Robust modified projective synchronization of fractional-order chaotic systems with parameters perturbation and external disturbance*

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Based on fractional-order Lyapunov stability theory, this paper provides a novel method to achieve robust modified projective synchronization of two uncertain fractional-order chaotic systems with external disturbance. Simulation of the fractional-order Lorenz chaotic system and fractional-order Chen's chaotic system with both parameters uncertainty and external disturbance show the applicability and the efficiency of the proposed scheme.

Keywords: fractional-order chaotic system, modified projective synchronization, uncertainty, disturbance

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1. Introduction

The theory of fractional-order derivatives, which was mainly developed in the 19th century, is a generalization of integration and differentiation to non-integer (fractional) order. Nowadays, it has been known that many systems in interdisciplinary fields, such as viscoelastic materials,[1] electrode-electrolyte polarization,[2] thermoelectric systems,[3] finance systems,[4] dielectric polarization,[5] etc., can be elegantly described by fractional differential derivatives. However, the application of fractional-order calculus to dynamic control is only a recent focus of interest.[6,7] Moreover, many studies have demonstrated that many fractional-order differential systems, such as the fractional-order Lorenz system,[8] fractional-order Duffling system,[9] fractional-order Chua's system,[10] and fractional-order Chen's system[11] exhibit chaotic behaviors.

Since Pecora and Carroll established a chaos synchronization scheme for two identical chaotic systems with different initial conditions,[12] chaos synchronization has attracted a great deal of interests[13–16] and it has a wide range of potential application, such as image encryption,[17,18] secure communication,[19,20] genetic networks,[21] and so on.[22,23] Recently, research on complex networks has attracted increasing attention because of it is extensive application.[24,25]The fractional-order chaotic system can be considered as a kind of complex network[26] that can describe many practical systems more accurately, and the research has been explored to the networked fractional-order chaotic systems[27] that has also attracted the attention of many scholars.[28–32] In practical situations, parameter uncertainty and external disturbance for fractional-order chaotic systems widely exist in fractional-order chaotic systems. Based on fractional-order stability theory, Xu[33] presented a novel adaptive controller and parameter updating rule to synchronize the fractional-order unified chaotic system with uncertain parameters. Zhou *et al.*[34] designed an adaptive controller to synchronize two entirely different fractional-order chaotic systems with uncertain parameters. Yang,[35] associated with system parameter estimating laws, proposed a method of single state proportional feedback to synchronize two nearly identical generalized Lorenz systems. Zhang *et al.*,[36] based on Lyapunov stability theory and a fractional-order differential inequality, developed a modified adaptive control scheme and adaptive laws of parameters to robustly synchronize coupled fractional-order chaotic systems with unknown parameters and uncertain perturbations. Xiang *et al.*[37] investigated the robust synchronization problem for a class of systems with external disturbances. However, the effects of both parameters perturbation and external disturbance of the fractional-order chaotic systems have seldom been considered in those approaches.

More recently, Li[38] put forward a new generalized synchronization method, called modified projective synchronization (MPS), where the responses of the synchronized dynamical states synchronize up to a constant scaling matrix. MPS is a more general definition than projective synchronization (PS) and could be used to make secure communications more secure in application, because it is obvious that the unpredictability of the scaling matrix in MPS can additionally enhance the security of communication.

In this paper, the problem of fractional-order chaos MPS to Lorenz system and Chen's system with parameter uncer-

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tainties and external disturbances is considered. For MPS of the systems, a novel robust control scheme has been proposed. Then, the fractional-order MPS of the systems can be guaranteed by the fractional-order Lyapunov theory.

The rest of this paper is organized as follows. In Section 2, the MPS law for fractional-order chaotic systems with uncertainty and external disturbance using fractional-order Lyapunov theory is designed. Numerical simulation results are given in Section 3 to illustrate the effectiveness of the proposed controller. Conclusions in Section 4 close the paper.

2. Modified projective synchronization of fractional-order chaotic systems with uncertainty and external disturbance

2.1. Fractional derivatives

Consider a fractional system,

$$ {}_{t_0}D_t^{\alpha} x(t) = f(x,\beta) \tag{1}$$

with initial condition $x(t_0)$, where ${}_{t_0}D_t^{\alpha}$ denotes the Caputo fractional operator, $\alpha \in (0,1)$. $f : [t_0,\infty] \times \Omega \to R^n$ is piecewise continuous in x and locally Lipschitz in x on $[t_0,\infty] \times \Omega$ ($\Omega \in R^n$ is a domain that contains the origin $x = 0$). β is the unknown parameter vectors of the system and $\beta \in R^p$.

Definition 1 The constant x_0 is an equilibrium point of the fractional dynamic system (1), if and only if $f(t,x_0) = {}_{t_0}D_t^{\alpha} x_0$. Without loss of generality, let the equilibrium point be $x_0 = 0$.

In this paper, we use Caputo fractional order operators as our main tool, which are shown in Podlubny's work.[6] The uniform formula of a fractional integral with $\alpha \in (0,1)$ is defined as

$$ {}_{t_0}I_t^{\alpha} x(t) = {}_{t_0}D_t^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{x(\tau)}{(t-\tau)^{1-\alpha}} \,\mathrm{d}\tau, \tag{2}$$

where $x(t)$ is an arbitrary integrable function, ${}_{t_0}I_t^{\alpha}$ is the fractional integral of order α on $[t_0,t]$, and $\Gamma(\cdot)$ denotes the Gamma function. For a arbitrary real number m, the Caputo fractional derivative is defined as

$$ {}_{t_0}D_t^{m} x(t) = \frac{1}{\Gamma(\lfloor m \rfloor + 1 - m)} \int_{t_0}^{t} \frac{x^{(\lfloor m \rfloor + 1)}(\tau)}{(t-\tau)^{m-\lfloor m \rfloor}} \,\mathrm{d}\tau, \tag{3}$$

where $\lfloor m \rfloor$ stands for the integer part of m, ${}_{t_0}D_t^{\alpha}$ denotes the Caputo fractional derivatives.

2.2. Asymptotic stability of fractional-order system

In nonlinear systems, Lyapunov's direct method (also called the Lyapunov's second method) provides a way to analyze the stability of a system without explicitly solving the differential equations. Although the Lyapunov stability theory for integer-order systems was proposed in 1892 and it has been studyied and modifyied by lots of expert researchers, the Lyapunov stability theory for fractional-order systems has been developed until recently.[39]

Definition 2 A continuous function g: $[0,t) \to [0,\infty)$ is said to belong to class-k if it is strictly increasing and $g(0) = 0$.

Theorem 1[39] Let $x = 0$ be an equilibrium point for the nonautonomous fractional order system (1). Assume that there exist a Lyapunov function $V(t,x(t))$ and class-k functions α_i $(i = 1,2,3)$ satisfying

$$ g_1(\|x\|) \le V(t,x) \le g_2(\|x\|), \tag{4}$$

$$ {}_0D_t^{\varphi} V(t,x(t)) \le -g_3(\|x\|), \tag{5}$$

where $\varphi \in (0,1)$. Then the equilibrium point of system (1) is asymptotically stable.

2.3. Design of modified projective synchronization scheme

Considering the following two dynamic systems: driving system

$$ {}_{t_0}D_t^{\alpha} x(t) = f(x,\beta) + d_1, \tag{6}$$

and response system

$$ {}_{t_0}D_t^{\alpha} y(t) = h(y,\gamma) + d_2 + u, \tag{7}$$

where $x, y \in R^n$ are the state vectors of the driving system and response system respectively; f and h are both $n \times 1$ bounded continuous functional matrices; d_1, $d_2 \in R^n$ are both unknown bounded external disturbance matrices; $\beta \in R^p$ and $\gamma \in R^q$ are the unknown parameter vectors of the driving and response systems respectively; $u \in R^n$ is a control input vector.

Assume that $\tilde{\beta}$, $\tilde{\gamma}$ are the rational pre-estimated valuse of β, γ respectively, then the approximate model $f(x,\tilde{\beta})$ and $h(y,\tilde{\gamma})$ will be used to design the MPS control law. The error between the pre-estimated models and the real systems are as follows respectively

$$ \Delta f(x,\beta,\tilde{\beta}) = f(x,\beta) - f(x,\tilde{\beta}), \tag{8}$$

$$ \Delta h(y,\gamma,\tilde{\gamma}) = h(y,\gamma) - h(y,\tilde{\gamma}). \tag{9}$$

The MPS error ($e = y - Jx$) dynamic system can be written as

$$ {}_{t_0}D_t^{\alpha} e(t) = h(y,\gamma) - Jf(x,\beta) + u + d_2 - Jd_1, \tag{10}$$

where $J = \mathrm{diag}(J_1,J_2,\ldots,J_n)$ is projective synchronization matrix and J_i $(i = 1,2,\ldots,n)$ are non-zero constant.

Our goal is to design a control vector u which should be robust enough such that the trajectory of the response system (7) with initial conditions y_0 and pre-estimated parameter $\tilde{\gamma}$ under unknown external disturbance d_2 can asymptotically approach the trajectory of the driving system (6) with initial conditions x_0 and the pre-estimated parameter $\tilde{\beta}$ as well as unknown external disturbance d_1. Finally the MPS can be implemented in the sense that $\lim_{t\to\infty} \|e(t)\| =$

$\lim_{t\to\infty}\|y(t,y_0,d_2)-Jx(t,x_0,d_1)\|$, where $\|\cdot\|$ is the Euclidean norm.

Remark 1 If $J=\sigma I, \sigma\in R$, the MPS problem will be reduced to projective synchronization (PS), where I is an $n\times n$ identity matrix with proper dimension. In particular, if $\sigma=1$ and $\sigma=-1$, the problem is further simplified to complete synchronization and anti-phase synchronization, respectively.

Remark 2 The approximate system model can be any form of model representation. For instance, the differential equation model of continuous time system as shown in Eqs. (8) and (9). And it could be also the nonparameter model or constructed intelligent model such as the neural network model and fuzzy model which are popularly used nowadays.[40]

Lemma 1 For a continuous function $f(x)=x^{\mathrm{T}}Ax$, where, $x\in R^{n\times 1}$, $A\in R^{n\times n}$ is a positive-definite matrix. Then

$$\lambda_{\min}(A)\|x\|^2\le f(x)\le\lambda_{\max}(A)\|x\|^2, \tag{11}$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are the maximum and minimum eigenvalues of matrix respectively.

Theorem 2 For systems (6) and (7), if the following control action u is used, then the systems can realize asymptotical MPS. That is to say, $\lim_{t\to\infty}e(t)=0$,

$$u=Jf(x,\tilde{\beta})-h(y,\tilde{\gamma})-Ke-K_0\frac{e}{\|e\|}, \tag{12}$$

where, $K=\mathrm{diag}\{k_1,k_2,\ldots,k_n\}$, $k_i>0$, $i=1,2,\ldots,n$, and

$$\begin{aligned}K_0 &= \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\eta_0,\eta_0\\ &= \sup_{x,y,\beta,\tilde{\beta},\gamma,\tilde{\gamma},d_1,d_2}\left\|\Delta h(y,\gamma,\tilde{\gamma})-J\Delta f(x,\beta,\tilde{\beta})+d_2-Jd_1\right\|,\end{aligned} \tag{13}$$

in which, $\mathrm{diag}(\cdot)$ is a diagonal matrix; P is the positive-definite solution of the following Riccati equation

$$K^{\mathrm{T}}P+PK=Q, \tag{14}$$

in which, Q is a positive-definite matrix and usually can be chosen as identity matrix I.

Proof We choose the candidate Lyapunov function V as $V=e^{\mathrm{T}}Pe/2$. It follows from Lemma 1 that

$$\lambda_{\min}(P)\|e\|^2\le V\le\lambda_{\max}(P)\|e\|^2. \tag{15}$$

By using the control action (12) and Eq. (13), the time derivative of V along trajectories of error dynamic (10) is

$$\begin{aligned}{}_0D_t^{\alpha}V &= \frac{1}{2}\left(\left(\frac{\mathrm{d}^{\alpha}e}{\mathrm{d}t^{\alpha}}\right)^{\mathrm{T}}Pe+e^{\mathrm{T}}P\left(\frac{\mathrm{d}^{\alpha}e}{\mathrm{d}t^{\alpha}}\right)\right)\\ &= \frac{1}{2}\left(\Delta h-J\Delta f+d_2-Jd_1-Ke-K_0\frac{e}{\|e\|}\right)^{\mathrm{T}}Pe\\ &\quad+\frac{1}{2}e^{\mathrm{T}}P\left(\Delta h-J\Delta f+d_2-Jd_1-Ke-K_0\frac{e}{\|e\|}\right)\\ &= -\frac{1}{2}e^{\mathrm{T}}Qe+(\Delta h-J\Delta f+d_2-Jd_1)^{\mathrm{T}}Pe-K_0\frac{e^{\mathrm{T}}}{\|e\|}Pe.\end{aligned}$$

During the above deduction, the following equation is used

$$\begin{aligned}&(\Delta h-J\Delta f+d_2-Jd_1)^{\mathrm{T}}Pe\\ &= e^{\mathrm{T}}P(\Delta h-J\Delta f+d_2-Jd_1).\end{aligned}$$

Due to

$$\begin{aligned}&(\Delta h-J(x)\Delta f+d_2-J(x)d_1)^{\mathrm{T}}Pe\\ &\le \eta_0\cdot\|P\|\cdot\|e\|\\ &\le \eta_0\cdot\lambda_{\max}(P)\cdot\|e\|,\end{aligned}$$

we have

$$\begin{aligned}{}_0D_t^{\alpha}V &\le -\frac{1}{2}e^{\mathrm{T}}Qe+\eta_0\cdot\lambda_{\max}(P)\cdot\|e\|\\ &\quad-K_0\lambda_{\min}(P)\|e\|.\end{aligned}$$

According to Eq. (13) and notice that Q is a positive-definite matrix, we obtain

$${}_0D_t^{\alpha}V\le-\frac{1}{2}e^{\mathrm{T}}Qe. \tag{16}$$

It follows from Eqs. (15), (16), and Theorem 1 that the robust MPS is realized and the error dynamic system is globally and asymptotically stable.

Remark 3 The MPS tracking speed mainly depends on the matrix K. $-K$ is similar to the poles in closed loop control systems. So large k_i $(i=1,2,\ldots,n)$ can improve MPS performance. But in practice, too large k_i $(i=1,2,\ldots,n)$ are harmful to the control devices. So K should be properly chosen. In despite of those aspects, a wide range of k_i $(i=1,2,\ldots,n)$ can work well.

Remark 4 The robustness to the uncertainty and disturbance mainly depends on the matrix K_0, and K_0 is dependent on K and the bounds of the uncertainty and disturbance. So K_0 is usually estimated to be close to the maximum bounds of the uncertainty and disturbance.

Remark 5 Though the control law (12) can guarantee the asymptotical stabilization of the error system, the control algorithm is not continuous in the origin of error plane. So during the implementation, $e/\|e\|$ is replaced by $e/[\|e\|+b\exp(-at)]$ which can guarantee chattering free and still keep asymptotical MPS, and this can be easily proved by using the similar method as demonstrated above. Then the following control law is used instead of Eq. (12)

$$\begin{aligned}u &= Jf(x,\tilde{\beta})-h(y,\tilde{\gamma})-Ke\\ &\quad-K_0\frac{e}{\|e\|+b\exp(-at)},\end{aligned} \tag{17}$$

where parameters a and b are adjustable positive parameters.

Remark 6 As good performance can be guaranteed with a wide range of a and b in practice. The parameters a and b in Eq. (17) can be easily determined. In this paper, $a=1$ and $b=0.1$ will be used in the following example.

3. Illustrative example

In this section, an illustrative example is presented to show the applicability and efficiency of the proposed synchronization method. The numerical simulations were carried out using MATLAB software.

The driving fractional-order Lorenz chaotic system[41,42] is described as follows:

$$ {}_{t_0}D_t^{\alpha}x(t) = \begin{bmatrix} \beta_1(x_2 - x_1) \\ \beta_2 x_1 - x_1 x_3 - x_2 \\ x_1 x_2 - \beta_3 x_3 \end{bmatrix} + d_1. \tag{18} $$

According to Eq. (6),

$$ f(x,\beta) = \begin{bmatrix} \beta_1(x_2 - x_1) \\ \beta_2 x_1 - x_1 x_3 - x_2 \\ x_1 x_2 - \beta_3 x_3 \end{bmatrix}, \quad d_1 = \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix}. \tag{19} $$

The response fractional-order Chen's chaotic system is described as follows:[43,44]

$$ {}_{t_0}D_t^{\alpha}y(t) = \begin{bmatrix} \gamma_1(y_2 - y_1) \\ (\gamma_2 - \gamma_1)y_1 - y_1 y_3 + \gamma_2 y_2 \\ y_1 y_2 - \gamma_3 y_3 \end{bmatrix} + d_2 + u. \tag{20} $$

According to Eq. (7),

$$ h(y,\gamma) = \begin{bmatrix} \gamma_1(y_2 - y_1) \\ (\gamma_2 - \gamma_1)y_1 - y_1 y_3 + \gamma_2 y_2 \\ y_1 y_2 - \gamma_3 y_3 \end{bmatrix}, \quad d_2 = \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix}. \tag{21} $$

When $\alpha = 0.993$, $\beta_1 = 10$, $\beta_2 = 28$, $\beta_3 = 8/3$, $\gamma_1 = 35$, $\gamma_2 = 28$, $\gamma_3 = 3$, both the fractional-order Lorenz system and fractional-order Chen's system are in chaos.[8] In the simulation, we stochastically choose $x_0 = [-20, -14, 30]^{\mathrm{T}}$ and $y_0 = [16, 15, -5]^{\mathrm{T}}$ as initial conditions. The control parameters for the control law (Eq. (17)) are chosen as follows: First, $K = \mathrm{diag}(10, 10, 10)$ is chosen to guarantee a good robust stability degree that can be determined in a wide range according to Remark 3; Second, calculating matrix P according to Eq. (14) and estimating the uncertain upper bound η_0 according to Eq. (13), and then parameter K_0 is chosen as $K_0 = 80$ that it can meet the condition of Theorem 2, with the projective synchronization matrix $J = I$ or $J = \mathrm{diag}(2, 3, 5)$ that will be used in Subsection 3.1 and Subsection 3.2 respectively.

Suppose that the pre-estimated model parameters are $\tilde{\beta}_i = 1.05\beta$, $\tilde{\gamma}_i = 0.95\gamma$ $(i = 1, 2, 3)$, and for simplicity but without loss of generality, we suppose the external disturbances to the driving system and the response system are $d_1 = [0, 0, 0]^{\mathrm{T}}$ and $d_2 = [1 + \sin(0.1t), 1 + \cos(0.01t), 1 + \cos(0.1t)]^{\mathrm{T}}$ respectively, i.e., there is not any external disturbance acting on the driving system, and the external disturbances on the response system are time-varying.

3.1. Simulation with $J = I$

If $J = I$, the problem is simplified to complete synchronization. Figure 1 shows the original chaotic curves of x_i and y_i $(i = 1, 2, 3)$. Figure 2 demonstrates the curves of x_i and y_i $(i = 1, 2, 3)$ and the error between them. It is clear that after a finite period of time, the error converges to be small enough when the control scheme is used. Figure 3 shows the trajectories of the fractional-order Lorenz and Chen's systems, from which we can see that the complete synchronization between the driving and response systems is achieved.

As noise exists widely in the real world, so the research of the noise effect on the synchronization process is necessary.[45] Figure 4 illustrates the synchronization phenomenon in noisy environment. The sub-figures 4(a)–4(c) in Fig. 4 are the noises that pollute on the states y_1, y_2, and y_3 of the response system. The sub-figures 4(d)–4(j) in Fig. 4 show the trajectories of the fractional-order Lorenz and Chen's systems, from which we can see that the synchronization between the driving and response systems is achieved successfully.

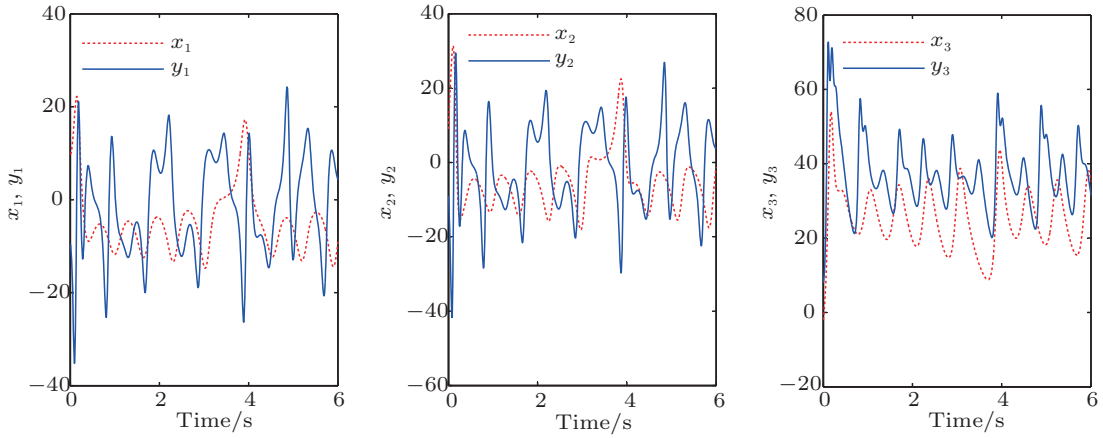


Fig. 1. (color online) Chaotic curves of the driving (x_1, x_2, x_3) and response (y_1, y_2, y_3) systems with uncertainty and disturbance.

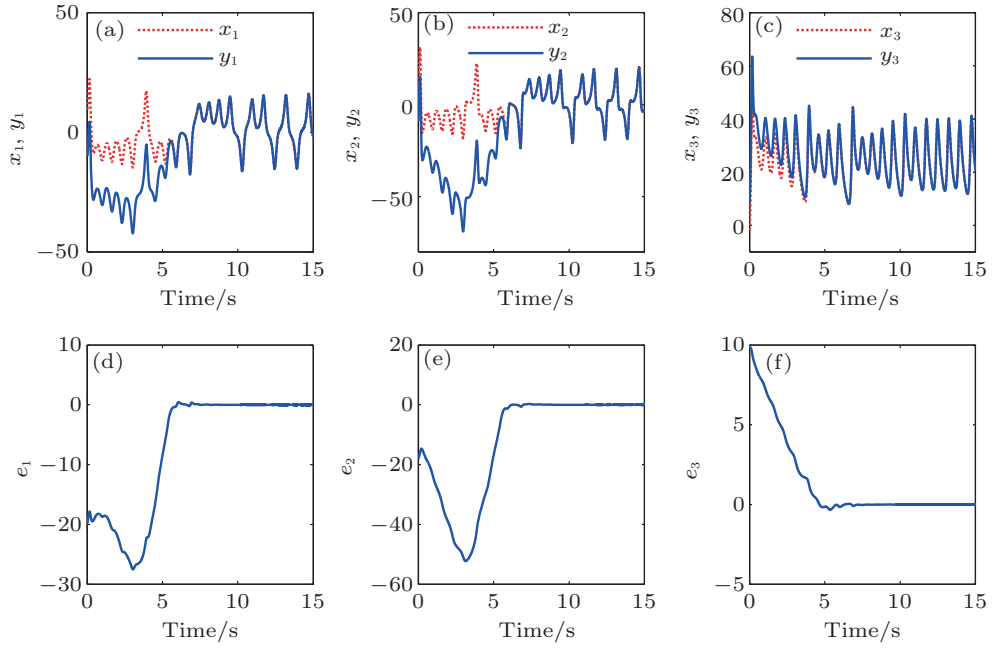


Fig. 2. (color online) Complete synchronization results of the driving and response systems with uncertainty and disturbance: (a) x_1, y_1; (b) x_2, y_2; (c) x_3, y_3; (d) e_1; (e) e_2; (f) e_3.

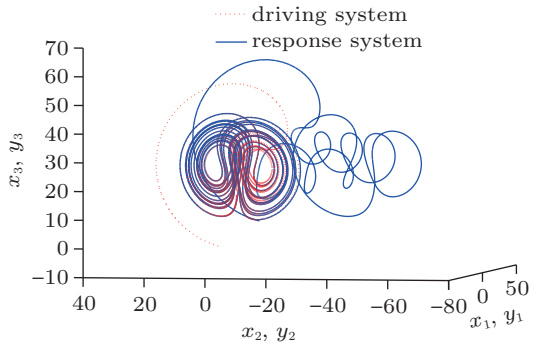
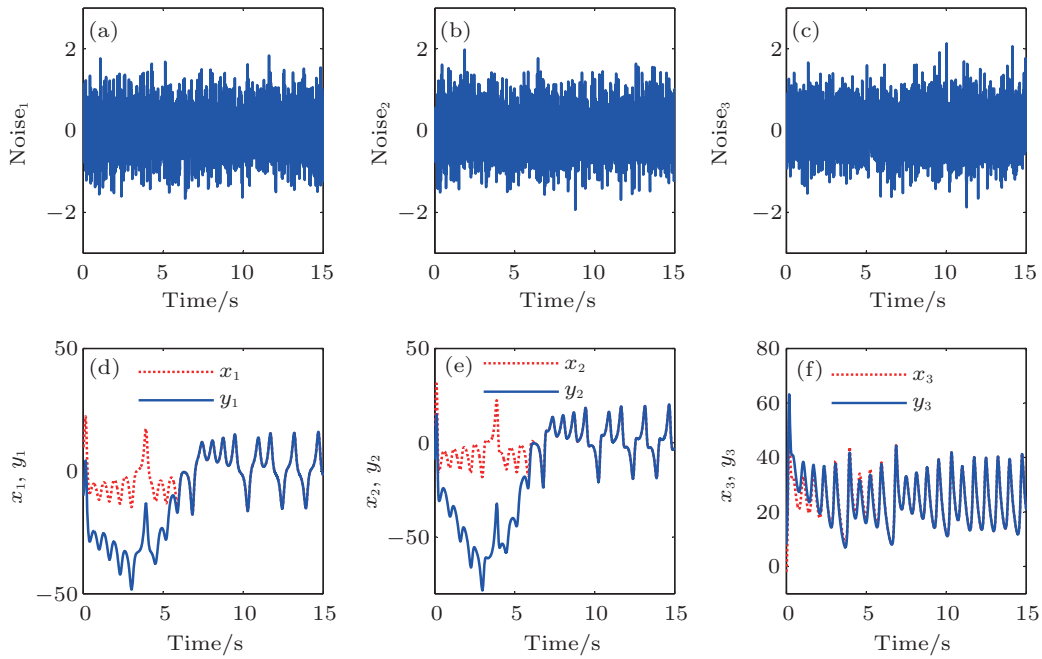


Fig. 3. (color online) Complete synchronization curves of the driving (x_t) and response (y_t) systems with uncertainty and disturbance.



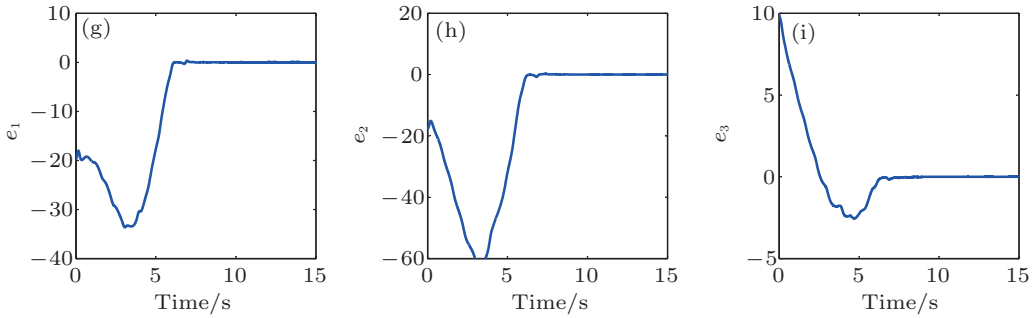


Fig. 4. (color online) Complete synchronization curves of the driving (x_t) and response (y_t) systems with uncertainty and white noise.

3.2. Simulation with $J = \mathrm{diag}(2,3,5)$

If $J = \mathrm{diag}(2,3,5)$, the problem is MPS. Figure 5 demonstrates the MPS error between the fractional-order Lorenz and Chen's systems. We can see that after a finite period of time, the error converges to be small enough when the control law is used. The trajectories of the driving and response systems are shown in Fig. 6, which also indicates that the two systems with uncertainty and disturbance are indeed MPS.

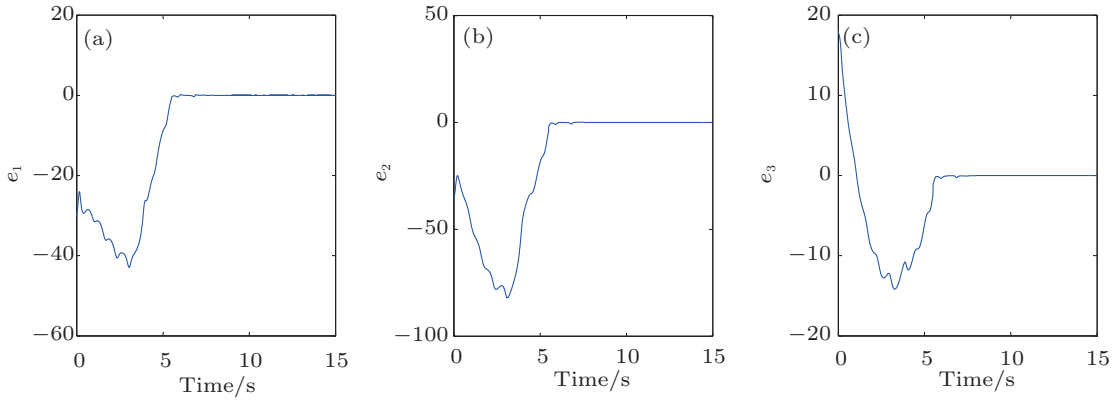


Fig. 5. (color online) MPS results of the driving and response systems with uncertainty and disturbance: (a) e_1; (b) e_2; (c) e_3.

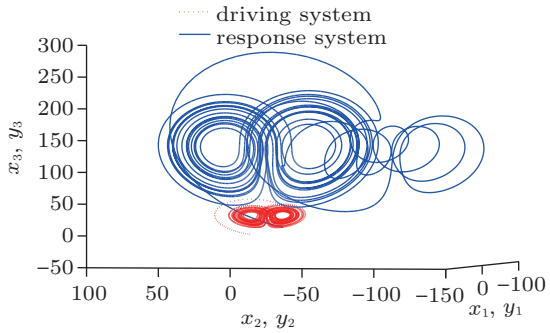


Fig. 6. (color online) MPS curves of the driving (x_t) and response (y_t) systems with uncertainty and disturbance.

4. Conclusions

In this paper, based on the fractional-order Lyapunov stability theory, a novel controller is proposed to achieve robust MPS of two fractional-order chaotic systems with parameter uncertainty and external disturbances. Simulation results for the synchronization of fractional-order Lorenz and Chen's systems show the applicability of the approach. In particular, the proposed synchronization method is applicable for a large class of fractional-order chaotic systems with parameter uncertainty and external disturbances.

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