Trial function method and exact solutions to the generalized nonlinear Schrödinger equation with time-dependent coefficient*

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In this paper, the trial function method is extended to study the generalized nonlinear Schrödinger equation with time-dependent coefficients. On the basis of a generalized traveling wave transformation and a trial function, we investigate the exact envelope traveling wave solutions of the generalized nonlinear Schrödinger equation with time-dependent coefficients. Taking advantage of solutions to trial function, we successfully obtain exact solutions for the generalized nonlinear Schrödinger equation with time-dependent coefficients under constraint conditions.

Keywords: generalized nonlinear Schrödinger equation, exact solution, trial function method

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1. Introduction

The construction of an exact solution for the nonlinear evolution equation (NLEE) is an important topic in the study of nonlinear physical phenomena. Exact solutions can help one to well understand the mechanism of the complicated physical phenomena and dynamical processes modeled by nonlinear evolution equations. The nonlinear Schrödinger equation as an important nonlinear model has been studied due to its importance in many branches of physics, such as nonlinear optics, plasma physics, superconductivity, and quantum mechanics.[1–3] With the development of soliton theory and computer algebraic system like Mathematica, much research has been devoted to exact solutions of nonlinear evolution equations, especially traveling wave solutions. Various effective methods of searching for exact solutions to NLEE have been presented. Özis and Yildirim[4] employed the He's modified Lindstedt–Poincare method to study the nonlinear Schrödinger (NLS) equation and obtained exact soliton solutions. Zhang *et al.*[5] investigated a generalized Vakhnenko equation via the extended Riccati mapping method and obtained novel single-loop-like and double-loop-like solitons. Other methods include the Bäcklund transformation,[6,7] Hirota bilinear method,[8] Li group method,[9] homogeneous balance method,[10] Darboux transformation,[11] the variable separation method,[12,13] the variational iteration method,[14] the extended tanh-function method,[15–17] the Jacobi elliptic function expansion method,[18–20] the auxiliary equation method,[21] the trial function method,[22] etc. Among them, the trial function method is one of the powerful methods of constructing exact solutions of NLEE.[23] In the present paper, we investigate some exact solutions for a generalized nonlinear Schrödinger (NLS) equation with a time-dependent coefficient using the trial function method.

During the past decade, the main attention has been paid to the constant coefficient NLS equation.[24–26] However, nonlinear evolution equations with variable coefficients are more realistic in physical phenomena than their constant coefficient counterparts. Solving variable coefficient NLEEs is much more difficult than their constant coefficient counterparts because of the existence of their coefficient function. Recently, much attention has been paid to variable coefficient NLS equations,[27–32] and many techniques and approaches have been extended to solve these physical models.

The generalized nonlinear Schrödinger equation in dimensionless form reads[33]

$$\mathrm{i}\psi_t + \gamma(t)\psi_{xx} + \alpha(t)\psi\,|\,\psi\,|^2 + \beta(t)x^2\psi = 0, \tag{1}$$

where $\psi = \psi(x,t)$ is the complex wave envelope in a comoving frame, $\gamma(t)$ is the time-modulated dispersion coefficient, $\alpha(t)$ is the nonlinearity, and $\beta(t)$ is the external potential coefficient, the subscripts x and t denote the spatial and temporal partial derivative, respectively. Zheng and Li[33] studied Eq. (1) by a direct self-similarity mapping approach. Generally speaking, equation (1) is difficult to solve because of the presence of the time-dependent dispersion, the nonlinear interaction parameter, and external potential, but it is very important in mathematical physics to find new methods to discover exact solutions of NLEEs. Under some special parameters of $\gamma(t)$, $\alpha(t)$, and $\beta(t)$, equation (1) can be converted into some Schrödinger-type equations. For $\gamma(t) = 1/2$, $\alpha(t) \neq 0$, $\beta(t) \neq 0$, Zhong *et al.*[34] established an integrability condition and discussed the propagation properties of periodic

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solitons. For $\gamma(t) = p, \alpha(t) = q, \beta(t) = 0$, equation (1) becomes the cubic Schrödinger equation with constant coefficients. Taghizadeh *et al.*[35] constructed exact solutions of this model by using the first integral method. In the present paper, we are concerned with the following generalized nonlinear Schrödinger equation with time-dependent coefficient

$$\mathrm{i}\psi_t + \gamma\psi_{xx} + \alpha(t)\psi \mid \psi \mid^2 + \beta x^2\psi = 0, \tag{2}$$

where β and γ are constants. It is worth noting that in the present work we make use of a mathematical technique different from those in previous work.[33–35] Furthermore, to our knowledge, there has been no report on the achievement of exact solutions to the generalized nonlinear Schrödinger (NLS) equation (2) using the trial function method.

The rest of the paper is organized as follows. In Section 2, we briefly describe the trial function method. In Section 3, we first make a transformation to reduce Eq. (2) into an ordinary differential equation, then we apply the trial function method to Eq. (2) and obtain exact solutions successfully under certain conditions. Finally, the concluding remarks are presented in Section 4.

2. Description of the trial function method

In the following, we describe the trial function method for a given nonlinear partial differential equation with a physical field $u(x,t)$ in two independent variables x,t with variable coefficients

$$F(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0, \tag{3}$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in u and its partial derivatives in which a nonlinear term is involved.

Step 1 Making use of the generalized wave transformation

$$u(x,t) = u(\xi), \quad \xi = p(t)x + q(t), \tag{4}$$

where $p(t)$ and $q(t)$ are differential functions to be determined. Then equation (3) is reduced into the following ordinary differential equation (ODE)

$$\begin{aligned} &Q(t, x, [p'(t)x + q'(t)]u'(\xi), p(t)u'(\xi), \\ &[p'(t)x + q'(t)]^2 u''(\xi), p(t)^2 u''(\xi), \ldots) = 0, \end{aligned} \tag{5}$$

where Q is a polynomial in u and its derivatives, while $'$ denotes the derivative with respect to the variable.

Step 2 Assuming the solution of Eq. (5) to satisfy the following trial equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^2 = G(u) = \sum_{i=0}^{n} a_i u^i(\xi), \tag{6}$$

where n and a_i are constants to be determined later.

Step 3 Substituting Eq. (6) into Eq. (5) and equating the coefficients of u to zero, we obtain an ordinary differential equation. Solving the ODE by Maple or Mathematica, we determine the values of a_i, $p(t)$, and $q(t)$.

Step 4 Equation (6) can be rewritten as

$$\pm(\xi - \xi_0) = \int \frac{1}{\sqrt{G(u)}}\,\mathrm{d}u. \tag{7}$$

According to the complete discrimination system of polynomial,[10] we solve the integral (7). Therefore exact solutions of Eq. (2) are obtained immediately.

3. Exact solutions to nonlinear Schrödinger equation (2)

Assume that the solution of Eq. (2) can be written in the form of

$$\begin{aligned} &\psi(x,t) = u(\xi)\exp(\mathrm{i}\zeta), \quad \xi = k(t)x + \omega(t), \\ &\zeta = p(t)x^2 + r(t)x + s(t), \end{aligned} \tag{8}$$

where $k(t)$, $\omega(t)$, $p(t)$, $r(t)$, and $s(t)$ are differential functions to be determined later. Substituting Eq. (8) into Eq. (2) and separating the real part from imaginary part, we obtain

$$(k'(t)x + \omega'(t))u' + 2\gamma k(t)(2p(t)x + r(t))u' = 0, \tag{9}$$

$$\begin{aligned} &\gamma k^2(t)u'' - [p'(t)x^2 + r'(t)x + s'(t) \\ &+ \gamma(2p(t)x + r(t))^2 - \beta x^2]u + \alpha(t)u^3 = 0. \end{aligned} \tag{10}$$

Considering the homogenous balance in Eq. (10) yields that $n = 4$. Therefore we suppose that the exact solution of Eq. (2) satisfies the following equation

$$\begin{aligned} G(u) \;=\; &[u'(\xi)]^2 = a_0 + a_1 u(\xi) + a_2 u^2(\xi) + a_3 u^3(\xi) \\ &+ a_4 u^4(\xi). \end{aligned} \tag{11}$$

Substituting Eq. (11) into Eq. (10) and equating all coefficients of $x^j u^i$ to zero, we obtain

$$\begin{aligned} &k'(t) + 4\gamma k(t)p(t) = 0, \\ &w'(t) + 2\gamma k(t)r(t) = 0, \\ &p'(t) + 4\gamma p^2(t) - \beta = 0, \\ &r'(t) + 4\gamma p(t)r(t) = 0, \\ &a_1 = 0, \quad a_3 = 0, \\ &\gamma k^2(t)a_2 - (s'(t) + \gamma r^2(t)) = 0, \\ &2\gamma k^2(t)a_4 + \alpha(t) = 0. \end{aligned} \tag{12}$$

After applying some careful and direct algebraic operations to the above equations, we obtain

$$\begin{aligned} &p(t) = p, \quad \beta = 4p^2\gamma, \quad a_1 = 0, \quad a_3 = 0, \\ &k(t) = r(t) = \exp(-4p\gamma t), \\ &\omega(t) = \frac{1}{4p}\exp(-8p\gamma t), \end{aligned}$$

$$a_2 = c_1 + 1, \quad a_4 = c_2, \quad a_0 = a_0,$$
$$\alpha(t) = -2c_2\gamma\exp(-8p\gamma t),$$
$$s(t) = -\frac{c_1 - 1}{8p}\exp(-8p\gamma t), \tag{13}$$

where a_0, c_1, and c_2 are arbitrary constants. Making use of the above results, equation (11) can be transformed into the following integral

$$\pm(8c_2)^{1/3}(\xi - \xi_0) = \int \frac{1}{\sqrt{\omega(\omega^2 + b_1\omega + b_0)}}\mathrm{d}\omega, \tag{14}$$

where $b_1 = (c_1 + 1)c_2^{-2/3}$ and $b_0 = a_0 c_2^{-1/3}$.

Taking advantage of the complete discrimination system of the polynomial,[22] we obtain a solution to Eq. (14). Meanwhile we derive the exact solutions for Eq. (2) with the help of Eqs. (8) and (13), which are listed as follows.

Family 1 $\Delta = b_1^2 - 4b_0 > 0$, $b_0 \neq 0$, $G(\omega) = (\omega - \alpha)(\omega - \beta)(\omega - \gamma)$ we suppose that $\alpha < \beta < \gamma$. When $\alpha < \omega < \beta$, we obtain the following elliptic-type solution

$$u = \pm c_2^{-1/6}[\alpha + (\beta - \alpha)\,\mathrm{sn}^2(\sqrt{\gamma - \alpha}(c_2)^{1/3} \times (\xi - \xi_0), m)]^{1/2}\exp(\mathrm{i}\zeta), \tag{15}$$

when $\omega > \gamma$, we obtain

$$u = \pm c_2^{-1/6}\frac{[\gamma - \beta\,\mathrm{sn}^2(\sqrt{\gamma - \alpha}(c_2)^{1/3}(\xi - \xi_0), m)]^{1/2}}{\mathrm{cn}(\sqrt{\gamma - \alpha}(c_2)^{1/3}(\xi - \xi_0), m)} \times \exp(\mathrm{i}\zeta), \tag{16}$$

where $m^2 = (\beta - \alpha)/(\gamma - \alpha)$.

Family 2 $\Delta = b_1^2 - 4b_0 > 0$, $b_0 = 0$, when $\omega > -b_1$, if $c_1 + 1 > 0$ and $c_2 > 0$, we obtain

$$u = \pm\sqrt{(c_1 + 1)(c_2)^{-2/3}}\left[\frac{1}{2}\tanh^2\left(\sqrt{\frac{(c_1 + 1)(c_2)^{-1/3}}{2}}(\xi - \xi_0)\right) - 1\right]^{1/2}\exp(\mathrm{i}\zeta), \tag{17}$$

$$u = \pm\sqrt{(c_1 + 1)(c_2)^{-2/3}}\left[\frac{1}{2}\coth^2\left(\sqrt{\frac{(c_1 + 1)(c_2)^{-1/3}}{2}}(\xi - \xi_0)\right) - 1\right]^{1/2}\exp(\mathrm{i}\zeta). \tag{18}$$

If $c_1 + 1 < 0$ and $c_2 > 0$, we obtain

$$u = \pm\sqrt{-(c_1 + 1)(c_2)^{-2/3}}\left[\frac{1}{2}\tan^2\left(\sqrt{-\frac{(c_1 + 1)(c_2)^{-1/3}}{2}}(\xi - \xi_0)\right) - 1\right]^{1/2}\exp(\mathrm{i}\zeta). \tag{19}$$

Family 3 $\Delta = b_1^2 - 4b_0 < 0$, we obtain

$$u = \pm(a_0/c_2)^{1/4} \times \left[\frac{1 - \mathrm{cn}(2(a_0c_2)^{1/4}(\xi - \xi_0), m)}{1 + \mathrm{cn}(2(a_0c_2)^{1/4}(\xi - \xi_0), m)}\right]^{1/2}\exp(\mathrm{i}\zeta), \tag{20}$$

where

$$m^2 = \frac{1}{2}\left(1 - \frac{c_1 + 1}{2\sqrt{a_0}}\right).$$

Family 4 $\Delta = b_1^2 - 4b_0 = 0$, when $\omega > 0, G(\omega) = \omega(\omega - \alpha)^2$, $\alpha \neq 0$. If $c_1 + 1 < 0$ and $c_2 > 0$, we obtain

$$u = \pm\sqrt{-\frac{(c_1 + 1)(c_2)^{-2/3}}{2}} \times \tanh\left(\sqrt{-\frac{(c_1 + 1)(c_2)^{-1/3}}{2}}(\xi - \xi_0)\right)\exp(\mathrm{i}\zeta), \tag{21}$$

$$u = \pm\sqrt{-\frac{(c_1 + 1)(c_2)^{-2/3}}{2}} \times \coth\left(\sqrt{-\frac{(c_1 + 1)(c_2)^{-1/3}}{2}}(\xi - \xi_0)\right)\exp(\mathrm{i}\zeta). \tag{22}$$

If $c_1 + 1 > 0$ and $c_2 > 0$, we obtain

$$u = \pm\sqrt{\frac{(c_1 + 1)(c_2)^{-2/3}}{2}} \times \tan\left(\sqrt{\frac{(c_1 + 1)(c_2)^{-1/3}}{2}}(\xi - \xi_0)\right)\exp(\mathrm{i}\zeta). \tag{23}$$

4. Discussion and conclusion

In the present paper, we investigated the generalized nonlinear Schrödinger equation with time-dependent coefficients based on a trial function and a generalized traveling wave transformation. As a result, we obtained exact solutions of the generalized nonlinear Schrödinger equation under constraint conditions. We believe that the work here is valuable and may be helpful for studying in nonlinear science. The nonlinear Schrödinger equation is worthy to be further studied because of its importance in nonlinear science. Finally, it is worth while to note that the trial function method can be applied to other variable coefficient NLEEs in mathematical physics.

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