxial crystals orthogonal to the optical axis[14] have been examined, respectively. The Wigner distribution function (WDF) was first proposed to account for quantum corrections to classical statistical mechanics.[15] Now, the WDF provides a powerful tool for the description of coherent and partially coherent beams and their propagations in linear and nonlinear media.[16–26] To further master the propagation properties of SLG beams, therefore, here we investigate the WDF of SLG beams through a paraxial *ABCD* optical system. Moreover, we pay attention to the simplest case of the SLG beams, where there are four mutually orthogonal SLG beams, namely SLG_{00}, SLG_{01}, SLG_{10}, and SLG_{11} beams. The SLG_{00} beam is just the Lorentz–Gauss beam, and the WDF of an SLG_{00} beam through a paraxial *ABCD* optical system has been derived.[27] In the remainder of this paper, therefore, we consider the WDF of an SLG_{11} beam. By making use of the WDFs of SLG_{00} and SLG_{11} beams, the WDFs of SLG_{01} and SLG_{10} beams can be easily obtained.

2. The WDF of an SLG_{11} beam through a paraxial *ABCD* optical system

In the Cartesian coordinate system, the z axis is taken to be the propagation axis. The SLG_{11} beam in the source plane $z = 0$ takes the form as

$$E(x,y,0) = E(x,0)E(y,0), \tag{1}$$

with $E(j,0)$ being given by

$$E(j,0) = \sqrt{\frac{2w_{0j}}{\pi}}\frac{j}{j^2+w_{0j}^2}\exp\left(-\frac{j^2}{w_0^2}\right), \tag{2}$$

where $j = x$ or y (hereafter), w_{0x} and w_{0y} are the parameters related to the beam widths of the super-Lorentzian part in the x and y directions, respectively, and w_0 is the waist of the Gaussian part. To obtain the analytical expression of the WDF, the Lorentz distribution should be expanded into the linear superposition of finite terms of Hermite–Gaussian functions:[28]

$$\frac{1}{j^2+w_{0j}^2} = \sqrt{\frac{\pi}{2}}\frac{1}{w_{0j}^2}\sum_{m=0}^{N}\sigma_{2m}\mathrm{H}_{2m}\left(\frac{j}{w_{0j}}\right)\exp\left(-\frac{j^2}{2w_{0j}^2}\right), \tag{3}$$

where N is the number of terms in the expansion. The weight

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coefficient σ_{2m} is given by[28]

$$\sigma_{2m} = \frac{(-1)^m\sqrt{2}}{2^{2m}}\left\{\frac{1}{m!}\mathrm{erfc}\left(\frac{\sqrt{2}}{2}\right)\exp\left(\frac{1}{2}\right) + \sum_{n_1=1}^{m}\frac{2^{2n_1}}{(2n_1)!(m-n_1)!}\left[\mathrm{erfc}\left(\frac{\sqrt{2}}{2}\right)\exp\left(\frac{1}{2}\right) + \sqrt{\frac{2}{\pi}}\sum_{n_2=1}^{n_1}(-1)^{n_2}(2n_2-3)!!\right]\right\}, \quad (4)$$

where erfc(·) is the complementary error function. $\mathrm{H}_{2m}(\cdot)$ is the $2m$-th order Hermite polynomial. Therefore, equation (2) can be rewritten as

$$E(j,0) = \frac{1}{w_{0j}^{3/2}}\sum_{m=0}^{N}\sigma_{2m} j\,\mathrm{H}_{2m}\left(\frac{j}{w_{0j}}\right)\exp\left(-\frac{j^2}{w_0^2u_j^2}\right), \quad (5)$$

where $u_j = e_j\sqrt{2}/(1+2e_j^2)^{1/2}$, and $e_j = w_{0j}/w_0$. Since the SLG_{11} beam has the same expression in the x and y directions, we consider the Wigner distribution function of an SLG_{11} beam in the j direction. The Wigner distribution function that corresponds to a one-dimensional optical field in an observation plane is defined as follows:[15]

$$W(x,v_x) = \int_{-\infty}^{\infty}E\left(x+\frac{x'}{2}\right)E^*\left(x-\frac{x'}{2}\right)\times\exp(-\mathrm{i}2\pi v_x x')\,\mathrm{d}x', \quad (6)$$

where v_x denotes the spatial-frequency variable in the phase space, and the asterisk means the complex conjugation. Substituting Eq. (5) into Eq. (7) and using the following integral formulae:[29]

$$\mathrm{H}_{2m}(x) = \sum_{l=0}^{m}\frac{(-1)^l(2m)!}{l!(2m-2l)!}(2x)^{2m-2l}, \quad (7)$$

$$(x+y)^{2m} = \sum_{s=0}^{2m}\frac{(2m)!}{s!(2m-s)!}x^{2m-s}y^s, \quad (8)$$

$$\int_{-\infty}^{\infty}x^{2n}\exp(-\alpha x^2+2\beta x)\,\mathrm{d}x = (2n)!\sqrt{\frac{\pi}{\alpha}}\left(\frac{\beta}{\alpha}\right)^{2n}\exp\left(\frac{\beta^2}{\alpha}\right)\times\sum_{\tau=0}^{n}\frac{1}{\tau!(2n-2\tau)!}\left(\frac{\alpha}{4\beta^2}\right)^{\tau}, \quad (9)$$

one can obtain the WDF of an SLG_{11} beam in the j direction of the source plane:

$$W(j,v_j) = \frac{\sqrt{2\pi}u_j}{e_j}\exp\left[-\frac{2}{u_j^2}\left(\frac{j}{w_0}\right)^2 - 2\pi^2u_j^2(w_0v_j)^2\right]\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}\sigma_{2m_1}\sigma_{2m_2}\sum_{l_1=0}^{m_1}\sum_{s_1=0}^{2m_1-2l_1}\frac{(-1)^{l_1}(2m_1)!}{l_1!s_1!(2m_1-2l_1-s_1)!}$$
$$\times\sum_{l_2=0}^{m_2}\sum_{s_2=0}^{2m_2-2l_2}\frac{(-1)^{l_2+s_2}(2m_2)!}{l_2!s_2!(2m_2-2l_2-s_2)}\left(\frac{2j}{w_0e_j}\right)^{2m_1+2m_2-2l_1-2l_2-s_1-s_2}\left[\frac{j^2}{w_0^2e_j^2}(s_1+s_2)!\left(\frac{-2\pi\mathrm{i}u_j^2w_0v_j}{e_j}\right)^{s_1+s_2}\right.$$
$$\times\sum_{\tau_1=0}^{(s_1+s_2)/2}\frac{1}{\tau_1!(s_1+s_2-2\tau_1)!}\left(\frac{-1}{8\pi^2u_j^2w_0^2v_j^2}\right)^{\tau_1} - \frac{1}{4}(s_1+s_2+2)!\left(\frac{-2\pi\mathrm{i}u_j^2w_0v_j}{e_j}\right)^{s_1+s_2+2}$$
$$\left.\times\sum_{\tau_2=0}^{(s_1+s_2+2)/2}\frac{1}{\tau_2!(s_1+s_2+2-2\tau_2)!}\left(\frac{-1}{8\pi^2u_j^2w_0^2v_j^2}\right)^{\tau_2}\right], \quad (10)$$

where $[(s_1+s_2)/2]$ gives the greatest integer less than or equal to $(s_1+s_2)/2$.

The propagation of an SLG_{11} beam through a paraxial *ABCD* optical system is described by the Collins integral formula:

$$E(j,z) = \sqrt{\frac{1}{\mathrm{i}\lambda B}}\exp(\mathrm{i}kz)\int_{-\infty}^{\infty}E(j_0,0)\times\exp\left[\frac{\mathrm{i}k}{2B}(Aj_0^2-2j_0j+Dj^2)\right]\mathrm{d}j_0, \quad (11)$$

where A, B, C, and D are the transfer matrix elements of the paraxial optical system. By using the integral formula[29]

$$x\mathrm{H}_M(x) = \frac{1}{2}\mathrm{H}_{M+1}(x) + M\mathrm{H}_{M-1}(x), \quad (12)$$

$$\int_{-\infty}^{\infty}\mathrm{H}_m(p)\exp[-\gamma(p-q)^2]\,\mathrm{d}p = \sqrt{\frac{\pi}{\gamma}}\left(1-\frac{1}{\gamma}\right)^{m/2}\mathrm{H}_m\left[q\left(1-\frac{1}{\gamma}\right)^{-1/2}\right], \quad (13)$$

the propagation of an SLG_{11} beam in j direction through a paraxial *ABCD* optical system is

$$E(j,z) = \sqrt{\frac{ke_jw_0}{2\mathrm{i}\gamma_jB}}\exp\left(-\frac{k^2e_j^2w_0^2j^2}{4\gamma_jB^2} + \frac{\mathrm{i}kDj^2}{2B} + \mathrm{i}kz\right)\times\sum_{m=0}^{N}\sigma_{2m}\left\{\frac{1}{2}\left(1-\frac{1}{\gamma_j}\right)^{(2m+1)/2}\mathrm{H}_{2m+1}(\xi_jj) + 2m\left(1-\frac{1}{\gamma_j}\right)^{(2m-1)/2}\mathrm{H}_{2m-1}(\xi_jj)\right\}, \quad (14)$$

where γ_j and ξ_j are defined as

$$\gamma_j = e_j^2 + \frac{1}{2} - \frac{\mathrm{i}kAe_j^2w_0^2}{2B}, \quad (15)$$

$$\xi_j = -\frac{\mathrm{i}ke_jw_0}{2B(\gamma_j^2-\gamma_j)^{1/2}}. \quad (16)$$

Substituting Eq. (14) into Eq. (6) and using Eqs. (7)–(9), the WDF of an SLG_{11} beam in the j direction through a paraxial $ABCD$ optical system is found to be

$$W(j,v_j) = \frac{ke_jw_0}{2B}\sqrt{\frac{\pi}{\gamma_j\gamma_j^*\alpha_{1j}}}\exp\left(\frac{\beta_{1j}^2}{\alpha_{1j}}-4\alpha_{1j}j^2\right)\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}\sigma_{2m_1}\sigma_{2m_2}\left[\frac{1}{4}\Omega(2m_1+1,2m_2+1)+m_1\Omega(2m_1-1,2m_2+1)\right.$$
$$\left.+m_2\Omega(2m_1+1,2m_2-1)+4m_1m_2\Omega(2m_1-1,2m_2-1)\right], \tag{17}$$

with $\Omega(m_1,m_2)$ being given by

$$\Omega(m_1,m_2) = \left(1-\frac{1}{\gamma_j}\right)^{m_1/2}\left(1-\frac{1}{\gamma_j^*}\right)^{m_2/2}\sum_{l_1=0}^{[m_1/2]}\sum_{s_1=0}^{m_1-2l_1}\sum_{l_2=0}^{[m_2/2]}\sum_{s_2=0}^{m_2-2l_2}\frac{(-1)^{l_1+l_2+s_2}m_1!m_2!}{l_1!s_1!(m_1-2l_1-s_1)!l_2!s_2!(m_2-2l_2-s_2)!}$$
$$\times\xi_j^{m_1-2l_1}(\xi_j^*)^{m_2-2l_2}(2j)^{m_1+m_2-2l_1-s_1-2l_2-s_2}(s_1+s_2)!\left(\frac{\beta_{1j}}{\alpha_{1j}}\right)^{s_1+s_2}\sum_{\tau=0}^{[(s_1+s_2)/2]}\frac{1}{\tau!(s_1+s_2-2\tau)!}\left(\frac{\alpha_{1j}}{4\beta_{1j}^2}\right)^{\tau}, \tag{18}$$

where the auxiliary parameters α_{1j} and β_{1j} are defined as follows:

$$\alpha_{1j} = \frac{k^2e_j^2w_0^2}{16\gamma_jB^2}+\frac{k^2e_j^2w_0^2}{16\gamma_j^*B^{*2}}-\frac{\mathrm{i}kD}{8B}+\frac{\mathrm{i}kD^*}{8B^*}, \tag{19}$$

$$\beta_{1j} = \frac{k^2e_j^2w_0^2j}{8\gamma_j^*B^{*2}}-\frac{k^2e_j^2w_0^2j}{8\gamma_jB^2}+\frac{\mathrm{i}kDj}{4B}+\frac{\mathrm{i}kD^*j}{4B^*}-\mathrm{i}\pi v_j. \tag{20}$$

$B=0$ corresponds to an image-forming system. In this case, the output field of an SLG_{11} beam in the j direction reduces into

$$E(j,z) = \frac{1}{A^{3/2}e_j^{3/2}w_0^{3/2}}\sum_{m=0}^{N}\sigma_{2m}j\mathrm{H}_{2m}\left(\frac{j}{Ae_jw_0}\right)\times\exp\left(-\frac{j^2}{Aw_0^2u_j^2}+\frac{\mathrm{i}kCj^2}{2A}+\mathrm{i}kz\right). \tag{21}$$

The corresponding WDF of an SLG_{11} beam in the j direction through an image-forming system is

$$W(j,v_j) = \frac{\pi^{1/2}}{(AA^*)^{3/2}\alpha_{2j}^{1/2}e_j^3w_0^3}\exp\left(\frac{\beta_{2j}^2}{\alpha_{2j}}-4\alpha_{2j}j^2\right)\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}\sigma_{2m_1}\sigma_{2m_2}\sum_{l_1=0}^{m_1}\frac{(-1)^{l_1}(2m_1)!}{l_1!(2m_1-2l_1)!}\sum_{s_1=0}^{2m_1-2l_1}\frac{(2m_1-2l_1)!}{s_1!(2m_1-2l_1-s_1)!}$$
$$\times\left(\frac{1}{Ae_jw_0}\right)^{2m_1-2l_1}\sum_{l_2=0}^{m_2}\frac{(-1)^{l_2}(2m_2)!}{l_2!(2m_2-2l_2)!}\sum_{s_2=0}^{2m_2-2l_2}\frac{(-1)^{s_2}(2m_2-2l_2)!}{s_2!(2m_2-2l_2-s_2)!}\left(\frac{1}{A^*e_jw_0}\right)^{2m_2-2l_2}(2j)^{2m_1+2m_2-2l_1-s_1-2l_2-s_2}$$
$$\times\left[j^2(s_1+s_2)!\left(\frac{\beta_{2j}}{\alpha_{2j}}\right)^{s_1+s_2}\sum_{\tau=0}^{[(s_1+s_2)/2]}\frac{1}{\tau!(s_1+s_2-2\tau)!}\left(\frac{\alpha_{2j}}{4\beta_{2j}^2}\right)^{\tau}-\frac{1}{4}(s_1+s_2+2)!\left(\frac{\beta_{2j}}{\alpha_{2j}}\right)^{s_1+s_2+2}\right.$$
$$\left.\times\sum_{\tau=0}^{[(s_1+s_2+2)/2]}\frac{1}{\tau!(s_1+s_2+2-2\tau)!}\left(\frac{\alpha_{2j}}{4\beta_{2j}^2}\right)^{\tau}\right], \tag{22}$$

where the auxiliary parameters α_{2j} and β_{2j} are defined as

$$\alpha_{2j} = \frac{1}{4Aw_0^2u_j^2}+\frac{1}{4A^*w_0^2u_j^2}-\frac{\mathrm{i}kC}{8A}+\frac{\mathrm{i}kC^*}{8A^*}, \tag{23}$$

$$\beta_{2j} = -\frac{j}{2Aw_0^2u_j^2}+\frac{j}{2A^*w_0^2u_j^2}+\frac{\mathrm{i}kCj}{4A}+\frac{\mathrm{i}kC^*j}{4A^*}-\mathrm{i}\pi v_j. \tag{24}$$

The value of σ_{2m} dramatically decreases with even number $2m$ increasing. $\sigma_0=0.7399$, $\sigma_2=0.9298\times10^{-2}$, and $\sigma_{10}=0.3008\times10^{-6}$. Therefore, equations (10), (17), and (22) converge quickly. The WDF of a two-dimensional SLG_{11} beam through a paraxial $ABCD$ optical system can be readily obtained by virtue of the above results:

$$W(x,y,v_x,v_y) = W(x,v_x)W(y,v_y). \tag{25}$$

By making use of the Wigner distribution functions of Lorentz–Gauss beam and SLG_{11} beam, the Wigner distribution functions for SLG_{01} and SLG_{10} beams are obtained as follows:

$$W^{01}(x,y,v_x,v_y) = W^0(x,v_x)W^1(y,v_y), \tag{26}$$

$$W^{10}(x,y,v_x,v_y) = W^1(x,v_x)W^0(y,v_y), \tag{27}$$

where $W^0(x,v_x)$ and $W^0(y,v_y)$ are the WDFs of a Lorentz–Gauss beam in the x and y directions, respectively. $W^1(x,v_x)$ and $W^1(y,v_y)$ are the WDFs of an SLG_{11} beam in the x and y directions.

Now, we consider the second-order moments of the WDF

of an SLG_{11} beam in the j direction of the source plane:

$$\langle j^2\rangle = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j^2 W(j,v_j)\,\mathrm{d}j\,\mathrm{d}v_j}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(j,v_j)\,\mathrm{d}j\,\mathrm{d}v_j} = \frac{\Phi_j(2,0)}{\Phi_j(0,0)}, \tag{28}$$

$$\begin{aligned}\langle \theta_j^2\rangle &= \lambda^2 \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_j^2 W(j,v_j)\,\mathrm{d}j\,\mathrm{d}v_j}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(j,v_j)\,\mathrm{d}j\,\mathrm{d}v_j} \\ &= \lambda^2 \frac{\Phi_j(0,2)}{\Phi_j(0,0)},\end{aligned} \tag{29}$$

$$\begin{aligned}\langle j\theta_j\rangle &= \langle \theta_j^W\rangle = -\lambda \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j v_j W(j,v_j)\,\mathrm{d}j\,\mathrm{d}v_j}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(j,v_j)\,\mathrm{d}j\,\mathrm{d}v_j} \\ &= -\lambda \frac{\Phi_j(1,1)}{\Phi_j(0,0)} = 0,\end{aligned} \tag{30}$$

with $\Phi_j(0,0)$, $\Phi_j(1,1)$, $\Phi_j(0,2)$, and $\Phi_j(2,0)$ being given by

$$\begin{aligned}&\Phi_j(\eta,\varepsilon) \\ &= \sum_{m_1=0}^{N}\sum_{m_2=0}^{N} \sigma_{2m_1}\sigma_{2m_2} \sum_{l_1=0}^{m_1}\sum_{s_1=0}^{2m_1-2l_1} \frac{(-1)^{l_1}(2m_1)!}{l_1!s_1!(2m_1-2l_1-s_1)!} \\ &\times \sum_{l_2=0}^{m_2}\sum_{s_2=0}^{2m_2-2l_2} \frac{(-1)^{l_2+s_2}(2m_2)!}{l_2!s_2!(2m_2-2l_2-s_2)} \left(\frac{2}{w_0 e_j}\right)^{\chi} \\ &\times \left(\frac{-\sqrt{2}\mathrm{i}u_j}{e_j}\right)^{s_1+s_2+\varepsilon} [1+(-1)^{\chi+\eta}][1+(-1)^{s_1+s_2+\varepsilon}] \\ &\times \left(\frac{2}{u_j^2 w_0^2}\right)^{-(\chi+\eta+1)/2} \left[(s_1+s_2)!\Gamma\left(\frac{\chi+\eta+3}{2}\right)\right. \\ &\times \sum_{\tau_1=0}^{(s_1+s_2)/2} \frac{(-1)^{\tau_1}}{4^{\tau_1}\tau_1!(s_1+s_2-2\tau_1)!}\Gamma\left(\frac{s_1+s_2+\varepsilon-2\tau_1+1}{2}\right) \\ &+ (s_1+s_2+2)!\Gamma\left(\frac{\chi+\eta+1}{2}\right) \\ &\times \sum_{\tau_2=0}^{(s_1+s_2+2)/2} \frac{(-1)^{\tau_2}}{4^{\tau_2}\tau_2!(s_1+s_2+2-2\tau_2)!} \\ &\left.\times \Gamma\left(\frac{s_1+s_2+\varepsilon-2\tau_2+3}{2}\right)\right],\end{aligned} \tag{31}$$

where χ is defined as

$$\chi = 2m_1+2m_2-2l_1-2l_2-s_1-s_2. \tag{32}$$

In the derivation of Eq. (31), the following integral formula is used

$$\begin{aligned}&\int_{-\infty}^{\infty} j^m \exp(-aj^2)\,\mathrm{d}j \\ &= \frac{1}{2}[1+(-1)^m]a^{-(m+1)/2}\Gamma\left(\frac{m+1}{2}\right),\end{aligned} \tag{33}$$

where $\Gamma(\cdot)$ is a Gamma function. $\langle j^2\rangle^{1/2}$ and $\langle \theta_j^2\rangle^{1/2}$ are just the beam waist and the divergence angle in the j direction. The second-order moments of the WDF of an SLG_{11} beam in the j direction through a paraxial *ABCD* optical system yield

$$\begin{aligned}&\begin{pmatrix}\langle j_z^2\rangle & \langle j_z\theta_{jz}\rangle \\ \langle \theta_{jz}j_z\rangle & \langle \theta_{jz}^2\rangle\end{pmatrix} \\ &= \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\langle j^2\rangle & \langle j\theta_j\rangle \\ \langle \theta_j j\rangle & \langle \theta_j^2\rangle\end{pmatrix}\begin{pmatrix}A & C\\ B & D\end{pmatrix}.\end{aligned} \tag{34}$$

Similarly, the second-order moments of the WDFs of SLG_{01} and SLG_{10} beams through a paraxial *ABCD* optical system can be obtained by the same procedure.

3. Numerical calculations and analyses

As a numerical example, the properties of the WDF of an SLG_{11} beam propagating in free space are demonstrated. As the WDFs in the x and y directions have the same variational law, we first only consider the WDF in the x direction. Figures 1–3 show the normalized WDFs of an SLG_{11} beam in several different observation planes in free space. The normalized WDF is given by $W(x,v_x)/W_{\max}(x,v_x)$ where the subscript denotes taking the maximum value. $e_x = 0.5$, 1, and 3 in Figs. 1–3, respectively. In Figs. 1–3, the transversal spatial coordinate, the spatial frequency variable, and the axial propagation distance are scaled in proportion to $1/w_0$, w_0, and w_0^2/λ. In the source plane, the pattern of the WDF of an SLG_{11} beam with small e_x is modulated by the Lorentz profile. The pattern of the WDF of an SLG_{11} beam with large e_x in the source plane is modulated by a Gaussian profile. When $e_x = 1$, the pattern of the WDF of an SLG_{11} beam in the source plane is jointly modulated by the Gaussian and the Lorentz distributions. Therefore, the different values of e_x result in the different distributions of the WDF of an SLG_{11} beam in the source plane. Upon propagation in free space, the pattern of the WDF of an SLG_{11} beam twists clockwise. Moreover, the pattern of the WDF shrinks in the direction of the spatial frequency variable and elongates in the direction of the transversal spatial coordinate. Normalized WDFs in the x direction of an SLG_{11} beam at the point $x = 0$ of different observation planes of free space are shown in Figs. 4 and 5. $e_x = 0.5$ in Fig. 4 and $e_x = 3$ in Fig. 5. Upon propagation in free space, the normalized WDF in the x direction of an SLG_{11} beam at the point $x = 0$ contracts in the direction of the spatial frequency variable. The WDF offers the properties of optical beams both in the spatial domain and in the frequency domain. Upon propagation, an SLG_{11} beam expands in the spatial domain. As a result, the normalized WDF becomes compact in the frequency domain. With the value of e_x decreasing, the shrinkage of the normalized WDF increases in the direction of the spatial frequency variable.

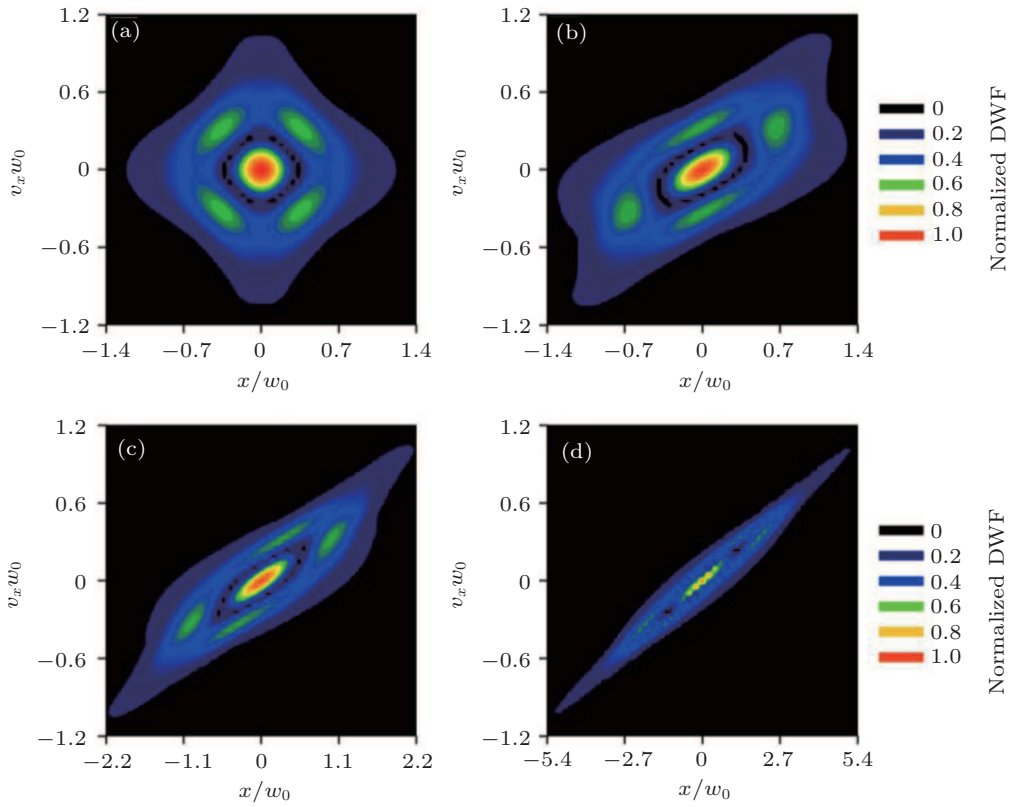


Fig. 1. (color online) Contours of the normalized WDF in the x direction of an SLG_{11} beam in different observation planes in free space. $e_x = 0.5$. (a) $z = 0$, (b) $z = w_0^2/\lambda$, (c) $z = 2w_0^2/\lambda$, (d) $z = 5w_0^2/\lambda$.

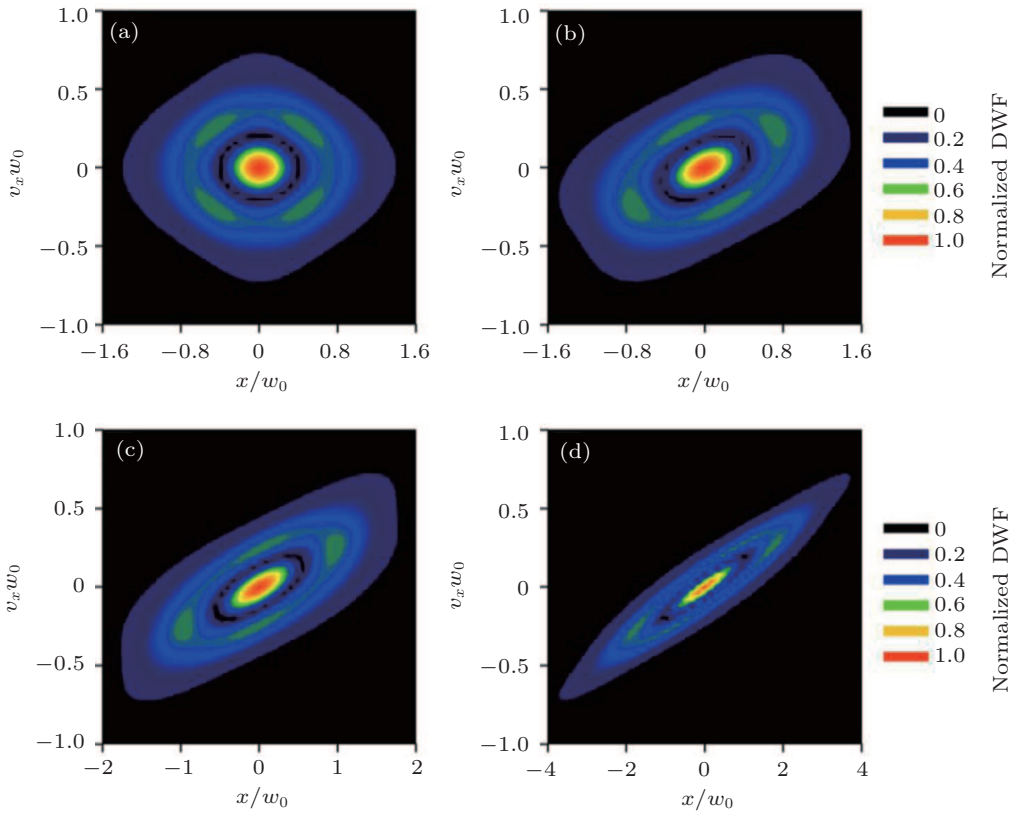


Fig. 2. (color online) Contours of the normalized WDF in the x direction of an SLG_{11} beam in different observation planes in free space. $e_x = 1$. (a) $z = 0$, (b) $z = w_0^2/\lambda$, (c) $z = 2w_0^2/\lambda$, (d) $z = 5w_0^2/\lambda$.

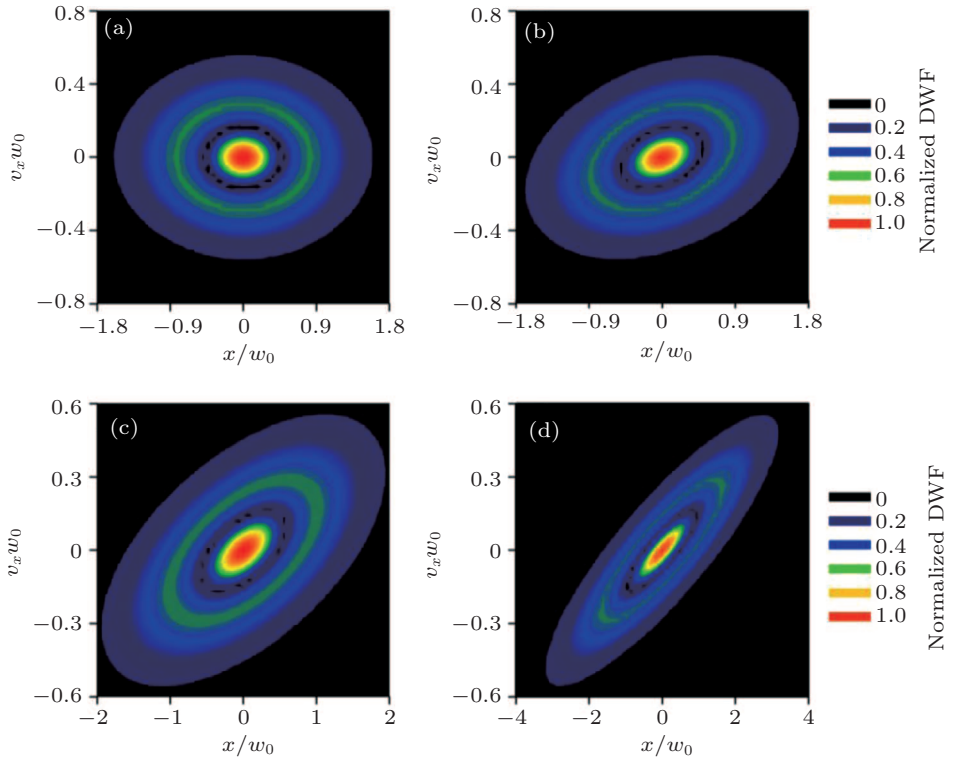


Fig. 3. (color online) Contours graph of the normalized WDF in the x direction of an SLG_{11} beam in different observation planes in free space. $e_x = 3$. (a) $z = 0$, (b) $z = w_0^2/\lambda$, (c) $z = 2w_0^2/\lambda$, (d) $z = 5w_0^2/\lambda$.

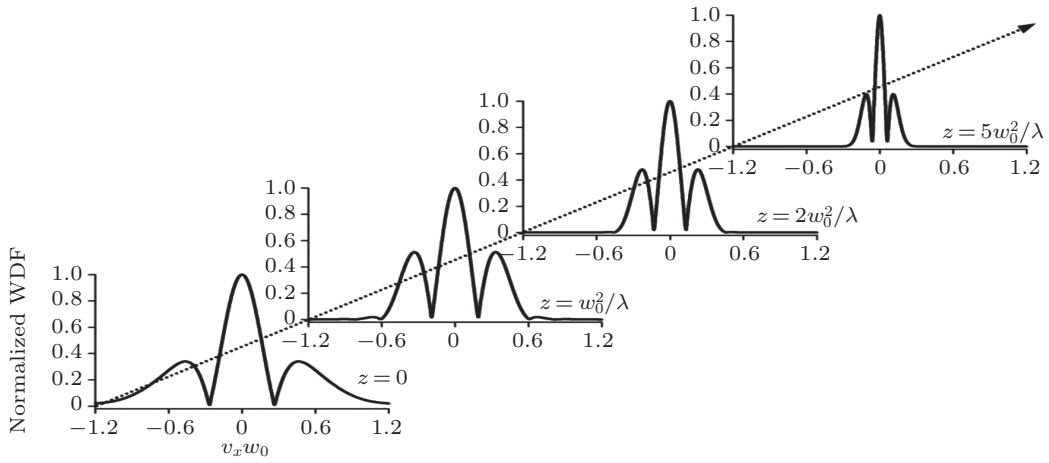


Fig. 4. Normalized WDFs in the x direction of an SLG_{11} beam at the point $x = 0$ of different observation planes in free space. $e_x = 0.5$.

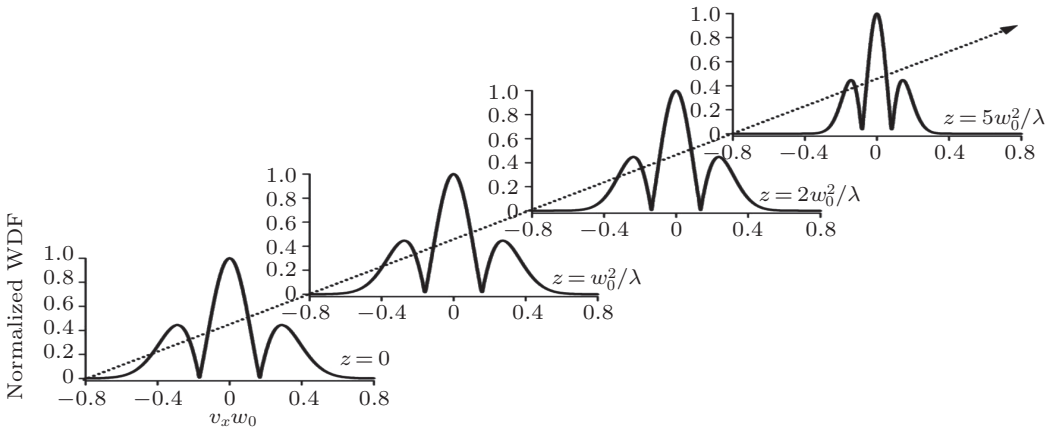


Fig. 5. Normalized WDFs in the x direction of an SLG_{11} beam at the point $x = 0$ of different observation planes in free space. $e_x = 3$.

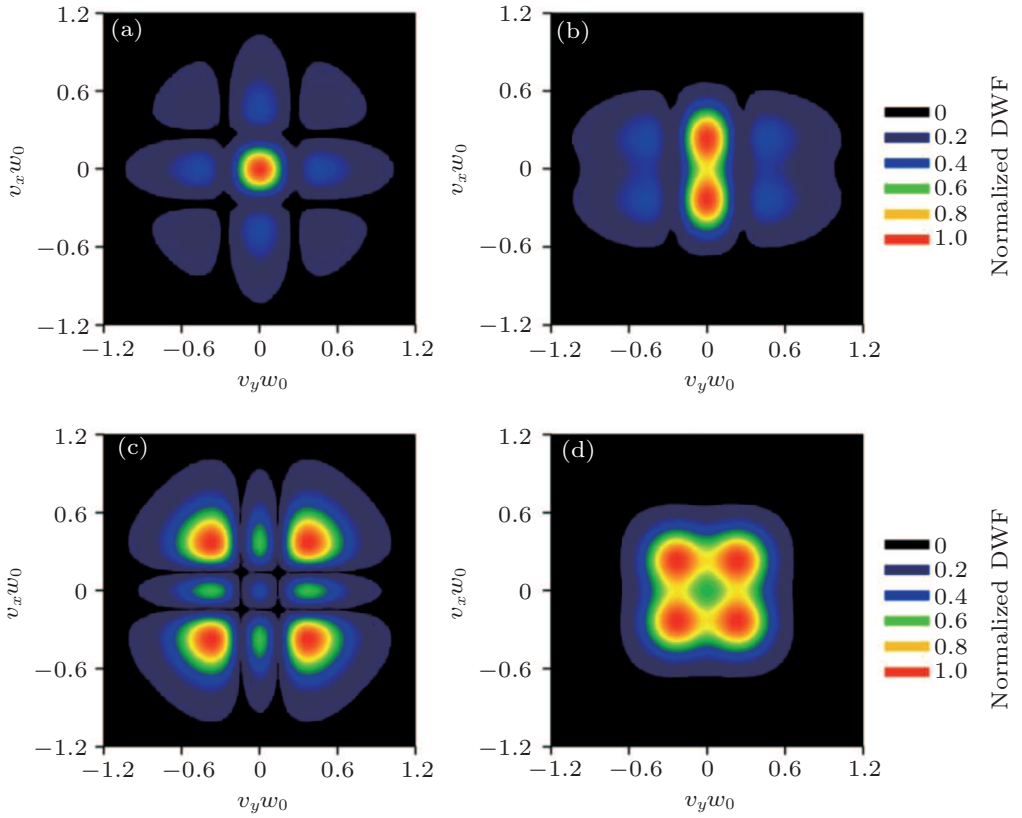


Fig. 6. (color online) Contours of the normalized WDF of an SLG_{11} beam at several different points in the source plane. $e_x = e_y = 0.5$. (a) $x = y = 0$, (b) $x = 0$ and $y = 0.5w_0$, (c) $x = y = 0.25w_0$, (d) $x = y = -0.5w_0$.

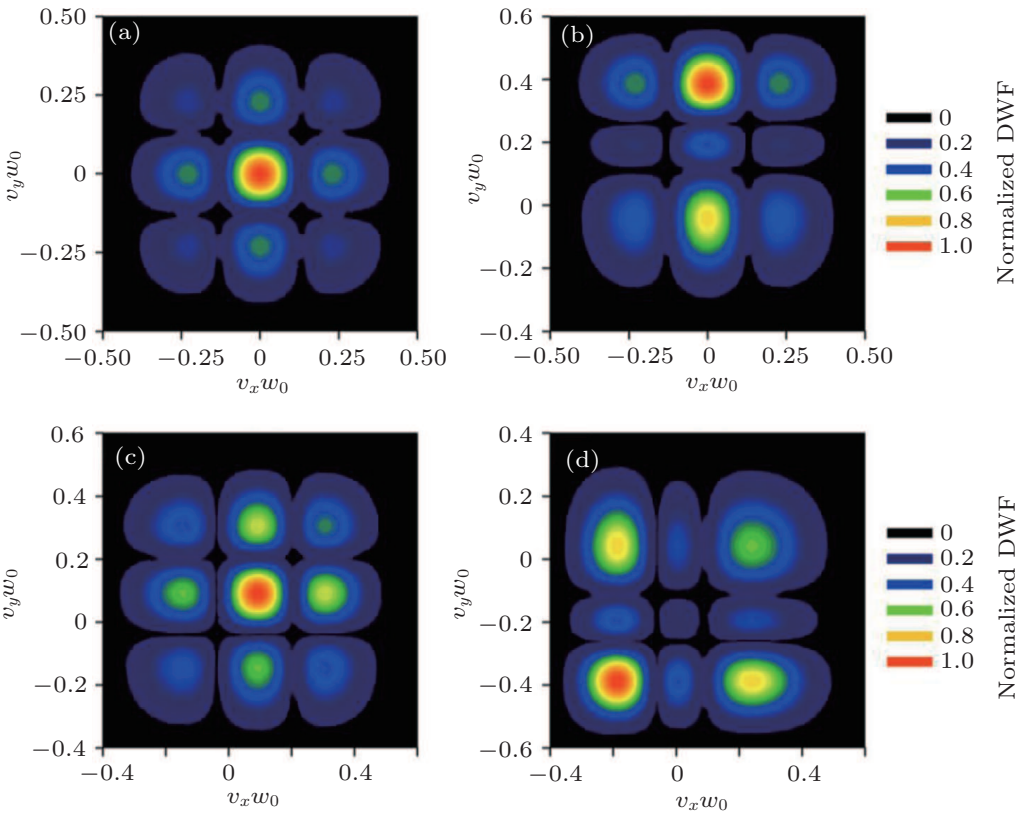


Fig. 7. (color online) Contours of the normalized WDF of an SLG_{11} beam at the several points in the observation plane of $z = 2w_0^2/\lambda$. $e_x = e_y = 0.5$. (a) $x = y = 0$, (b) $x = 0$ and $y = 0.5w_0$, (c) $x = y = 0.25w_0$, (d) $x = y = -0.5w_0$.

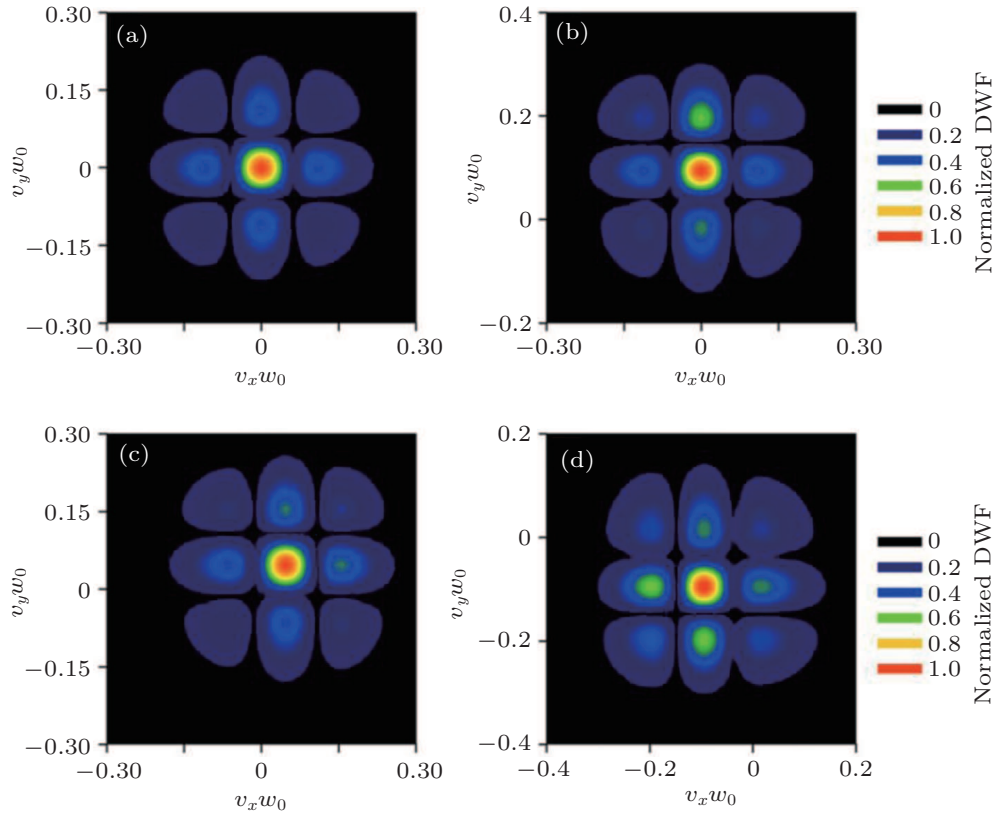


Fig. 8. (color online) Contours of the normalized WDF of an SLG_{11} beam at the several points in the observation plane of $z = 5w_0^2/\lambda$. $e_x = e_y = 0.5$. (a) $x = y = 0$, (b) $x = 0$ and $y = 0.5w_0$, (c) $x = y = 0.25w_0$, (d) $x = y = -0.5w_0$.

The normalized WDFs of an SLG_{11} beam as a function of v_x and v_y at several different positions (x, y) are depicted in Figs. 6–8. The reference planes in Figs. 6–8 are at $z = 0$, $z = 2w_0^2/\lambda$, and $z = 5w_0^2/\lambda$, respectively. The WDF of an SLG_{11} beam at the origin (0, 0) of the source plane is composed of a central dominant lobe and 8 side lobes. The WDF of an SLG_{11} beam at the point $(0, 0.5w_0)$ of the source plane is composed of a dominant lobe and 2 side lobes. Moreover, each lobe is the superposition of two lobes. The WDF of an SLG_{11} beam at the point $(0.25w_0, 0.25w_0)$ of the source plane is composed of 4 dominant lobes, 4 subdominant lobes, and a central weakest lobe. The orientations of 4 dominant lobes are 45°, 135°, 225°, and 315° with respect to the x axis, respectively, and 4 subdominant lobes are located at the x and y axes. The WDF of an SLG_{11} beam at the point $(-0.5w_0, -0.5w_0)$ of the source plane is only one lobe, which looks like the superposition of 4 lobes. These 4 points are representative points. Besides the shape of the side lobe, the WDF of an SLG_{11} beam at the origin (0, 0) of the plane $z = 2w_0^2/\lambda$ is similar to that of the source plane. The WDFs of an SLG_{11} beam at other points of the plane $z = 2w_0^2/\lambda$ are still composed of a dominant lobe and 8 side lobes. However, the dominant lobe is not located at the point (0, 0). The position of the dominant lobe depends on the values of x and y. If x or y is positive, the dominant lobe moves towards the positive direction of the corresponding spatial frequency variable. Otherwise, the opposite is happening. The displacement that the dominant lobe moves stops with the values of x and y increasing. The shapes of the side lobes vary with the position, i.e., the values of x and y, changing. When the observation plane is far enough from the source plane, the WDF of an SLG_{11} beam is nearly stable, which means that the side lobes change little with values of x and y varying.

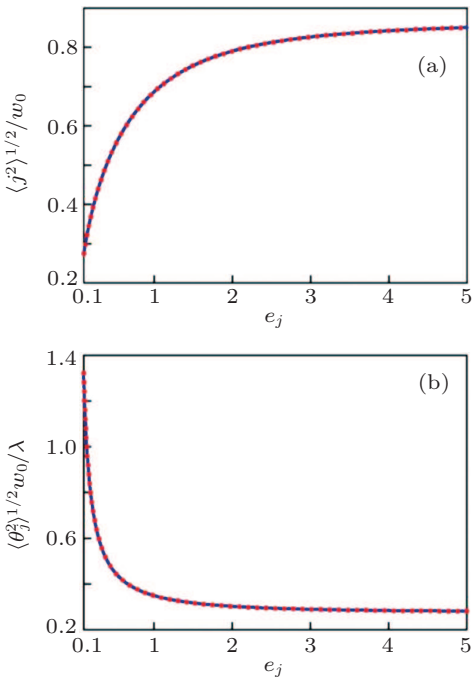


Fig. 9. (color online) Beam waist (a) and the divergence angle (b) in the j direction of an SLG_{11} beam, each as a function of e_j.

Figure 9 shows the beam waist and the divergence angle in the j direction of an SLG_{11} beam propagating in free space. The solid curve denotes the result obtained in this paper, and the dotted curve means the result given in Ref. [11]. They are identical, which confirms that our derivation is correct. The increase of e_j means that the beam width of the super-Lorentzian part in the j direction augments. With e_j increasing, therefore, the beam waist increases and the divergence angle decreases. The second-order moments of the WDF of an SLG_{11} beam in the reference plane other than the source plane can be calculated by Eq. (34).

4. Conclusions

Based on the Collins integral formula and the expansion of a Lorentz distribution, an analytical expression for the WDF of an SLG_{11} beam through a paraxial *ABCD* optical system is derived. The properties of the WDF of an SLG_{11} beam propagating in free space are demonstrated. The WDF of an SLG_{11} beam propagating in free space depends on w_0, e_x, e_y, and the spatial position (x, y, z). The influences of the beam parameter on the WDF of an SLG_{11} beam in free space are also analyzed at different propagation distances. Upon propagation in free space, the pattern of the WDF of an SLG_{11} beam twists clockwise. Moreover, the pattern of the WDF shrinks in the direction of the spatial frequency variable and elongates in the direction of the transversal spatial coordinate. As e_x decreases, the shrinkage of the WDF increases in the direction of the spatial frequency variable. The patterns of the WDF of an SLG_{11} beam at the different points of the source plane are different. When the observation plane is far from the source plane, the patterns of the WDF of an SLG_{11} beam at the different points become similar. Of course, there exists the difference in the pattern. The position of the dominant lobe is different and depends on the values of x and y. Also, the different points result in slight difference between the patterns of the side lobes. The second-order moments of the WDF of an SLG_{11} beam propagating in free space obtained here are consistent with the previously reported results, which implies that the derivation in this paper is correct. This research reveals the propagation properties of an SLG_{11} beam from another perspective, which is useful to comprehensively master the propagation properties of SLG beams. Also, this research is beneficial to the practical applications involving in the higher-order mode diode laser.

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